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Technical Brief

A Simple Approach to Volterra-Fredholm Integral Equations

Ji-Huan He[1,2,3]

[1] School of Science, Xi'an University of Architecture and Technology, Xi'an, China
[2] School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China
[3] National Engineering Laboratory for Modern Silk, College of Textile and Engineering, Soochow University, Suzhou, China

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Corresponding author: J.H. He (Hejihuan@suda.edu.cn)

Abstract. This paper suggests a simple analytical method for Volterra-Fredholm integral equations, the solution process is similar to that by variational-based analytical method, e.g., Ritz method, however, the method requires no establishment of the variational principle for the discussed problem, making the method much attractive for practical applications. The examples show the method is straightforward and effective, and the method can also be extended to other nonlinear problems.

Keywords: Integral equation, Series solution, Variational principle.

Mathematics Subject Classification: 49J21 · 45G10 · 33C47 · 33C90

1. Introduction

As integral equations arise everywhere from the architectural engineering to nanotechnology [1-5], analytical methods for such problems have been caught much attention. Ghorbani and Saberi-Nadjafi suggested a modification of the homotopy perturbation method [6], Novin & Araghi also suggested a modified homotopy perturbation method for hypersingular integral equations [7], Deniz gave an optimal perturbation iteration technique [8], which was a development of the iteration perturbation method [9]. Tian found that the Monte Carlo method is an effective tool to integral equations [10, 11]. A review on various analytical methods for integral equations is available in Ref. [12]. This paper focuses on Volterra-Fredholm integral equations, and a simple analytical method is suggested.

2. Volterra-Fredholm Integral Equations

This paper adopts two examples in Ref. [8] to show the solution process.

$$y(x) = g(x) + \int_0^x f(x,s)ds + \int_0^1 h(x,s)ds \quad 0 \le x \le 1 \tag{1}$$

Similar to various analytical methods in the variational theory [13, 14], e.g., the Ritz method, we can choose to a suitable trial solution for the problem. The most used trial solution is the series form

$$y(x) = \sum_{n=0}^{N} a_n x^n \tag{2}$$

where a_n (n=0~N) are unknown constants.

Submitting Eq. (2) and expanding g(x) into a series of x, combining the like terms, and setting the coefficients of x^n (n=0~N) to zero, we obtain algebraic equations for a_n (n=0~N).

Alternatively, we can assume the solution has the form

$$y(x) = \sum_{n=0}^{N} A_n(x) \tag{3}$$

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where $A_n(x)$ (n=0~N) are unknown functions. The choice of $A_n(x)$ depends upon the $g(x)$. For example $g(x)=e^x$, we can assume the solution has the form $y(x)=a_0+a_1e^x+a_2e^{2x}+a_3e^{3x}+\cdots$, where a_n are unknown constants.

Example 1 [8]. Consider the following Volterra-Fredholm integral equation

$$y(x)=-\frac{1}{30}x^6+\frac{1}{3}x^4-x^2+\frac{5}{3}x-\frac{5}{4}+\int_0^x(x-s)y^2(s)ds+\int_0^1(x+s)y(s)ds \tag{4}$$

To elucidate the solution process, we assume the approximate solution can be expressed as

$$y(x)=a_0+a_1x+a_2x^2 \tag{5}$$

Putting Eq. (5) into Eq. (4) results in

$$a_0+a_1x+a_2x^2=-\frac{1}{30}x^6+\frac{1}{3}x^4-x^2+\frac{5}{3}x-\frac{5}{4}+\int_0^x(x-s)(a_0+a_1s+a_2s^2)^2ds+\int_0^1(x+s)(a_0+a_1s+a_2s^2)ds \tag{6}$$

Using some a mathematical software, we can solve the unknown constants in Eq. (6), which read

$$a_0=-2,\quad a_1=0 \quad \text{and} \quad a_2=1 \tag{7}$$

An approximate solution is obtained as

$$y(x)=x^2-2 \tag{8}$$

which happens to be the exact solution.

Example 2 [8]. Consider the following Volterra-Fredholm integral equation

$$y(x)=e^x-\frac{1}{3}e^{3x}+\frac{1}{3}+\int_0^x y^3(s)ds \tag{9}$$

We assume that the solution can be approximated as

$$y(x)=a_0+a_1x+a_2x^2+a_3x^3 \tag{10}$$

Putting Eq. (10) into Eq. (9) results in

$$a_0+a_1x+a_2x^2+a_3x^3=e^x-\frac{1}{3}e^{3x}+\frac{1}{3}+\int_0^x(a_0+a_1s+a_2s^2+a_3s^3)^3ds \tag{11}$$

Expanding e^x and e^{3x}

$$e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots \tag{12}$$

$$e^{3x}=1+3x+\frac{9}{2}x^2+\frac{27}{6}x^3+\cdots \tag{13}$$

and ignoring the terms $x^n(n\geq 4)$, we re-write Eq. (11) in the form

$$a_0+a_1x+a_2x^2+a_3x^3=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{3}(1+3x+\frac{9}{2}x^2+\frac{27}{6}x^3)+\frac{1}{3}+a_0^3x+\frac{3}{2}a_0^2a_1x^2+(a_0^2a_2+\frac{2}{3}a_0a_1^2)x^3+\cdots \tag{14}$$

From above equation, we identify that

$$a_0=1 \tag{15}$$

$$a_1=a_0^3 \tag{16}$$

$$a_2=\frac{1}{2}-\frac{3}{2}+\frac{3}{2}a_0^2a_1 \tag{17}$$

The constants can be determined as $a_0=1$, $a_1=1$, and $a_2=1/2$, as a result an approximate solution is obtained, which is

$$y(x)=1+x+\frac{1}{2}x^2 \tag{18}$$

The exact solution is $y(x)=e^x$.

Considering $y(0)=1$, we can assume the solution has the form

$$y(x)=e^{bx} \tag{19}$$

From Eq. (9), we have

$$e^{bx} = e^x - \frac{1}{3}e^{3x} + \frac{1}{3} + \int_0^x e^{3bs} ds \tag{20}$$

or

$$e^{bx} = e^x - \frac{1}{3}e^{3x} + \frac{1}{3} + \frac{1}{3b}(e^{3bx} - 1) \tag{21}$$

Solving b from Eq. (21), we obtain

$$b = 1 \tag{22}$$

We, therefore, obtain the following approximate solution

$$y(x) = e^x \tag{23}$$

which is the exact one.

3. Conclusion

This paper suggests a simple analytical method for integral equation, a suitable choice of a trial solution always leads to an ideal result.

Conflict of Interest

The author declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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References

[1] Ji FY, He CH, Zhang JJ, et al. A fractal Boussinesq equation for nonlinear transverse vibration of a nanofiber-reinforced concrete pillar, *Applied Mathematical Modelling*, 82, 2020, 437-448.

[2] Negarchi N, Nouri K. A New Direct Method for Solving Optimal Control Problem of Nonlinear Volterra-Fredholm Integral Equation via the Muntz-Legendre Polynomials, *Bulletin of the Iranian Mathematical Society*, 45(3), 2019, 917-934.

[3] Li XX, Li YY, Li Y, et al. Gecko-like adhesion in the electrospinning process, *Results in Physics*, 16, 2020, 102899.

[4] He JH, Ain QT. New promises and future challenges of fractal calculus: from two-scale Thermodynamics to fractal variational principle, *Thermal Science*, 24(2A), 2020, 659-681.

[5] He JH. Taylor series solution for a third order boundary value problem arising in architectural engineering, *Ain Shams Engineering Journal*, 2020, DOI: 10.1016/j.asej.2020.01.016.

[6] Ghorbani A, Saberi-Nadjafi J. Exact solutions for nonlinear integral equations by a modified homotopy perturbation method, *Computers & Mathematics with Applications*, 56(4), 2008, 1032-1039.

[7] Novin R, Araghi MAF. Hypersingular integral equations of the first kind: A modified homotopy perturbation method and its application to vibration and active control, *Journal of Low Frequency Noise Vibration and Active Control*, 38(2), 2019, 706-727.

[8] Deniz S. Optimal perturbation iteration technique for solving nonlinear Volterra-Fredholm integral equations, *Mathematical Methods in the Applied Sciences*, 2020; 1-7. https://doi.org/10.1002/mma.6312

[9] He JH. Iteration perturbation method for strongly nonlinear oscillations, *Journal of Vibration and Control*, 7(5), 2001, 631-642.

[10] Tian Y. Markov chain Monte Carlo method to solve Fredholm integral equations, *Thermal Science*, 22(4), 2018, 1673-1678.

[11] Tian Y. Monte Carlo method with control variate for integral equations, *Thermal Science*, 22(4), 2018, 1765-1771.

[12] He JH. A short review on analytical methods for to a fully fourth-order nonlinear integral boundary value problem with fractal derivatives, *International Journal of Numerical Methods for Heat and Fluid Flow*, 2020, DOI: 10.1108/HFF-01-2020-0060

[13] He JH. Variational principle and periodic solution of the Kundu–Mukherjee–Naskar equation, *Results in Physics*, 17, 2020, 103031.

[14] He JH. The simpler, the better: Analytical methods for nonlinear oscillators and fractional oscillators, *Journal of Low Frequency Noise, Vibration and Active Control*, 38, 2019, 1252-1260.

ORCID iD

Ji-Huan He https://orcid.org/0000-0002-1636-0559

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