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Technical Brief

A Simple Approach for Dealing with Autonomous Conservative Oscillator under Initial Velocity

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Abstract. The current study is involved to analytical solution of nonlinear oscillators under initial velocity. By using energy conservation principle, system initial condition converts to condition which oscillator's velocity become zero. When oscillator's speed is zero and placed out of movement's origin, the relation between frequency and amplitude could be extracted. By paying attention to energy conservation principle and relation between the initial velocity and amplitude, the frequency-amplitude relation is extended to frequency-initial velocity relationship. In order to demonstrate the effectiveness of proposed method, Duffing oscillator with cubic nonlinearity and oscillator with discontinuity are considered. Comparison of results with numerical solution shows good agreement. The proposed method is simple and efficient enough to achieve the analytical approximation of nonlinear autonomous conservative oscillator with initial velocity.

Keywords: Nonlinear oscillator, Analytical solution, initial velocity, Duffing oscillator.

1. Introduction

Nonlinear factors usually appear in the mathematical modeling of different physical systems. In some cases, the nonlinear factor effect is neglected and the governing equations is linearized and analyzed with acceptable accuracy. A well-known example in this context is pendulum oscillation with small amplitude which considers linear approximation and has an appropriate accuracy. In some cases, like dynamics of systems and structures with large amplitude, the nonlinear factor effect is huge and can't be neglected. In solid mechanics, the large deformations and strains amplifies the nonlinear behavior of such systems. For example, large amplitude vibration of beams with various conditions due to the axial load or boundary conditions [1-8], oscillation of bifilar pendulum [9], fluid flow induced vibration [10], tsunami motion modeling [11], static and dynamic analysis of micro and nano scale structures [12-15] and different kinds of nonlinear systems [16-22] are affected by the nonlinear factors and their governing equations are nonlinear.

Different methods are currently used for explicit solution of nonlinear equations. Some of them are only applicable to solve the governing equations of conservative system. Some methods like differential transform method [23, 24] and Adomian decomposition method [25] have no limitation in the equation form but give solution in Taylor series form which sometimes needs very much terms for accurate solution and so the employment of such solution is not beneficial.

Several methods have been proposed to obtain the analytical solution of governing equations in nonlinear autonomous/conservative systems in physics and mechanic which have oscillatory behavior. The methods are applicable when the initial condition is in the form of displacement and give an explicit relation between the frequency and amplitude, however, are not applicable in systems with initial velocity conditions. For instance, the energy balance method [26-29], Hamiltonian approach [30], Max-Min approach [31], He's frequency formulation [32] and other similar approaches [33-39] are used to derive the frequency-amplitude relation. There are some methods to solve the nonlinear equations under desired initial conditions but the solution is in the form of time series and hence don't give the explicit relation between the frequency and initial conditions of nonlinear systems.

In this research, by using the energy conservation principle in autonomous conservative systems, the initial condition from velocity type with a time shift is changed to the initial displacement form and then other methods are employed to approximate the relation between the frequency and initial condition. In other words, the proposed method converted the initial condition from velocity type to initial amplitude one and then takes the advantageous of previous methods to extract the nonlinear frequency of motion. The accuracy of this method is also verified by studying well-known nonlinear problems.

2. The Basic Idea

Consider the autonomous conservative nonlinear oscillator as follows:

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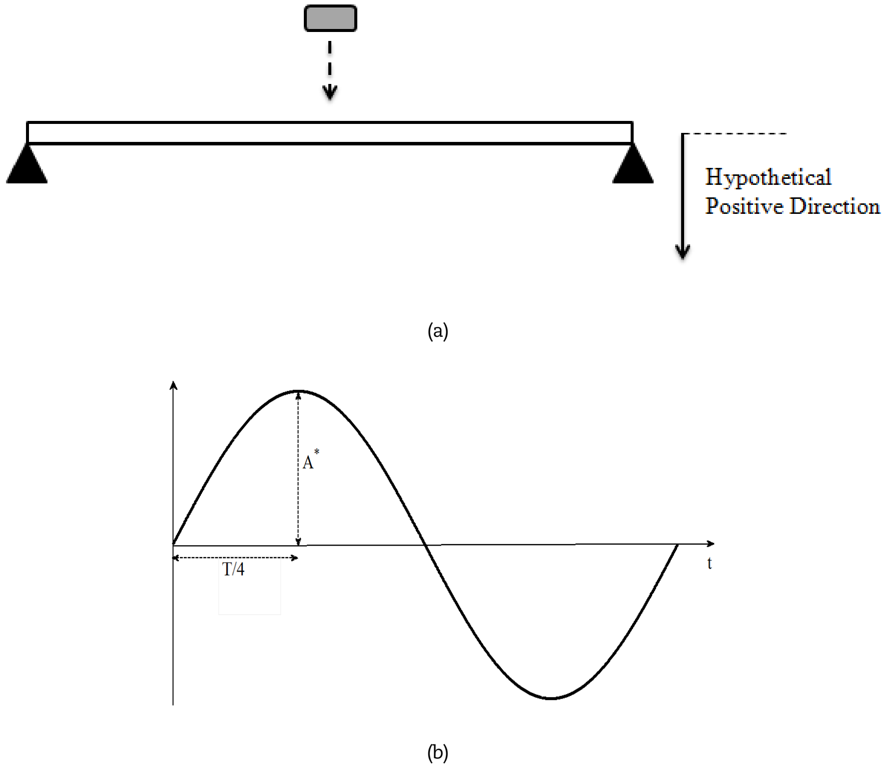

Fig. 1. (a) Low velocity impact on a beam, (b) schematic behavior of beam midpoint due to impact which is similar to excitation by initial velocity.

$$\ddot{u}+f(u)=0, \tag{1}$$

with initial conditions

$$u(0)=0, \dot{u}(0)=B. \tag{2}$$

where $f(u)$ is the restoring force. Several practical examples exist in which the system is excited by initial velocity; low velocity impact on structures is one of them. In modal analysis of structure's identification, the system is excited by a hammer and exposed to initial velocity. Low velocity impact of a mass on a beam is shown in Fig. 1. (a) and the corresponding time history response of beam midpoint is plotted in Fig. 2. (b). It is assumed that the amplitude of oscillation is A^*, which for the first time occurs at $T/4$ after the beginning of motion, where $T=2\pi/\omega$ and ω is the circular frequency of motion. Therefore, to describe the considered motion, two unknown parameters exist in the problem; A^* and ω.

In order to determine the maximum amplitude A^*, we use the energy conservation law which states that in the conservative systems, the total energy is a constant quantity and energy only changes form one type to another. In the conservative oscillator system, energy periodically changed from potential form to kinetic one and vice versa; although, the total energy is conserved. In autonomous conservative nonlinear oscillator under initial velocity, at the beginning of motion, the initial energy is totally in the form of kinetic energy and gradually converts to the potential energy until $t=T/4$, at this time, the energy of the system is completely in the form of potential energy. We can determine A^* by equalization of initial kinetic energy with potential energy at $t=T/4$. For the considered nonlinear oscillator in Eq. (1), the initial kinetic energy is:

$$\text{K.E.}: \frac{1}{2}B^2, \tag{3}$$

and the potential energy at time $T/4$ is as follows:

$$\text{P.E.}: \int_0^{A^*} f(u)du, \tag{4}$$

Thereby, the conservation of energy yields:

$$\int_0^{A^*} f(u)du=\frac{1}{2}B^2. \tag{5}$$

In Eq. (5), $f(u)$ and B are known, therefore, after performing the integration, A^* will be obtained. Consider Fig. 2 in which the oscillator has the amplitude equal to A^* and zero velocity at $t=T/4$. If we suppose that the motion starts at this point, it is considered that the oscillator has the initial amplitude and zero velocity. In other words, it is assumed the motion of oscillator caused by initial amplitude A^*, when time measure from $t=T/4$. There are various methods proposed to determine ω in nonlinear oscillators in terms of initial amplitude and any of them could be applied to obtain the frequency of motion. When the amplitude and frequency of oscillating motion is determined, it is necessary to determine the time domain of solution. A trial solution can be considered as follows:

$$u(t) = \frac{B}{\omega}\sin(\omega t). \tag{6}$$

The solution, however, satisfies the initial conditions in Eq. (2), but not confirm the constraint applied at time $t = T/4$ where $u(T/4) = A^*$ and so it's not applicable. In order to solve this problem, the below trial solution is considered:

$$u(t) = \frac{\alpha}{\omega}\sin(\omega t) + \frac{B-\alpha}{3\omega}\sin(3\omega t), \tag{7}$$

It is clear that Eq. (7) satisfied the initial conditions. To determine the unknown parameters in the proposed solution, we employed the constraint of motion at $T/4$, which leads to:

$$u(\frac{T}{4}) = A^* \rightarrow \frac{\alpha}{\omega} - \frac{B-\alpha}{3\omega} = A^* \rightarrow \alpha = \frac{3}{4}\omega A^* + \frac{B}{4} \tag{8}$$

when α determines the oscillating motion in which the time domain is valid.

3. Applications

3.1 Example 1

Consider the nonlinear Duffing oscillator with cubic term and specified initial conditions as following:

$$\ddot{u} + u^3 = 0,\ u(0) = 0,\ \dot{u}(0) = B \tag{9}$$

restoring force is $f(u) = u^3$ in the Eq. (9). Substituting restoring force into Eq. (5) and integrating yields:

$$A^* = \sqrt[4]{2B^2} \tag{10}$$

Duffing oscillator's motion frequency under initial amplitude is calculated and reported in different references. A first order approximation which is obtained from homotopy perturbation method [40] for this oscillator is as follows:

$$\omega = \sqrt{\frac{3}{4}A^{*2}}, \tag{11}$$

by substituting Eq. (10) into Eq. (11), the frequency- initial velocity relation is derived for oscillator in Eq. (9) as following:

$$\omega = \sqrt{\frac{3}{4}\sqrt{2B^2}}, \tag{12}$$

On the other hand, by substituting A^* and ω into Eq. (8), the unknown parameter in the assumed solution is determined as follows:

$$\alpha = (2 + 3\sqrt{6})\frac{B}{8}, \tag{13}$$

Finally, the analytical solution for nonlinear Duffing oscillator under initial velocity can be obtained as:

$$u(t) = \frac{(9\sqrt{2} + 2\sqrt{3})B}{12\sqrt{\sqrt{2B^2}}}\sin(\sqrt{\frac{3}{4}\sqrt{2B^2}}\,t) + \frac{(2\sqrt{3} - 3\sqrt{2})B}{12\sqrt{\sqrt{2B^2}}}\sin(3\sqrt{\frac{3}{4}\sqrt{2B^2}}\,t). \tag{14}$$

The comparison between the analytical solution obtained in Eq. (14) in conjunction with numerical solution presented in Fig. 2 and Fig. 3 demonstrates good agreement between the solutions.

3.2 Example 2

Consider the nonlinear oscillator with discontinuity under initial velocity as follows:

$$\ddot{u} + u|u| = 0,\ u(0) = 0,\ \dot{u}(0) = B \tag{15}$$

restoring force is $f(u) = u|u|$ and in the first quarter of motion by taking the absolute value it is equal to $f(u) = u^2$. By substituting the restoring force into Eq. (5) and after integrating the equation, we have:

$$A^* = \sqrt[3]{\frac{3}{2}B^2} \tag{16}$$

An approximation for the frequency-amplitude is obtained from homotopy perturbation method [41] for this oscillator as follows:

$$\omega = \sqrt{\frac{8}{3}\frac{A^*}{\pi}}, \tag{17}$$

by substituting Eq. (16) into Eq. (17), the frequency- initial velocity relation is derived for oscillator with nonlinear discontinuity described in Eq. (15) as follows:

$$\omega = \sqrt[3]{\frac{16B}{3\sqrt{\pi^3}}} \tag{18}$$

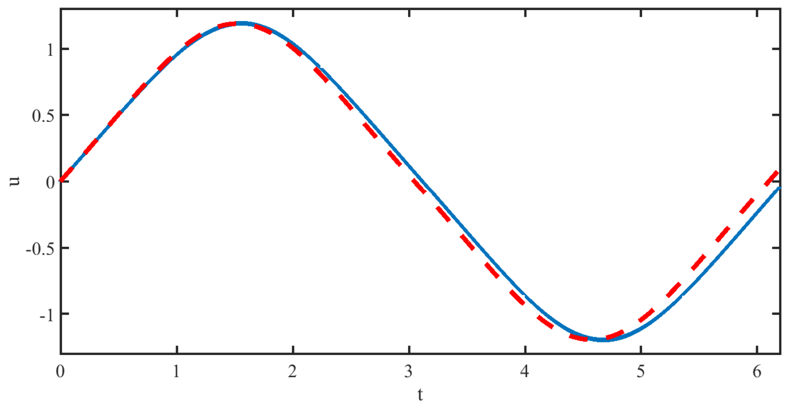

Fig. 2. Comparison between numerical (solid line) and analytical solutions (dashed line) in the case of Duffing oscillator for $B = 1$.

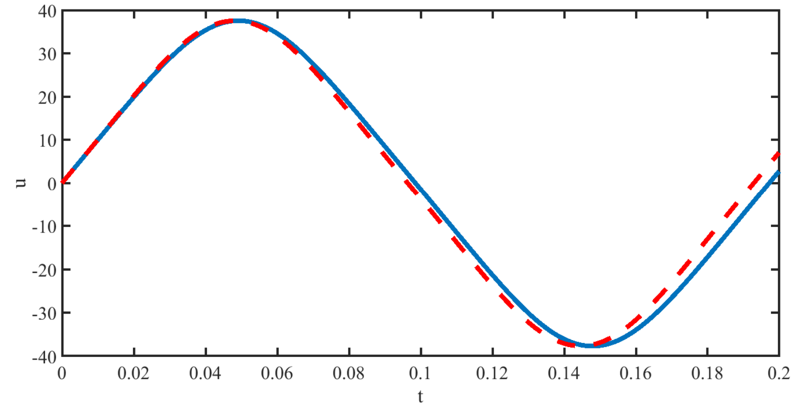

Fig. 3. Comparison between numerical (solid line) and analytical solutions (dashed line) in the case of Duffing oscillator for $B = 1000$.

By substituting A^* and ω into Eq. (8), the unknown parameter in the assumed solution is determined and then the analytical solution is obtained in the following form:

$$u(t) = \frac{\sqrt[3]{12B^2}}{16}\{(6+\sqrt{\pi})\sin(\sqrt[3]{\frac{16B}{3\sqrt{\pi^3}}}t) + (\sqrt{\pi}-2)\sin(3\sqrt[3]{\frac{16B}{3\sqrt{\pi^3}}}t)\}. \tag{19}$$

The comparison between the analytical solution obtained by Eq. (19) in conjunction with the numerical solution is presented in Fig. 4 and Fig. 5 and shows acceptable agreement.

3.3 Higher Order Approximation

The accuracy of frequency-amplitude relation plays an important role to derive the solution of treated nonlinear oscillators under initial velocity. In the mentioned numerical examples, first order approximation is used. In this section, we obtain the second order approximation for Example 1. One more time, consider Duffing oscillator in Eq. (9). The second order homotopy perturbation solution yields the frequency-amplitude relation as follows [40]:

$$\omega_{SPHM} = \sqrt{\frac{23}{32}A^{*2}}, \tag{20}$$

by substituting Eq. (10) into Eq. (20), the second order frequency - initial velocity relation is derived for Duffing oscillator in Eq. (9) as:

$$\omega = \sqrt{\frac{23}{32}\sqrt{2B^2}}, \tag{21}$$

By substituting A^* and ω into Eq. (8), the unknown parameter in the assumed solution is determined and after that the second order analytical solution is obtained for Duffing oscillator as follows:

$$u(t) = \frac{(69\sqrt{2}+4\sqrt{46})B}{92\sqrt{\sqrt{2B^2}}}\sin(\sqrt{\frac{23}{32}\sqrt{2B^2}}t) + \frac{(4\sqrt{46}-23\sqrt{2})B}{92\sqrt{\sqrt{2B^2}}}\sin(3\sqrt{\frac{23}{32}\sqrt{2B^2}}t). \tag{22}$$

The comparison between second order analytical solution obtained by Eq. (22) in conjunction with the numerical solution is presented in Fig. 6. and Fig. 7. As can be seen, the accuracy of solution increased very much and excellent agreement is observed.

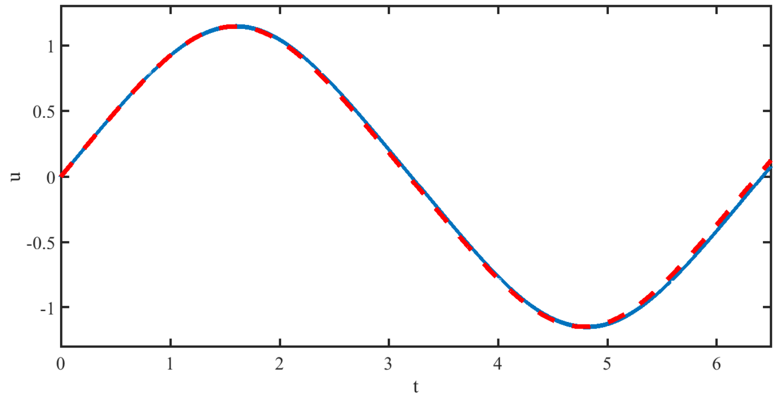

Fig. 4. Comparison between numerical (solid line) and analytical solutions (dashed line) in the case of oscillator with discontinuity for $B=1$.

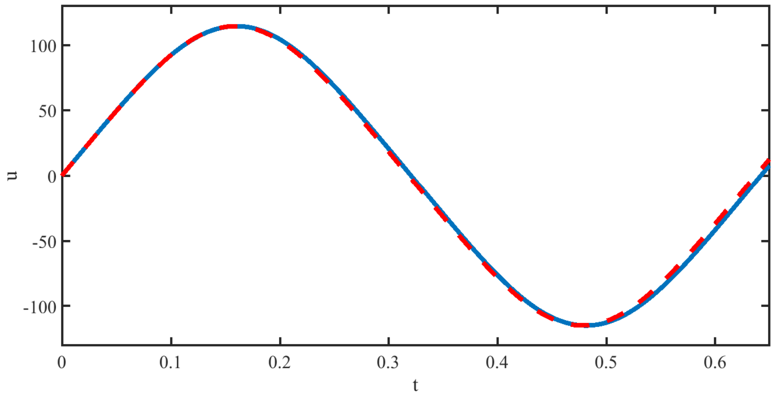

Fig. 5. Comparison between numerical (solid line) and analytical solutions (dashed line) for oscillator with discontinuity at $B=1000$.

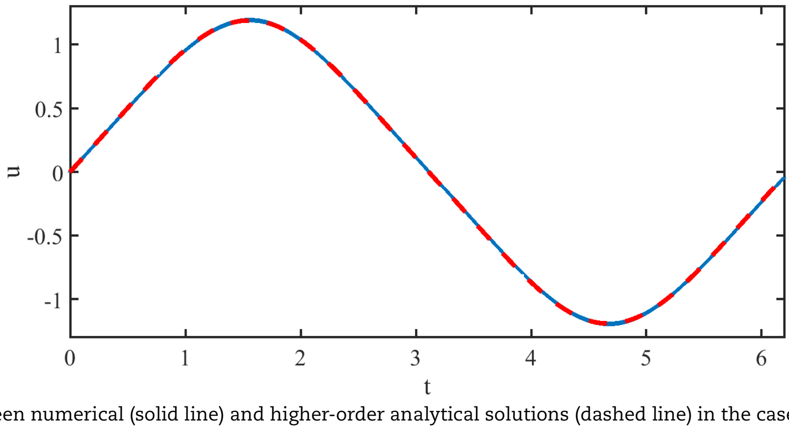

Fig. 6. Comparison between numerical (solid line) and higher-order analytical solutions (dashed line) in the case of Duffing oscillator for $B=1$.

4. Conclusion

In this study, a simple approach was proposed to analytically solve the governing equations of nonlinear oscillators under initial velocity. Previous methods could present an analytical relation between the frequency and amplitude of oscillator's motion but at initial velocity condition wasn't applicable. Therefore, the amplitude extraction in zero velocity state of the mentioned oscillators in the previous methods could be utilized. At start time, the total energy of the system is in the type of kinetic energy which when oscillator's velocity become zero, the system's energy converts to its potential form. By using the potential energy when the velocity becomes zero, the amplitude of motion could be calculated. By using the amplitude of motion, the frequency-amplitude relation extracted by previous methods is converted to frequency – initial velocity relation. The assumed solution should satisfy the initial conditions and also should have the value equal to the amplitude of motion when the oscillator's velocity becomes zero at first quarter of its motion. Therefore, an unknown parameter enters to the solution which must confirm the mentioned constraint. The soundness of the obtained solutions are related to the accuracy of utilized frequency – amplitude relations. It was exhibited that the accuracy of proposed method increases by using the higher order approximations.

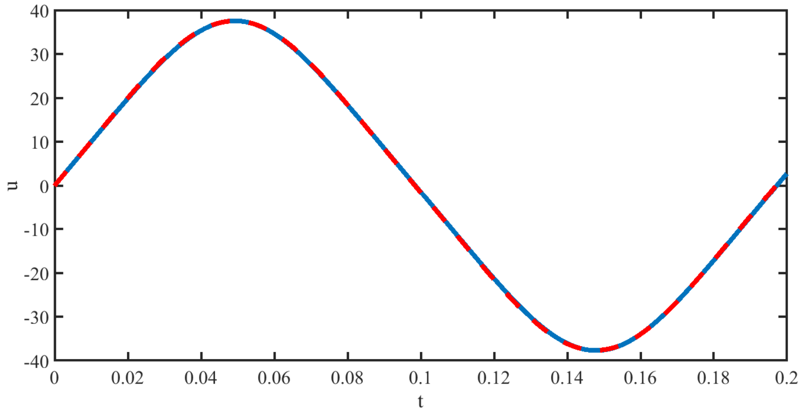

Fig. 7. Comparison between numerical (solid line) and higher-order analytical solutions (dashed line) for Duffing oscillator at $B = 1000$.

Author Contributions

Not applicable.

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Not applicable.

Conflict of Interest

The author declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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