J. Appl. Comput. Mech., 7(3) (2021) 1676-1686
DOI: 10.22055/JACM.2020.33790.2287

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

A Modified Thermoelastic Fractional Heat Conduction Model with a Single-Lag and Two Different Fractional-Orders

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Received May 31 2020; Revised June 20 2020; Accepted for publication June 20 2020.
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Abstract. Recently, fractional calculus theory has been successfully employed in generalized thermoelasticity theory and several models with fractional order have been introduced. In this work, a fractional thermoelastic modified Fourier's Law with phase lag and two different fractional-orders has been constructed. The previous fractional models of thermoelasticity introduced by Sherief et al. [1], Ezzat [2] and Lord and Shulman [3] as well as classical coupled thermoelasticity [4] follow as limiting cases. This proposed model is applied to an infinitely annular cylinder that is subject to time-dependent surface temperatures, and whose surfaces are free of traction. The method of the Laplace transform is employed to get the solutions of the field variables. A numerical technique is utilized to invert the Laplace transforms. Some results are presented in tables and figures to extract the effects of fractional order parameters on all variables studied. The theory's predictions have been checked and compared to previous models.

Keywords: Two fractional thermoelasticity, different order, time-dependent surface temperature, generalized thermoelasticity theory, annular cylinder.

1. Introduction

Since 1967, many theories and models have been developed and presented, are often called generalized theories of thermoelasticity [3-5]. The main objective of proposing these theories is to take care of the imperfection of the uncoupled and coupled thermoelasticity theories. The main objective of proposing these theories was to take care of the shortcomings of unconnected and associated thermal elasticity theories [6]. Recently, other attempts have been attained to modify the classic Fourier law to generalize previous models based on introducing higher-order derivatives into governing equations [7-11].

The branch of the fractional calculus (FC) is very important and is useful in describing the development of systems with memory, in which these systems are usually dispersed and complex. It can be said that the complete theory of fractional derivatives and fractional integrals is not established recently but was founded in the nineteenth century.

In the last few years, this important branch has been successfully applied in many fields of physical processes such as biopolymers, chemistry, porous materials, biology, semiconductors, biological cells, electronics, and viscoelasticity [12]. Padlubny's book [13] can be referred to as an important reference in this field and is also a survey of FC applications. In recent decades, the definition of the fractional derivative and fractional integration has been generalized in different approaches, and some different alternative concepts of the fractional derivative have been developed [14-17]. Based on the new heat-conduction model with the fractional derivative defined by Caputo is introduced by Povstenko [18] to study the thermoelastic interactions in an infinite body cylindrical cavity. Among the most important contributions are in the field of thermoelasticity with fractional orders were investigated in [19-26].

Over the past decades, several analytical/approximate methods have been developed to solve nonlinear ordinary and partial fractional differential equations. For initial and boundary-value problems in ordinary and partial differential equations, some of these techniques include the perturbation method [27, 28], the variational iteration method [29-33], residual power series method [34], expansion methods [35-49]. He's variational iteration method is based on the use of restricted variations and correction functional which has found a wide application for the solution of linear and nonlinear ordinary and partial differential equations, e.g., [33]. This method does not require the presence of small parameters in the differential equation and provides the solution (or an approximation to it as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives.

In the current work, a fractional heat conduction model with a single-phase lag is derived based on the concepts of the fractional calculus [50]–[52]. In the proposed model, the heat equation including two fractional parameters, unlike the previous models. This model introduced is employed to investigate thermoelastic interaction due to time-dependent varying heat in an annular cylinder.

Published online: March 26 2021

Some numerical values are presented in tables and figures to assess the influences of the fractional coefficients on all distributions of the physical fields.

2. Derivation of Fractional thermoelasticity with two fractional derivatives

In this section, we intend to derive a modified heat conduction model with two fractional derivatives of different orders. We will apply the Caputo time-fractional derivative definition [1] of fractional-order α, $(0<\alpha\leq 1)$ of any function $f(t)$ which is absolutely continuous is defined as

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}f(t)=I^{1-\alpha}f'(t) \tag{1}$$

where the operator I^α is defined by [53]

$$I^\alpha f(t)=\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)\mathrm{d}\tau, \tag{2}$$

and the function $f(t)$ satisfying the equation

$$\lim_{\alpha\to 1}\left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}f(t)\right)=f'(t). \tag{3}$$

The classical Fourier law is given by states the linear dependence between the heat flux vector $\boldsymbol{q}$ and the gradient of temperature $\nabla\theta$:

$$\boldsymbol{q}(\boldsymbol{x},t)=-K\nabla\theta(\boldsymbol{x},t) \tag{4}$$

where $\boldsymbol{q}$ denotes the heat flux vector, $\nabla\theta$ is the temperature gradient and K is thermal conductivity.
The mathematical model of Fourier's law (4) is in a parabolic type and its investigation shows that the disturbance of temperature propagates at infinite speed.
To overcome this contradiction, Cattaneo's [54] has improved the heat-conduction law to be in the form

$$q(\boldsymbol{x},t)+\tau_0\frac{\partial}{\partial t}q(\boldsymbol{x},t)=-K\nabla\theta(\boldsymbol{x},t) \tag{5}$$

The τ_0 relaxation time appearing. The energy equation is given by [6]

$$\rho C_E\frac{\partial\theta}{\partial t}+\gamma T_0\frac{\partial e}{\partial t}=-\nabla\cdot\boldsymbol{q}+Q \tag{6}$$

During the last decades, theories of non-classical thermoelasticity have been developed, where Eqs. (1) and (3) are replaced by more general and appropriate formulas. The generalization of the derivative and integral model of a non-integer order has been subjected to several methods and different alternative concepts have emerged for the derivative of the fractional order.
One of these generalizations was provided by Sherief et al. in [1] where the Fourier's law is replaced by the formula

$$q(\boldsymbol{x},t)+t_0\frac{\partial^\alpha}{\partial t^\alpha}q(\boldsymbol{x},t)=-K\nabla\theta(\boldsymbol{x},t),\quad (0<\alpha\leq 1) \tag{7}$$

Also, Ezzat [2] established another fractional heat conduction model using the definition proposed by Jumarie [17] as

$$\boldsymbol{q}(\boldsymbol{x},t)+\frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\boldsymbol{q}(\boldsymbol{x},t)=-K\nabla\theta(\boldsymbol{x},t) \tag{8}$$

The present work is an effort to construct a generalized model of thermoelasticity with fractional derivative. This modification depends on replacing the time derivative founding equation (2) with a fractional derivative. The resulting generalized heat conduction equation will be in the form:

$$\boldsymbol{q}(\boldsymbol{x},t)+\tau_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\boldsymbol{q}(\boldsymbol{x},t)=-K\nabla\theta(\boldsymbol{x},t),\ \ 0<\alpha\leq 1. \tag{9}$$

Most of the researchers did not address the energy equation of modification or development, but the advances they made were based on Fourier law. In this generalization as well, there will be another improvement, not by Fourier's law, but by modifying the energy equation. The time derivative in the energy equation (3) is replaced by a fractional derivative of a different order β. The modified energy equation with fractional order, in this case, has the form

$$\tau_1^{\beta-1}\frac{\partial^\beta}{\partial t^\beta}(\rho C_E\theta+\gamma T_0 e)=-\nabla\cdot\boldsymbol{q}+Q,\ \ 0<\beta\leq 1 \tag{10}$$

In the above equation, the parameter $\tau_1^{\beta-1}$ is presented to retain the matching dimensions.
Combining the two equations (9) and (10), we get the new heat conduction equation with two different fractional orders (**2FTE**) in the form

$$\left(1+\tau_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho C_E\tau_1^{\beta-1}\frac{\partial^\beta\theta}{\partial t^\beta}+\gamma T_0\tau_1^{\beta-1}\frac{\partial^\beta e}{\partial t^\beta}\right)=K\nabla^2\theta+\left(1+\tau_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)Q \tag{11}$$

In addition, the field equations, the constitutive relations and the train–displacement relation for thermoelastic isotropic materials at uniform environmental temperature T_0 are:

$$\sigma_{ij}=2\mu e_{ij}+\delta_{ij}\left[\lambda e_{ij}-\gamma\theta\right] \tag{12}$$

$$2e_{ij}=u_{j,i}+u_{i,j} \tag{13}$$

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ij} - \gamma\theta_{,i} + F_i = \rho\ddot{u}_i \tag{14}$$

Equation (11) and equation (14) constitute the whole system for governing equations for thermoelasticity theory with two fractional orders.

3. Special cases

3.1 Fractional thermoelasticity model suggested by Sherief et al. (SFTE)[1]:

In the limiting case $\beta \to 1$ and $t_0 = \tau_0^\alpha$, the fractional heat equations of the SFTE-theory can be obtained as

$$\left(1 + t_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho C_E\frac{\partial\theta}{\partial t} + \gamma T_0\frac{\partial e}{\partial t}\right) = K\nabla^2\theta + \left(1 + t_0\frac{\partial^\alpha}{\partial t^\alpha}\right)Q \tag{15}$$

3.2 Fractional order thermoelasticity theory suggested by Ezzat (EFTE) [2]:

The fractional heat equation (11), when $\beta \to 1$ and $t_0 = \sqrt[\alpha]{\alpha!\,\tau_0^\alpha}$ converts to **EFTE** model as

$$\left(1 + \frac{t_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho C_E\frac{\partial\theta}{\partial t} + \gamma T_0\frac{\partial e}{\partial t}\right) = K\nabla^2\theta + \left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)Q \tag{16}$$

3.3 Lord-Shulman theory of generalized thermoelasticity (LS) [3]:

If $\beta \to 1$ and $\alpha \to 1$ in the Eq. (11), we get the heat equation in the context of **LS** –theory in the form

$$\left(1 + t_0\frac{\partial}{\partial t}\right)\left(\rho C_E\frac{\partial\theta}{\partial t} + \gamma T_0\frac{\partial e}{\partial t}\right) = K\nabla^2\theta + \left(1 + t_0\frac{\partial}{\partial t}\right)Q \tag{17}$$

3.4 Classical thermoelasticity theory (CTE) [4]

If $\tau_0 = 0$ and $\beta \to 1$, the equation (11) reduces to classical heat conduction equation **(CTE)**

$$C_E\frac{\partial\theta}{\partial t} + \gamma T_0\frac{\partial e}{\partial t} - Q = K\nabla^2\theta \tag{18}$$

4. Application to the modified model

The Now to investigate and verify the new model we will study a boundary value problem. We study thermoelastic vibration in a hollow cylinder of inner r_1 and outer radii r_2 respectively.
The boundary and initial conditions are assumed to be:

- The surfaces are subjected to time-dependent or constant surrounding temperature.
- The surfaces are traction free.
- The initial conditions are supposed to be quiescent.
- There are no body forces or external heat sources applied to the cylinder.

The cylindrical polar coordinates (r,ϑ,z) are used. Due to the cylindrical symmetry, the field variables are considered be depending only on the radial r and the instant time t.
The displacement vector has the components

$$u_r = u(r,t), \qquad u_\vartheta(r,t) = u_z(r,t) = 0 \tag{19}$$

The non- vanishing components of strain are

$$e_{rr} = \frac{\partial u}{\partial r}, \qquad e_{\vartheta\vartheta} = \frac{u}{r} \tag{20}$$

The constitutive equations (12) have the forms

$$\begin{aligned}\sigma_{rr} &= 2\mu\frac{\partial u}{\partial r} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - \gamma\theta\\ \sigma_{\vartheta\vartheta} &= 2\mu\frac{u}{r} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - \gamma\theta\end{aligned} \tag{21}$$

The motion equation (14) will be in the form

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\vartheta\vartheta}) = \rho\frac{\partial^2 u}{\partial t^2} \tag{22}$$

Using Eq. (16), the motion equation (17) can be written as

$$(\lambda + 2\mu)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - \gamma\frac{\partial\theta}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2} \tag{23}$$

The fractional heat conduction (11) can be reduced to

$$\left(1 + \tau_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho C_E\tau_1^{\beta-1}\frac{\partial^\beta\theta}{\partial t^\beta} + \gamma T_0\tau_1^{\beta-1}\frac{\partial^\beta}{\partial t^\beta}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\right) = K\nabla^2\theta \tag{24}$$

We introduce the following non-dimensional quantities:

$$\begin{gathered}\{R, U, r_1, r_2\} = c_0\eta\{r, u, R_1, R_2\}, \quad \{\tau, t_0, t_1\} = c_0^2\eta\{\tau, \tau_0, \tau_1\},\\ \Theta = \frac{\theta}{T_0}, \quad \Sigma_{ij} = \frac{\sigma_{ij,}}{\mu}, \quad \eta = \frac{\rho C_E}{K}, \quad c_0 = \sqrt{\frac{\lambda + 2\mu}{\rho}}.\end{gathered} \tag{25}$$

Using Eq. (20), Eqs. (16), (18) and (19) in the non-dimensional forms will be in the forms

$$a^2\frac{\partial}{\partial R}\left(\frac{\partial U}{\partial R}+\frac{U}{R}\right)-b\frac{\partial\Theta}{\partial R}=a^2\frac{\partial^2 U}{\partial\tau^2} \tag{26}$$

$$\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}\right)\Theta=\left(1+t_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(t_1^{\beta-1}\frac{\partial^\beta\Theta}{\partial\tau^\beta}+gt_1^{\beta-1}\frac{\partial^\beta}{\partial\tau^\beta}\left(\frac{\partial U}{\partial R}+\frac{U}{R}\right)\right) \tag{27}$$

$$\begin{aligned}\Sigma_{rr}&=a^2\frac{\partial U}{\partial R}+(a^2-2)\frac{U}{R}-b\Theta\\ \Sigma_{\vartheta\vartheta}&=a^2\frac{U}{R}+(a^2-2)\frac{\partial U}{\partial R}-b\Theta\end{aligned} \tag{28}$$

where

$$a^2=\frac{\lambda+2\mu}{\mu},\quad g=\frac{\gamma^2T_0}{\rho^2C_Ec_0^2},b=\frac{\gamma T_0}{\mu}. \tag{29}$$

Now we can define the potential function φ by

$$U=\frac{\partial\varphi}{\partial R} \tag{30}$$

Introducing the function φ into Eqs. (21)-(23), we get

$$a^2\left(\nabla^2-\frac{\partial^2}{\partial\tau^2}\right)\varphi=b\Theta \tag{31}$$

$$\left(\nabla^2-t_1^{\beta-1}\left(1+t_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^\beta}{\partial\tau^\beta}\right)\Theta=gt_1^{\beta-1}\left(1+t_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^\beta}{\partial\tau^\beta}\nabla^2\varphi \tag{32}$$

$$\begin{aligned}\Sigma_{rr}&=a^2\frac{\partial^2\varphi}{\partial R^2}+\frac{(a^2-2)}{R}\frac{\partial\varphi}{\partial R}-b\Theta\\ \Sigma_{\vartheta\vartheta}&=(a^2-2)\frac{\partial^2\varphi}{\partial R^2}+\frac{a^2}{R}\frac{\partial\varphi}{\partial R}-b\Theta\end{aligned} \tag{33}$$

5. Initial and boundary conditions

The system of governing equations has to be solved subject to the homogeneous initial conditions. The initial conditions are assumed to be

$$U(R,0)=\frac{\partial U(R,0)}{\partial R}=0,\quad \Theta(R,0)=\frac{\partial\Theta(R,0)}{\partial R}=0 \tag{34}$$

- The heat flow and convection boundary conditions can be expressed as:

$$q(R_1,\tau)=h_1(f_1-\Theta),\ \text{at}\quad R=R_1,\ \tau>0 \tag{35}$$

$$q(R_2,\tau)=h_2(\Theta-f_2),\ \text{at}\quad R=R_2,\ \tau>0 \tag{36}$$

where h_1 and h_2 are constant coefficients of surface heat transfer of inner and outer surroundings sometimes known as a Biot number and f_1 and f_2 inner and outer surrounding temperatures.

- The surfaces of the annular cylinder are subjected to traction free conditions i.e.

$$\Sigma_{rr}(R,\tau)=0\quad\text{at}\quad R=R_1,R_2,\ \ \tau>0. \tag{37}$$

6. Solution in the transformed domain

Applying the technique of the Laplace transform, Eqs. (20)-(23) can be converted to

$$a^2(\nabla^2-s^2)\bar{\varphi}=b\bar{\Theta} \tag{38}$$

$$(\nabla^2-\omega)\bar{\Theta}=g\omega\nabla^2\bar{\varphi} \tag{39}$$

$$\begin{aligned}\bar{\Sigma}_{rr}&=a^2\frac{\mathrm{d}^2\bar{\varphi}}{\mathrm{d}R^2}+\frac{(a^2-2)}{R}\frac{\mathrm{d}\bar{\varphi}}{\mathrm{d}R}-b\bar{\Theta}\\ \bar{\Sigma}_{\vartheta\vartheta}&=(a^2-2)\frac{\mathrm{d}^2\bar{\varphi}}{\mathrm{d}R^2}+\frac{a^2}{R}\frac{\mathrm{d}\bar{\varphi}}{\mathrm{d}R}-b\bar{\Theta}\end{aligned} \tag{40}$$

where $\omega=t_1^{\beta-1}s^\beta(1+t_0^\alpha s^\alpha)$. By eliminating $\bar{\Theta}$ from the Eqs. (33), and (34), we get:

$$(\nabla^2-m_1^2)(\nabla^2-m_2^2)\bar{\varphi}=0 \tag{41}$$

where m_1^2 and m_2^2 are the roots of the following equation

$$m^4-\left[\omega\left(1+\frac{gb}{a}\right)+s^2\right]m^2+s^2\omega=0 \tag{42}$$

Solving the differential equation (36), the general solution for $\bar{\varphi}$ can be written as

$$\bar{\varphi}=\sum_{i=1}^{2}[A_i I_0(m_i R)+B_i K_0(m_i R)] \tag{43}$$

where $I_0(z)$ and $K_0(z)$ are the well-known modified Bessel functions and A_i, B_i $(i=1,2)$ are integral parameters.
From Eq. (38) and Eq. (33), we get

$$\bar{\Theta}=\frac{a^2}{b}\sum_{i=1}^{2}(m_i^2-s^2)[A_i I_0(m_i R)+B_i K_0(m_i R)] \tag{44}$$

It follows from Eq. (38) and Eq. (25) that

$$\bar{U}=\sum_{i=1}^{2}[A_i m_i I_1(m_i R)-B_i m_i K_1(m_i R)] \tag{45}$$

Introducing the functions $\bar{\varphi}$ and $\bar{\Theta}$ into Eq. (35), we have

$$\bar{\Sigma}_{rr}=\sum_{i=1}^{2}\left\{A_i\left[a^2 s^2 I_0(m_i R)-\frac{2m_i}{R}I_1(m_i R)\right]+B_i\left[a^2 s^2 K_0(m_i R)+\frac{2m_i}{R}K_1(m_i R)\right]\right\} \tag{46}$$

$$\begin{aligned}\bar{\Sigma}_{\vartheta\vartheta}=\sum_{i=1}^{2}\Big\{&A_i\left[(a^2 s^2-2m_i^2)I_0(m_i R)+\frac{2m_i}{R}I_1(m_i R)\right]+\\ &B_i\left[(a^2 s^2-2m_i^2)K_0(m_i R)-\frac{2m_i}{R}K_1(m_i R)\right]\Big\}\end{aligned} \tag{47}$$

The boundary conditions (32) in the Laplace transform domain have the form

$$\bar{\Sigma}_{rr}(R,s)=0 \quad \text{at} \quad R=R_1,R_2 \tag{48}$$

To convert the boundary conditions (30) and (31), we will employ the modified Fourier's Law of heat conduction (6) in non-dimensional form, namely

$$\left(1+t_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)q_r=-\frac{\mathrm{d}\bar{\Theta}}{\mathrm{d}R} \tag{49}$$

Then the non-dimensional boundary conditions (30) and (31) in the Laplace transform domain have the forms

$$\frac{\mathrm{d}\bar{\Theta}}{\mathrm{d}R}=h_1\left(1+t_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)(\bar{\Theta}-f_1), \quad \text{at} \quad R=R_1 \tag{50}$$

$$\frac{\mathrm{d}\bar{\Theta}}{\mathrm{d}R}=h_2\left(1+t_0^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)(f_2-\bar{\Theta}), \quad \text{at} \quad R=R_2 \tag{51}$$

Substituting from Eqs. (39) and (40) into the boundary conditions (42), (44) and (45), we get a linear system of equations. Solving this system, we can obtain the constants A_i, B_i, $(i=1,2)$.

To obtain inverse transformations for different fields, an efficient and accurate numerical technique will be used to obtain the Laplace transform inversion [39]. In this method, any transformed function in the Laplace field can be reversed to the real domain using the formula

$$g(R,\tau)=\frac{e^{c\tau}}{\tau}\left(\frac{1}{2}\bar{g}(R,c)+Re\sum_{n=1}^{N_f}\bar{g}\left(R,c+\frac{in\pi}{\tau}\right)(-1)^n\right) \tag{52}$$

where N_f is a finite number. The parameter c satisfies $c\tau\cong 4.7$ in order to achieve the fastest convergence [55].

7. Numerical results

The effect of the time-fractional orders α and β on the thermoelastic materials has been investigated in the previous sections. The copper material was chosen for the numerical discussion purposes. The material constants of the copper material are [11]

$$\lambda=7.76\times10^{10}\text{kg m}^{-1}\text{s}^{-2},\ \ \mu=3.86\times10^{10}\text{kg m}^{-1}\text{s}^{-2},\ \ \rho=8954\text{ kg m}^{-3},$$
$$K=386\text{ W m}^{-1}\text{K}^{-1},\ \ C_E=3.381\text{ J kg K}^{-1},\ \ \varepsilon=0.0168,\ \ \beta=2.$$

For computational purposes, we take the functions f_1 and f_2 as [56]

$$f_1(R,t)=1.5+0.5\cos(13t),\ \ f_2(R,t)=1.0 \tag{53}$$

The numerical results of the studied fields for different values of fractional-order parameters α,β and also for the instant time τ are introduced along the radial distance R. Using the numerical procedure given in [55], the distributions of temperature Θ and displacement U as well as the stresses Σ_{rr} and $\Sigma_{\vartheta\vartheta}$ are acquired and explained in Tables 1–4 and Figures 1–8, respectively.

Case I: The effect of the time-fractional order parameters

The influence of the parameters of fractional orders α and β on the thermoelastic material has been investigated in the previous case. As mentioned earlier in section 3, some models submitted in this field can be obtained from the new model (**2FTE**) as special cases (**CTE, LS, SFTE, ESTE**).

Table 1. The effect of fractional order parameters α and β on Θ

R	CTE	LS	SFTE	EFTE	MFTE $((0<\alpha<1),\ (0<\beta<1))$				
		$\beta=1$ $\alpha=1$	$\beta=1$ $\alpha=0.5$	$\beta=1$ $\alpha=0.5$	$\beta=0.75$ $\alpha=0.75$	$\beta=0.5$ $\alpha=0.5$	$\beta=0.3$ $\alpha=0.3$	$\beta=0.75$ $\alpha=0.5$	$\beta=0.75$ $\alpha=0.3$
1	0.248149	0.094586	0.1685130	0.134861	0.1397720	0.212465	0.320537	0.1644690	0.175283
1.1	0.154216	0.0318511	0.085273	0.0591014	0.0714837	0.147784	0.269433	0.0913432	0.112582
1.2	0.10384	0.0111101	0.0460871	0.0272559	0.0392313	0.113389	0.244039	0.0545810	0.078647
1.3	0.0790839	0.00412814	0.0279088	0.0137675	0.0242768	0.098087	0.235317	0.0363102	0.061492
1.4	0.0732624	0.00181642	0.0211817	0.0091635	0.0189580	0.097946	0.240393	0.0295481	0.057276
1.5	0.0849577	0.0020704	0.0216923	0.0098516	0.0196291	0.111650	0.258017	0.0325825	0.065671
1.6	0.116175	0.00350852	0.0302073	0.0145813	0.0256098	0.139393	0.287771	0.0479078	0.088394
1.7	0.170966	0.0101339	0.0566096	0.025980	0.0449633	0.182431	0.329809	0.0812102	0.127879
1.8	0.254171	0.0254428	0.116796	0.0632962	0.0941547	0.242772	0.384691	0.1385910	0.186087
1.9	0.370381	0.0538702	0.217712	0.149632	0.1790790	0.322425	0.453210	0.2243780	0.265076
2.0	0.518045	0.194144	0.357339	0.280393	0.2944410	0.420842	0.536256	0.3394540	0.365007

Table 2. The effect of fractional order parameters α and β on U

R	CTE	LS	SFTE	EFTE	MFTE $((0<\alpha<1),\ (0<\beta<1))$				
		$\beta=1$ $\alpha=1$	$\boldsymbol{\beta=1}$ $\boldsymbol{\alpha=0.5}$	$\beta=1$ $\alpha=0.5$	$\beta=0.75$ $\alpha=0.75$	$\beta=0.5$ $\alpha=0.5$	$\beta=0.3$ $\alpha=0.3$	$\beta=0.75$ $\alpha=0.5$	$\beta=0.75$ $\alpha=0.3$
1	-0.485747	-0.0270063	-0.235478	-0.141843	-0.214805	-0.70828	-1.09578	-0.33112	-0.54947
1.1	-0.0611974	-0.00273262	-0.0291744	-0.0172706	-0.0267242	-0.0903011	-0.140289	-0.0416189	-0.069913
1.2	-0.0071778	-0.000100989	-0.0032136	-0.001801	-0.0029942	-0.0111571	-0.0177556	-0.0048523	-0.0085589
1.3	-0.0006367	6.96376E-05	-0.0001874	-6.39E-05	-0.0001972	-0.0012536	-0.0022552	-0.0004099	-0.0009187
1.4	-0.0001387	2.41065E-05	-1.259E-05	1.532E-05	-1.621E-05	-0.0002509	-0.0005537	-5.74E-05	-0.0001662
1.5	-0.0004102	-2.64353E-05	-0.0001858	-9.823E-05	-0.000158	-0.0003973	-0.000638	-0.0002173	-0.0002971
1.6	-0.0008347	-0.000112003	-0.000448	-0.0002817	-0.0003764	-0.0007238	-0.0009656	-0.0004823	-0.0005722
1.7	-0.0013703	-0.000256837	-0.0008187	-0.0005719	-0.0006756	-0.0011104	-0.0013045	-0.000844	-0.0009145
1.8	-0.0017804	-0.000624486	-0.0012275	-0.0009329	-0.0009922	-0.0013146	-0.0012677	-0.0011923	-0.0011491
1.9	0.0003193	-0.000315496	-0.0002541	-0.0003707	-0.0001853	0.0006561	0.0019042	-5.392E-05	0.0003294
2	0.0232269	0.00631432	0.0143314	0.0106335	0.0115863	0.0196321	0.0282616	0.0144075	0.0156911

Table 3. The effect fractional order parameters α and β on Σ_{rr}

R	CTE	LS	SFTE	EFTE	MFTE $((0<\alpha<1),\ (0<\beta<1))$				
		$\boldsymbol{\beta=1}$ $\boldsymbol{\alpha=1}$	$\beta=1$ $\alpha=0.5$	$\beta=1$ $\alpha=0.5$	$\beta=0.75$ $\alpha=0.75$	$\beta=0.5$ $\alpha=0.5$	$\beta=0.3$ $\alpha=0.3$	$\beta=0.75$ $\alpha=0.5$	$\beta=0.75$ $\alpha=0.3$
1	0	0	0	0	0	0	0	0	0
1.1	-0.0030095	-1.61222E-05	-0.0005985	-0.0001591	-0.0005218	-0.0051488	-0.0171114	-0.0010836	-0.0025233
1.2	-0.0062232	-0.000030257	-0.0013616	-0.0003834	-0.0012059	-0.010376	-0.0323231	-0.0022398	-0.0050615
1.3	-0.010571	-0.000126929	-0.0025462	-0.0008244	-0.0023005	-0.0167236	-0.0471996	-0.0037747	-0.0083007
1.4	-0.0172606	-0.00024148	-0.0043951	-0.0017496	-0.004027	-0.0251548	-0.0628869	-0.0061975	-0.0131888
1.5	-0.0279948	-0.000858765	-0.0072352	-0.0035236	-0.0065566	-0.036662	-0.0803081	-0.0106068	-0.0210585
1.6	-0.0449907	-0.00181437	-0.0124761	-0.0064977	-0.0104813	-0.0522884	-0.100235	-0.0190741	-0.0335296
1.7	-0.0705162	-0.00524664	-0.0247529	-0.012084	-0.0195663	-0.0729321	-0.123068	-0.0344343	-0.0519816
1.8	-0.104525	-0.0120347	-0.0495488	-0.027522	-0.03977	-0.0977021	-0.146839	-0.0579686	-0.0757495
1.9	-0.129441	-0.0152048	-0.0762575	-0.0522999	-0.0624637	-0.111822	-0.152682	-0.0786064	-0.0920319
2	0	0	0	0	0	0	0	0	0

Table 4. The effect of fractional order parameters α and β on $\Sigma_{\vartheta\vartheta}$

R	CTE	LS	SFTE	EFTE	MFTE $((0<\alpha<1),\ (0<\beta<1))$				
		$\beta=1$ $\alpha=1$	$\beta=1$ $\alpha=0.5$	$\beta=1$ $\alpha=0.5$	$\beta=0.75$ $\alpha=0.75$	$\beta=0.5$ $\alpha=0.5$	$\beta=0.3$ $\alpha=0.3$	$\beta=0.75$ $\alpha=0.5$	$\beta=0.5$ $\alpha=0.75$
1	-0.0068788	-0.00001148	-0.0008853	-0.0001984	-0.0008218	-0.0125104	-0.0564041	-0.0019323	-0.0064777
1.1	-0.0067542	-1.82196E-05	-0.0009906	-0.0002318	-0.0008938	-0.0125383	-0.0549008	-0.0020347	-0.0061883
1.2	-0.0082393	-2.82375E-05	-0.0015096	-0.000396	-0.0013533	-0.0147655	-0.0579542	-0.0026817	-0.007128
1.3	-0.0115902	-0.000113954	-0.0025484	-0.0007933	-0.0023142	-0.0193076	-0.0647373	-0.0039291	-0.0094279
1.4	-0.0176556	-0.000215484	-0.0042736	-0.0016597	-0.0039303	-0.0266101	-0.0749453	-0.0061575	-0.013733
1.5	-0.0279345	-0.000768718	-0.0069662	-0.0033274	-0.0063338	-0.0373685	-0.0885987	-0.0103899	-0.0212024
1.6	-0.0445301	-0.00162784	-0.0120035	-0.006118	-0.0100975	-0.0524638	-0.105915	-0.0186463	-0.0333593
1.7	-0.0697762	-0.00477953	-0.0240391	-0.0114756	-0.0190125	-0.0728202	-0.127093	-0.0337953	-0.0516254
1.8	-0.104833	-0.0114471	-0.0492619	-0.0271229	-0.0396053	-0.0985165	-0.151206	-0.0578764	-0.0761901
1.9	-0.141607	-0.018923	-0.0836503	-0.0575085	-0.0686611	-0.122354	-0.168549	-0.0860925	-0.100906
2	-0.104351	-0.0396054	-0.0724416	-0.0569626	-0.059718	-0.0843141	-0.105912	-0.0686349	-0.0735342

To validate the obtained results, a comparison will be introduced between the different models of thermoelasticity. It is useful to organize the numerical results in figures and tabular forms to explain the comparisons between different fractional thermoelastic models. Also, the obtained results will be introduced in a tabular form to aid other investigators to compare and verify their results. Numerical results are presented in Tables (1-4) as well as graphically in Figures (1-4) for different distances R.

From the Tables, it is clear that the parameters of the fractional differentiation α and β have a clear influence on the distributions of the different fields. By checking also in the tables, we find that there is a difference in the values of the different physical distributions for all different models of thermoelasticity [57]. Although the values differ in the case of different models, there are similarities in the behavior of the distributions, and this is evident from the Tables (1-4) and figures (1-4). By comparing the results between different models, it can be theoretically said that there is no model of thermoelasticity with fractional order that can be more accurate than others.

From Table 1 and Fig. 1 we can see that:

- By increasing the parameters of fractional orders α and β together, the temperature Θ distribution values decrease.
- When one of the parameters of fractional order is constant and the other parameter increases, we find a decrease in the temperature distribution [45].
- The values of temperature at both boundaries of the hollow cylinder do not vanish due to the presence of time-dependent heat flow on both surfaces.
- The temperature values in the case of the modified model with two fractional orders (**2FTE** model) are large compared to other models of thermoelasticity with one fractional order (**SFTE** and **ESTE**).

Table 2 and Fig. 2 display the distribution of displacement u for different values of the fractional order parameters α and β along the radius of the cylinder. By noticing the numerical values, we can find that:

- The displacement begins with negative values from the inner surface that is affected by time-dependent heat flow, and then gradually increases to values approaching zero on the outer surface that is affected by a constant heat flow [58].
- The displacement is strongly affected as the fractional differential orders change.
- The displacement values at the inner boundary of the hollow cylinder in the SFTE and ESTE problems are greater than that of the 2FTE problem (see Fig. 2).

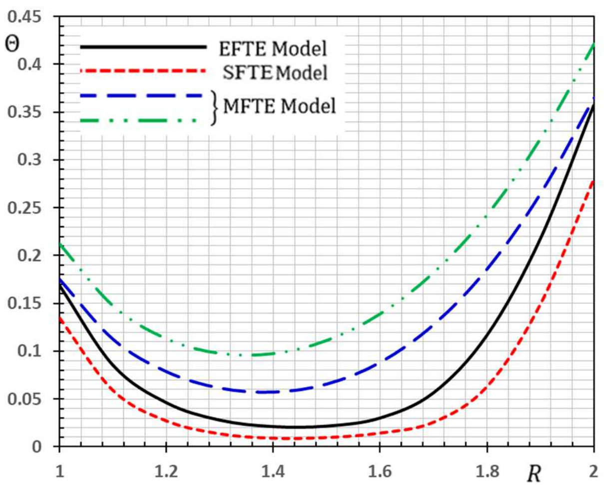

Fig. 1. The distribution of temperature Θ

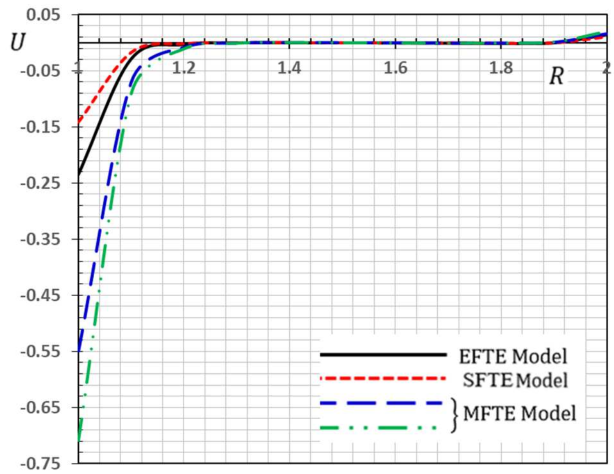

Fig. 2. The distribution of displacement U

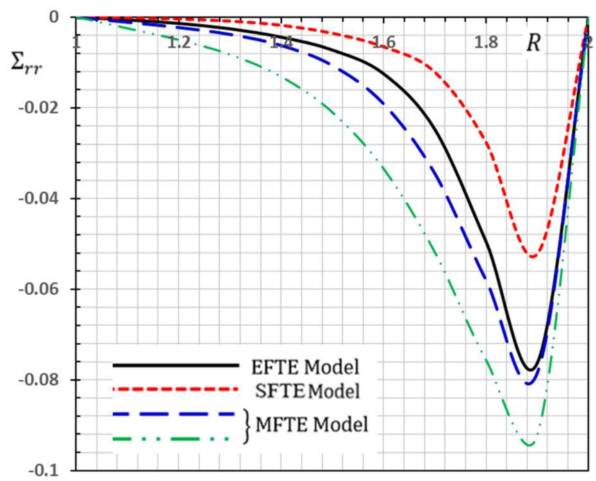

Fig. 3. The distribution of the stress Σ_{rr}

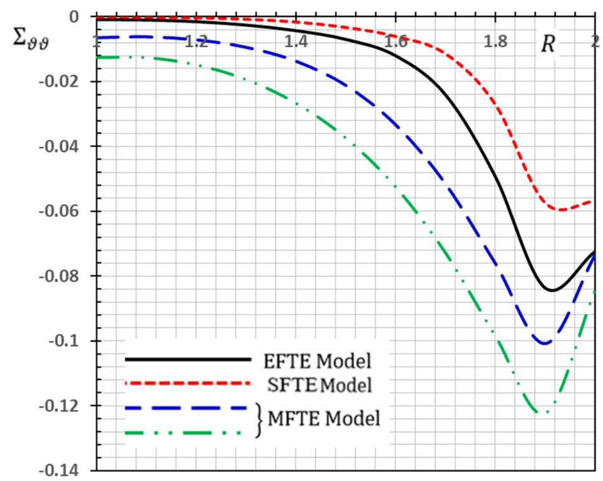

Fig. 4. The distribution of the stress $\Sigma_{\vartheta\vartheta}$

The distribution of the thermal stress Σ_{rr} in the hollow cylinder is shown in Table 3 and Fig. 3, respectively. From the thermal stress profile, we can see that:

- The thermal stress Σ_{rr} meets the boundary conditions (stress-free) Table 3 and Fig. 2 and has a coincident starting and vanishing points at zero for all models.
- The magnitude of the stress Σ_{rr} gradually increases slowly from the side of the inner surface until it reaches its largest value near the outer surface and then rapidly decreases to zero [59].
- With different values in the case of different models of thermoelasticity, the apparent influence of parameters of fractional orders α and β on the thermal stress Σ_{rr} is evident.
- There is a clear difference in Σ_{rr}values in different theories with fractional order and the amplitude of the thermal stress in the new model (**2FTE**) is larger than that in the **SFTE** and **ESTE** models.

Table 4 and Fig. 4 discuss the effect of fractional order parameters on the distributions of the thermal stress $\Sigma_{\xi\xi}$ against the radial distance R. It can be seen from the numerical results that the behavior of thermal stress $\Sigma_{\xi\xi}$ exhibits the same behavior as thermal stress Σ_{rr}, but it differs from it only at the starting point.

Case II: Effects of time instant

In this last case, we will study the influence of the time instant on the various distributions studied. To study the discussion of the effect more clearly on physical distributions, we will present a set of three-dimensional Figures (see Figs. 5-8). The change will be along with the radius r $(1 \le r \le 2)$ and also over the time τ $(0.05 \le \tau \le 0.15)$ when fractional order parameters α and β are fixed.

The study will also be in the case of using the new model of thermoelasticity with two fractional orders (**2FTE**). Through the Figs. 5-8 it is evident that the variations of the temperature, the thermal stresses and the displacement fields are very responsive to the time instant change τ. It was also detected that the time instant plays a vital role in formatting various physical fields. The influence increases with increasing time and it is more significant in the distributions of displacement and temperature as compared to the thermal stresses in the cylinder. It is detected from Fig. 5 that the distribution of temperature Θ at any fixed distance r increases with increasing time τ. Through Fig. 6, it was seen that the displacement distribution has different distortions when time and distance change. Also from Figs. 7 and 8 it is observed that the instant time τ causes an increase in the thermal stresses Σ_{rr} and $\Sigma_{\xi\xi}$. Figure 7 displays that the radial pressure Σ_{rr} meets the stress-free condition at $R = R_1$ and $R = R_2$ and has a behavior that may be different.

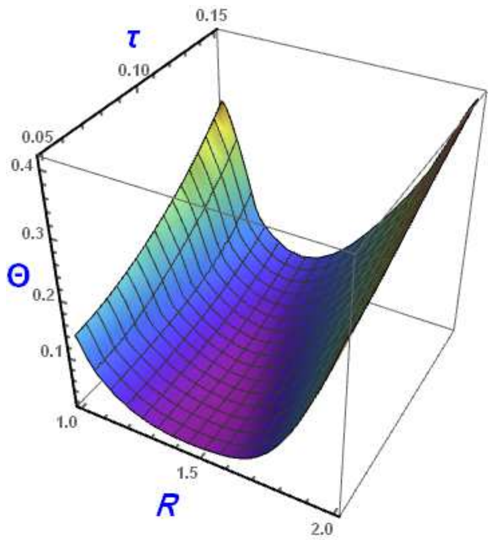

Fig. 5. The distribution of temperature Θ

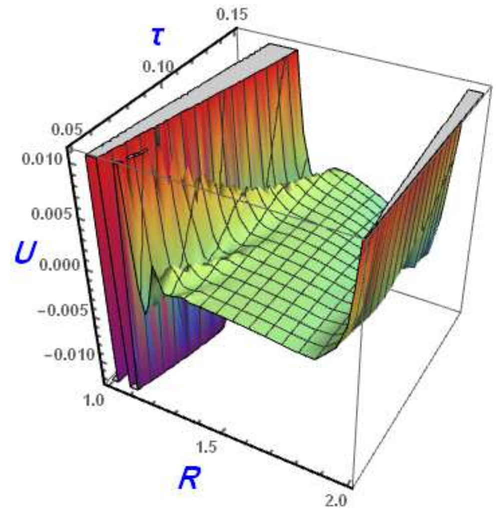

Fig. 6. The distribution of displacement U

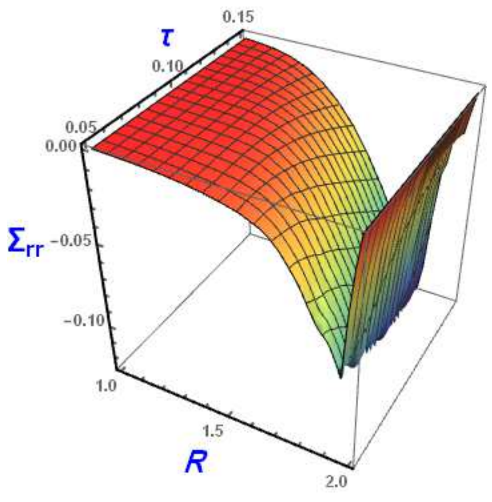

Fig. 7. The distribution of the stress Σ_{rr}

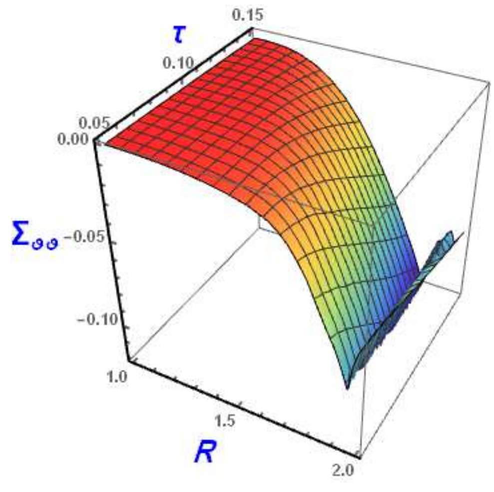

Fig. 8. The distribution of the stress $\Sigma_{\vartheta\vartheta}$

8. Conclusion

The present article is an effort to construct a modified novel model of fractional thermoelasticity by performing a fractional calculus. The heat conduction equation has been developed based on the Riemann-Liouville fractional integral operators. To clarify the presented model, a problem was examined for a hollow cylinder due to ambient time-dependent temperatures and traction free surfaces. The Laplace transform technique is employed to solve the system of the governing equations. Laplace inverse transformations are performed using a numerical approach technique. Some previous models and some earlier results can also be inferred from the current investigation as special cases. The influence of changing the parameters of fractional order α and β on the studied fields has been studied. To investigate the influences of the parameters of the fractional-order and instant time, some comparisons between different models are tabulated and illustrated graphically. From the discussions, it is clear that the parameters of the fractional order have a strong influence on the physical fields. The discussed fractional heat conduction model with a single-phase lag can be derived based on the concepts of the fractal calculus [51, 60]. It is clear from the study presented and the discussions above that the proposed model and the numerical results in this research are very significant for scientists, engineering and researchers, especially working in the thermodynamics and solid mechanics as well as thermoelasticity fields.

Nomenclature

λ, μ	Lam´e's constants	K	thermal conductivity
α_t	thermal expansion coefficient	∇^2	Laplacian operator
$\gamma = (3\lambda + 2\mu)\alpha_t$	coupling parameter	t	the time
T_0	environmental temperature	ρ	material density
$\theta = T - T_0$	temperature increment	u_i	displacement vector
T	absolute temperature	F_i	body force vector
C_E	specific heat	Q	heat source
σ_{ij}	stress tensor	τ_0, τ_1	relaxation times
$e = u_{k,k}$	dilatation	α	fractional derivative of order
e_{ij}	strain tensor	δ_{ij}	Kronecker's delta
R_1	inner radius	R_2	outer radius
f_1	inner surrounding temperature	f_2	outer surrounding temperature
h_1	dimensionless heat transfer	h_2	dimensionless heat transfer

Author Contributions

A.E. Abouelregal planned the scheme, initiated the project, suggested the problems and analyzed the empirical results developed the mathematical modeling and examined the theory validation; H. Ahmad writing—review, editing and performed formal analysis. The manuscript was written through the contribution of both the authors. Both the authors discussed the results, reviewed and approved the final version of the manuscript.

Acknowledgments

The authors thank the Editor-in-Chief and unknown referees for the fruitful comments that helped them in improving the quality and readability of the paper, which led to a significant improvement of the paper.

Conflict of Interest

Both authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

The authors received no financial support for the research, authorship and publication of this article.

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How to cite this article: Abouelregal A.E., Ahmad H. A Modified Thermoelastic Fractional Heat Conduction Model with a Single-Lag and Two Different Fractional-Orders, *J. Appl. Comput. Mech.*, 7(3), 2021, 1676–1686.
https://doi.org/10.22055/JACM.2020.33790.2287