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Approximate Solutions of Coupled Nonlinear Oscillations: Stability Analysis

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Abstract. The current article is concerned with a comprehensive investigation in achieving approximate solutions of coupled nonlinear oscillations with high nonlinearity. These equations are highly nonlinear second-order ordinary differential equations. Via a coupling of the Homotopy perturbation method and Laplace transforms, which is so-called the He-Laplace method, traditional approximate solutions involving the secular terms are accomplished. On the other hand, in order to cancel the secular terms, an expanded frequency technique is adapted to accomplish periodic approximate solutions. Therefore, a nonlinear frequency, for each differential equation, is achieved. Furthermore, for more convenience, these solutions are pictured to indicate their behavior. The multiple time-scales with the aid of the Homotopy concept are utilized to judge the stability criteria. The analyses reveal the resonance as well as the non-resonant cases. Additionally, numerical calculations are carried out, graphically, to address the regions that guaranteed the bounded solutions. It is found that the latter method, is the most powerful mathematical tool in extracting the stability analysis of the considered system.

Keywords: He-Laplace Method, Expanded Frequency Analysis, Multiple Time Scales Technique.

1. Introduction

The coupled nonlinear oscillators that governed by the generic Hamiltonian function of two-degrees of freedom may be written in the following form:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y), \tag{1}$$

where p_x and p_y are the momentum of the system and the function $V(x,y)$ consists of two-dimensional harmonic potentials, in addition, with the quartic terms as follows:

$$V(x, y) = \omega^2 x^2 + \sigma^2 y^2 + \alpha x^4 + \beta y^4 + \gamma x^2 y^2. \tag{2}$$

This potential is utilized in a model of quantum field theory; for instance, see Rajaraman and Weinberg [1]. Additionally, in the scalar field theory, as given by Friedberg et al. [2].

In the dynamics of galactic motion (AGK galactic potential), see El-Sabaa et al. [3], as given by:

$$V(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{a}{4}(x^2 + y^2)^2 - \frac{b}{2}x^2 y^2 \tag{3}$$

The Yang-Mills fields are vector fields belonging to the adjoint representation of the local symmetry group (Gauge group SU(2)). For instance, see Jiménez–Lara, and Llibre [4].

$$V(x, y) = Ax^2 + By^2 + \frac{1}{4}x^4 + \frac{c}{4}y^4 + \frac{b}{2}x^2 y^2. \tag{4}$$

Furthermore, consider the potential

$$V(x,y)=\frac{1}{((x+1)^2+y^2)^n}+\frac{1}{((x-1)^2+y^2)^n}. \tag{5}$$

when $n=-2$, the potential that is given in eq. (5) may takes the following form:

$$V(x,y)=2\left[x^4+y^4+6x^2+2y^2+2x^2y^2+1\right]. \tag{6}$$

In comparison with eq. (2), one finds

$$\omega^2=12,\ \sigma^2=4,\ \alpha=2,\ \beta=2 \ \text{ and } \ \gamma=4. \tag{7}$$

Our considered system is a conserved energy with two-degrees of freedom, which can be integrable in the Liouville sense, see Arnol'd [5], if there exist an additional first integral I besides the Hamiltonian function H , such that the gradient vectors of I and H are independent of all points in the phase space, except perhaps a set of zero measure and in the involution of $\{I,H\}=0$. Moreover, most problems in the dynamical system with nonlinearity cannot be exactly solvable, where there is no general method for determining whether the system is integrable or not. Painlevé (P) and his co-workers have been using the $P-$ property to integrate a class of nonlinear ordinary differential equations and to obtain their underling solution; for instance, see Ince [6]. The $P-$ .property requiring that the general solution has only movable singularities, on the complex plane are as poles, and the corresponding solution can be given in the Laurent series in the neighborhood of a movable singular point. Furthermore, the $P-$ property gives the necessary condition for the solution. It has been shown that the potential function that is given in eq. (1) can be integrated by using Ablowitz et al. [7] (ARS) conjecture for some values of the parameters; for instance, see Bountis et al. [8]. It should be noted that a survey of the relativistic theory of a rigid body was introduced by Ditchburn and Heavens [9].

As shown in the aforementioned aspects, the problems in these cases are reduced to quadrature, but some of them still non solvable. Therefore, no information about the behavior of the solutions is addressed. Simultaneously, the numerical treatment is sometimes available. Unfortunately, it is not adequate to yield complete information about the system. For this purpose, the perturbation methods are needed. These methods are widely used in obtaining analytically approximate solution of the nonlinear differential equations. Simply, they convert the nonlinear differential equation into linear ones. Regrettably, in many cases, these equations do not possess a small parameter. In this case, the application of the classical perturbation techniques is highly restricted. Consequently, they failed to handle the problems of strong nonlinearity behavior. As an early mathematical tool in several mathematical analysis is known as: the perturbation methods. These methods are concerned with the solution of the nonlinear equations in many practical physical situations. Some problems may be solved by a straightforward technique, which mainly depend on a small physical parameter; for instance, see Dyke [10]. Away from the existence of the small parameter, recently, a new technique has been suggested to obtain an analytical approximate solution. He [11-17], Yn and He [18], He and Jin [19] and He and Ain [20] were considered as the first researchers, who proposed and developed this new method; namely the Homotopy perturbation method (HPM). This heuristic algorithm is found to be working well in solving strongly nonlinear systems. It is considered as a promising and powerful technique in analyzing many problems in diverse areas in engineering and practical physical applications. Furthermore, the solution in this approach is addressed as an infinite series and usually converges to an accurate solution. Throughout the latter references, El-Dib, and Moatimid [21] adapted the HPM to accomplish exact solutions of a class of linear, nonlinear, singular and system of ordinary/partial differential equations. Simply, they choose an initial trial function, typically, in the form of a power series. The cancellation of the first order approximate solution warranties that all higher orders are, also, ignored. Consequently, the residual zero-order solution will be corroborated to achieve an exact one. In order to analyze the Duffing equation with a displacement time-delay, El-Dib [22] utilized two perturbation techniques to analyze the damped Duffing equation with a time delayed displacement variable; namely the HPM as well as the nonlinear frequency analysis. Additionally, based on the combination of the HPM along with the multiple scale, a uniform periodic equation was achieved. Recently, Sripacharasakullert et al. [23] investigated the fractional multi-dimensional Burgers equation by means of the Caputo fractional calculus. Utilizing the HPM, they achieved an approximate analytical solution. Furthermore, the convergence analysis and the error estimation were derived and obtained by the HPM. Recently, Moatimid et al. [24] made a coupling of the HPM together with the Laplace transforms (L_T) to obtain an approximate analytical solution of a cylindrical surface deflection between two hydro-magnetic Darcian fluids. Once more, Moatimid [25] examined the behavior of the motion of a sliding bead in a vertically rotated parabola. His analysis was based on the HPM. Furthermore, it reveals the stability profile of the system with different approaches. Moatimid [26] investigated the stability analysis of a parametric Duffing oscillator. Again, he utilized the HPM together with the L_T to obtain different analytical approximate solutions of this equation. It is worthy to mention, here, that his analysis recovers the exact solution of the Duffing equation. In other words, he utilized the He-Laplace approach to obtain an approximate solution of the interface profile. Recently, in the light of the potential applications in engineering, electronics, physics, chemistry, and biology, Ghaleba et al. [27] applied several techniques to achieve analytic approximate and numerical solutions of the cubic-quintic Duffing-Van der Pol equation with two external periodic forcing.

The current paper is concerned with analytical approximation solutions, along with the stability analysis, of the following governing equations of motion:

$$\ddot{x}+\omega^2x+\alpha x^3+\gamma x y^2=0 \tag{8}$$

$$\ddot{y}+\sigma^2y+\beta y^3+\gamma y x^2=0. \tag{9}$$

The potential of such consider system is sketched throughout Fig. 1. To crystallize the present work, the paper is organized as follows: The next Section is depicted to obtain approximate solutions of the equations (8) and (9) via the He-Laplace approach. Unluckily, these solutions include secular terms. Simultaneously, we do not have any authority to eliminate these terms. Of course, this situation is physically undesirable. An analytic periodic approximate solution, based on an expanded natural frequency concept, is depicted in Section 3. The stability analysis, based on the multiple-time scales with the Homotopy concept, is achieved in Section 4. Its Subsections involve the stability analysis of the resonance as well as the non-resonance cases. Finally, concluding remarks are summarized throughout Section 5.

Fig. 1. Depicts the potential of the considered system

2. Approximate Solutions by means of the He-Laplace Approach

In terms of the elliptic functions, the equation of the un-damped relativistic oscillator was solved by Hutten [28]. The results showed that the frequency was decreased in light of the total energy. Additionally, a consistent formula with the experimental features, under some restrictions, was achieved. The current governing equations of motion of the considered coupled nonlinear oscillations consist of high nonlinear second-order ordinary differential equations. Precisely, they have no exact solutions. Consequently, they are organized by making use of a perturbation technique along with the Laplace transforms. As well-known, there is an advantage in this couple; the Laplace transform is valid for solving all linear differential equations. Additionally, the Homotopy perturbation method (HPM) converts the nonlinear differential nonlinear equation into a series of linear ones. Therefore, the Laplace transform makes the solution process extremely simple and accessible to all audiences. For this purpose, see Refs. [29-39]. The following procedure depends mainly on the He-Laplace approach. The linear and nonlinear parts of equations (8) and (9) may be written as:

$$L_1 = \ddot{x} + \omega^2 x \quad \text{and} \quad N_1 = \alpha x^3 + \gamma x y^2 \tag{10I}$$

$$L_2 = \ddot{y} + \sigma^2 y \quad \text{and} \quad N_2 = \beta y^3 + \gamma y x^2. \tag{10II}$$

For this purpose, it is convenient to consider the following initial conditions:

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 1\,, \tag{11I}$$

$$y(0) = 0 \quad \text{and} \quad \dot{y}(0) = 1. \tag{11II}$$

It follows that the Homotopy equations may be written as follows:

$$\ddot{x} + \omega^2 x + \rho\left(\alpha x^3 + \gamma x y^2\right) = 0 \tag{12I}$$

$$\ddot{y} + \sigma^2 y + \rho\left(\beta y^3 + \gamma y x^2\right) = 0; \qquad \rho \in [0,1] \tag{12II}$$

where ρ is an artificial embedded parameter. Sometimes, it is called the Homotopy parameter. Along with this approach, the time-dependent parameters may be written as: $x(t) = x(t;\rho)$ and $y(t) = y(t;\rho)$.

Applying the Laplace transforms (L_T) on both sides of the equations (12I) and (12II), one finds

$$\left(s^2 + \omega^2\right) L_T\left\{x(t;\rho)\right\} - s x(0) - \dot{x}(0) = -\rho L_T\left\{N_1(x,y;\rho)\right\} \tag{13I}$$

$$\left(s^2 + \sigma^2\right) L_T\left\{y(t;\rho)\right\} - s y(0) - \dot{y}(0) = -\rho L_T\left\{N_2(x,y;\rho)\right\}. \tag{13II}$$

In accordance with the regular HPM, the dependent function $x(t;\rho)$, and $y(t;\rho)$ may be expanded as:

$$x(t;\rho) = \sum_{i=0}^{\infty} \rho^i x_i(t) \tag{14I}$$

$$y(t;\rho) = \sum_{j=0}^{\infty} \rho^j y_j(t). \tag{14II}$$

Accordingly, the nonlinear part Nr should be written as follows:

$$N\left(\sum_{k=0}^{\infty} \sum_{l=0}^{k} x_l \; y_{k-l} \rho^k\right) = N_0(x_0 y_0) + \rho N_1(x_0 y_1, x_1 y_0) + \rho^2 N_2(x_0 y_2, x_1 y_1, y_0 x_2) + \ldots . \rho^k N_k(x_0 y_k, x_1 y_{k-1}, y_0 x_k, \; y_1 x_{k-1}), \tag{15}$$

where

$$N_k\left(x_0y_k, x_1y_{k-1}, y_0x_k, y_1x_{k-1}\right)=\frac{1}{k!}\lim_{\rho\to 0}\frac{\partial^k}{\partial\rho^k}N\left(\sum_{k=0}^{\infty}\sum_{l=0}^{k}x_l\,y_{k-l}\rho^k\right). \tag{16}$$

Utilizing the initial conditions as given by equations (11I and 11II), one gets

$$L_T\left\{x(t;\rho)\right\}=\frac{1}{s^2+\omega^2}-\frac{\rho}{s^2+\omega^2}L_T\left\{N_1\left(x,y;\rho\right)\right\} \tag{17I}$$

$$L_T\left\{y(t;\rho)\right\}=\frac{1}{s^2+\sigma^2}-\frac{\rho}{s^2+\sigma^2}L_T\left\{N_2\left(x,y;\rho\right)\right\}. \tag{17II}$$

Acting on both sides by the inverse Laplace transforms of equations (17I) and (17II), one finds

$$x(t;\rho)=\frac{\sin\omega t}{\omega}-\rho L_T^{-1}\left\{\frac{1}{s^2+\omega^2}L_T\left[N_1(x,y;\rho)\right]\right\} \tag{18I}$$

$$y(t;\rho)=\frac{\sin\sigma t}{\sigma}-\rho L_T^{-1}\left\{\frac{1}{s^2+\sigma^2}L_T\left[N_2(x,y;\rho)\right]\right\}. \tag{18II}$$

Substituting from equations (14I) and (14II) into equations (18I) and (18II) and then equating the coefficients of like powers of ρ on both sides, one gets the following set of equations:

$$\rho^0:\ x_0=\frac{\sin\omega t}{\omega}, \tag{19I}$$

$$\rho^0:\ y_0=\frac{\sin\sigma t}{\sigma}, \tag{19II}$$

$$\rho:\quad x_1=-L_T^{-1}\left\{\frac{1}{s^2+\omega^2}L_T\left\{\alpha\,x_0^3+\gamma\,x_0y_0^2\right\}\right\}, \tag{20I}$$

$$\rho:\quad y_1=-L_T^{-1}\left\{\frac{1}{s^2+\sigma^2}L_T\left\{\beta\,y_0^3+\gamma\,y_0x_0^2\right\}\right\}, \tag{20II}$$

$$\rho^2:\quad x_2=-L_T^{-1}\left\{\frac{1}{s^2+\omega^2}L_T\left\{3\alpha\,x_1x_0^2+\gamma\,x_1y_0^2+2\gamma\,x_0y_0y_1\right\}\right\} \tag{21I}$$

and

$$\rho^2:\quad y_2=-L_T^{-1}\left\{\frac{1}{s^2+\sigma^2}L_T\left\{3\beta\,y_1y_0^2+\gamma\,y_1x_0^2+2\gamma\,x_0x_1y_0\right\}\right\}. \tag{21II}$$

As seen from the previous equations (20I), 20II), (21I) and (21II), the solutions of first, and second orders depend mainly on the zero-order solutions.
Using the Mathematica software (12.0.0.0), the first-order may be written as:

$$x_1(t)=\frac{1}{32\sigma^3\omega^5(\sigma^2-\omega^2)}\left\{\begin{array}{l}\left(4\gamma\sigma\omega^4-9\alpha\sigma^5+(9\alpha-8\gamma)\sigma^3\omega^2\right)\sin\omega t+\\ \left(4t\sigma\omega(\sigma^2-\omega^2)(3\alpha\sigma^2+2\gamma\omega^2)\right)\cos\omega t+\\ \alpha\sigma^3(\omega^2-\sigma^2)\sin 3\omega t+2\gamma\omega^4(\omega+\sigma)\sin((2\sigma-\omega)t)+\\ 2\gamma\omega^4(\omega-\sigma)\sin((2\sigma+\omega)t)\end{array}\right\},\ \sigma\neq\omega \tag{22I}$$

and

$$y_1(t)=\frac{1}{32\omega^3\sigma^5(\sigma^2-\omega^2)}\left\{\begin{array}{l}\left(-4\gamma\omega\sigma^4+9\beta\omega^5+(8\gamma-9\beta)\sigma^2\omega^3\right)\sin\sigma t+\\ \left(4t\sigma\omega(\sigma^2-\omega^2)(3\beta\omega^2+2\gamma\sigma^2)\right)\cos\sigma t+\\ \beta\omega^3(\omega^2-\sigma^2)\sin 3\sigma t+2\gamma\sigma^4(\omega+\sigma)\sin((\sigma-2\omega)t)+\\ 2\gamma\sigma^4(\omega-\sigma)\sin((\sigma+2\omega)t)\end{array}\right\},\ \sigma\neq\omega \tag{22II}$$

For an easier follow-up of the analysis, the functions $x_2(t)$ and $y_2(t)$ are known from the context. They will be omitted here. Simultaneously, they are obtainable under the memorial of the readers. For simplicity, the analysis is stopped at the second-order only. Therefore, the approximate solutions of equations (8) and (9) may be written as follows:

$$x(t)=\lim_{\rho\to 1}\left(x_0(t)+\rho\,x_1(t)+\rho^2\,x_2(t)\right) \tag{23I}$$

and

$$y(t)=\lim_{\rho\to 1}\left(y_0(t)+\rho\,y_1(t)+\rho^2\,y_2(t)\right). \tag{23II}$$

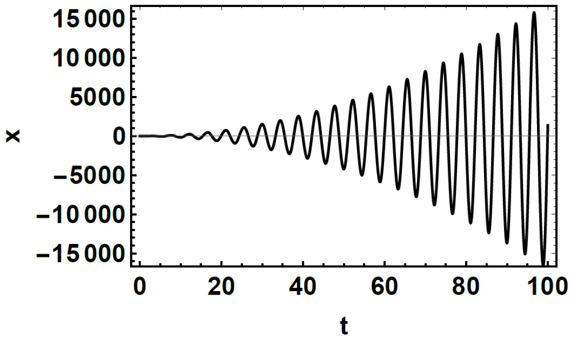

Fig. 2I. Plots the approximate solution that is given in eq. (23I), for a system having the particulars: $\alpha = 16$, $\gamma = 6, \beta = 1, \omega = \sqrt{2}, \sigma = 4$.

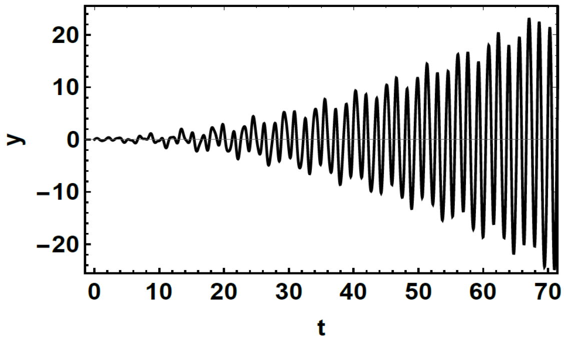

Fig. 2II. Plots the approximate solution that is given in eq. (23II), for a system having the particulars as given in Fig. (1I).

These solutions cannot be judged the bounded/unbounded behavior in light of the increasing of time by the presence of the secular terms. Actually, the unbounded solution comes in light of the existence of the secular terms. Unfortunately, physically it is not preferred. In reality, the previous classical method does not enable us to ignore the source of these secular terms. In this case, the stability of the given problem cannot be governed by this traditional approximate solution.

To this end, it is convenient to confirm the previous unbounded approximate solutions in a numerical manner. Therefore, in what follows, some figures are depicted to show the behavior of the approximate solutions $x(t)$ and $y(t)$ versus the independent variable t. Therefore, the figures (2I) and (2II) are plotted to indicate the behavior of the dependent variables $x(t)$ and $y(t)$. From these two figures, it is seen that for small values of the time, the solutions behave like a stable one. In contrast, for large values of the time, it is observed a growth rate of the amplitudes. Of course, this is occurring in light of the presence of the secular terms. Subsequently, the analysis in the next Section follows an adaptation of the previous analysis to avoid the existence of these secular terms, and consequently, to obtain bounded approximate solutions.

3. An Approximate Solution by means of an Expanded Frequency Analysis

The main objective of this Section is to achieve bounded analytical approximate solutions of equations (8) and (9). Unfortunately, the previous analysis does not enable us to do this. Therefore, another a new technique, to govern a periodic solution of the considered equations, must be in sight. For this purpose, as given before, the Homotopy formula of the considered equations is given as in (12I) and (12II) will be used.

Actually, the squares of the natural frequencies of the present model are given by ω^2 and σ^2. The following stability examination will be based on an expanded frequency analysis; for instance, see Moatimid [26]. In accordance with this approach, expanded artificial frequencies υ^2 and λ^2 may be formulated as follows:

$$\upsilon^2 = \omega^2 + \sum_{j=1}^{\infty} \rho^j \omega_j \tag{24I}$$

and

$$\lambda^2 = \sigma^2 + \sum_{j=1}^{\infty} \rho^j \sigma_j . \tag{24II}$$

Combining equations (12I), (12II), (24I) and (24II), the Homotopy equations may be rewritten as follows:

$$\ddot{x} + \upsilon^2 x + \rho\left(-(\omega_1 + \rho\omega_2)x + \alpha x^3 + \gamma x y^2\right) = 0 \tag{25I}$$

and

$$\ddot{y} + \lambda^2 y + \rho\left(-(\sigma_1 + \rho\sigma_2)y + \beta y^3 + \gamma y x^2\right) = 0. \tag{25II}$$

For simplicity, one may consider the same initial conditions as given in the previous case. Taking the Laplace transforms of equations (25I) and (25II), one gets

$$L_T\{x(t;\rho)\} = \frac{1}{s^2 + \upsilon^2} - \frac{\rho}{s^2 + \upsilon^2} L_T\left\{-(\omega_1 + \rho\omega_2)x + \alpha x^3 + \gamma x y^2\right\} \tag{26I}$$

and

$$L_T\{y(t;\rho)\} = \frac{1}{s^2 + \lambda^2} - \frac{\rho}{s^2 + \lambda^2} L_T\left\{-(\sigma_1 + \rho\sigma_2)y + \beta y^3 + \gamma y x^2\right\}. \tag{26II}$$

Employing the inverse Laplace transforms of both sides of the equations (16I) and (16II), one finds

$$x(t;\rho) = \frac{\sin \upsilon t}{\upsilon} - L_T^{-1}\left[\frac{\rho}{s^2 + \upsilon^2} L_T\left\{-(\omega_1 + \rho\omega_2)x + \alpha x^3 + \gamma x y^2\right\}\right] \tag{27I}$$

and

$$y(t;\rho) = \frac{\sin \lambda t}{\lambda} - L_T^{-1}\left[\frac{\rho}{s^2 + \lambda^2} L_T\left\{-(\sigma_1 + \rho\sigma_2)y + \beta y^3 + \gamma y x^2\right\}\right] \tag{27II}$$

Utilizing the expansion of the dependent parameters $x(t;\rho)$ and $y(t;\rho)$ as given in equations (14I) and (14II), and then equating the coefficients of like powers ρ on both sides, one gets the following system:

$$\rho^0:\ x_0 = \frac{\sin \upsilon t}{\upsilon}, \tag{28I}$$

$$\rho^0:\ y_0 = \frac{\sin \lambda t}{\lambda}, \tag{28II}$$

$$\rho:\quad x_1 = -L_T^{-1}\left\{\frac{1}{s^2+\upsilon^2} L_T\left\{-\omega_1 x_0 + \alpha x_0^3 + \gamma x_0 y_0^2\right\}\right\}, \tag{29I}$$

$$\rho:\quad y_1 = -L_T^{-1}\left\{\frac{1}{s^2+\lambda^2} L_T\left\{-\sigma_1 y_0 + \beta y_0^3 + \gamma y_0 x_0^2\right\}\right\}, \tag{29II}$$

$$\rho^2:\quad x_2 = -L_T^{-1}\left\{\frac{1}{s^2+\upsilon^2} L_T\left\{-\omega_2 x_0 - \omega_1 x_1 + 3\alpha x_1 x_0^2 + \gamma x_1 y_0^2 + 2\gamma x_0 y_0 y_1\right\}\right\} \tag{30I}$$

and

$$\rho^2:\quad y_2 = -L_T^{-1}\left\{\frac{1}{s^2+\lambda^2} L_T\left\{-\sigma_2 y_0 - \sigma_1 y_1 + 3\beta y_1 y_0^2 + \gamma y_1 x_0^2 + 2\gamma x_0 x_1 y_0\right\}\right\}. \tag{30II}$$

Typically, a uniformly valid expression turns up from the cancellation of the secular terms. For this purpose, the coefficients of the functions $\sin \upsilon t, \cos \upsilon t, \sin \lambda t$ and $\cos \lambda t$ must be excluded from both equations. This procedure enables us to find the parameters ω_i and σ_i as follows:

$$\omega_1 = \frac{2\gamma \upsilon^2 + 3\alpha\lambda^2}{4\upsilon^2\lambda^2},\quad \sigma_1 = \frac{2\gamma\lambda^2 + 3\beta\upsilon^2}{4\upsilon^2\lambda^2},\qquad \omega_2 = \frac{12\beta\gamma\upsilon^8 - 21\alpha^2\lambda^8 + 21\alpha^2\upsilon^2\lambda^6 - 4\gamma(3\beta+\gamma)\upsilon^6\lambda^2 + 8\gamma(3\alpha-2\gamma)\upsilon^4\lambda^4}{128\upsilon^6\lambda^6(\upsilon^2-\lambda^2)} \tag{31I}$$

and

$$\sigma_2 = \frac{-12\alpha\gamma\lambda^8 + 21\beta^2\upsilon^8 - 21\beta^2\lambda^2\upsilon^6 + 12\alpha\gamma\upsilon^2\lambda^6 - 8\gamma(3\beta-\gamma)\upsilon^4\lambda^4}{128\upsilon^6\lambda^6(\upsilon^2-\lambda^2)}. \tag{31II}$$

As seen from the previous results, to obtain uniform and bounded solutions, the nonlinear artificial frequencies must be different. It follows that the periodic solution of $x_1(t)$ and $y_1(t)$ becoming

$$x_1(t) = \frac{1}{32\lambda^3\upsilon^5(\upsilon^2-\lambda^2)}\left\{\begin{array}{l}\lambda\left(4\gamma\upsilon^4 + 3\alpha\upsilon^2\lambda^2 - 3\alpha\lambda^4\right)\sin \upsilon t + \\ \alpha\lambda^3(\lambda^2-\upsilon^2)\sin 3\upsilon t + 2\gamma\upsilon^4(\upsilon+\lambda)\sin((\upsilon-2\lambda)t) + \\ 2\gamma\upsilon^4(\lambda-\upsilon)\sin((\upsilon+2\lambda)t)\end{array}\right\} \tag{32I}$$

and

$$y_1(t) = \frac{1}{32\upsilon^3\lambda^5(\upsilon^2-\lambda^2)}\left\{\begin{array}{l}\upsilon\left(-4\gamma\lambda^4 - 3\beta\upsilon^2\lambda^2 + 3\beta\upsilon^4\right)\sin \lambda t + \\ \beta\upsilon^3(\lambda^2-\upsilon^2)\sin 3\lambda t + 2\gamma\lambda^4(\upsilon+\lambda)\sin((2\upsilon-\lambda)t) + \\ 2\gamma\lambda^4(\lambda-\upsilon)\sin((2\upsilon+\lambda)t)\end{array}\right\}. \tag{32II}$$

As mentioned before, for an easy follow-up of the analysis, the functions $x_2(t)$ and $y_2(t)$ are known from the context. To avoid the length of the paper, they will be excluded here. Simultaneously, they are available under the request of the readers. Therefore, the bounded approximate solutions of the equations of motion that are given in equations (8) and (9) may be written as follows:

$$x(t) = \lim_{\rho\to 1}\left(x_0(t) + \rho x_1(t) + \rho^2 x_2(t)\right) \tag{33I}$$

and

$$y(t) = \lim_{\rho\to 1}\left(y_0(t) + \rho y_1(t) + \rho^2 y_2(t)\right). \tag{33II}$$

In fact, the bounded approximate solution as given in equations (33I) and (33II) requires that the arguments of the trigonometric functions must be of real values. For this purpose, combining equations (31) into equations (24I) and (24II), it follows that the nonlinear artificial frequencies satisfy the following characteristic equations.

$$\begin{array}{l}128\upsilon^{10}\lambda^6 - \upsilon^8\left(12\gamma\beta + 64\gamma\lambda^4 + 128\lambda^8 + 128\omega^2\lambda^6\right) + \upsilon^6\lambda^2\left(12\gamma\beta + 4\gamma^2 - 96\alpha\lambda^4 + 64\gamma\lambda^4 + 128\omega^2\lambda^6\right) + \\ \upsilon^4\lambda^4\left(16\gamma^2 + 96\alpha\lambda^4 - 24\alpha\gamma\right) - 21\alpha^2\upsilon^2\lambda^6 + 21\alpha^2\lambda^8 = 0,\end{array} \tag{34I}$$

and

$$\begin{array}{l}128\upsilon^6\lambda^{10} - \lambda^8\left(12\alpha\gamma + 64\gamma\upsilon^4 + 128\upsilon^8 + 128\sigma^2\upsilon^6\right) + \lambda^6\upsilon^2\left(12\alpha\gamma - 96\beta\upsilon^4 + 64\gamma\upsilon^4 + 128\sigma^2\upsilon^6\right) + \\ \lambda^4\upsilon^4\left(8\gamma^2 + 96\beta\upsilon^4 - 24\beta\gamma\right) - 21\beta^2\lambda^2\upsilon^6 + 21\beta^2\upsilon^8 = 0\,.\end{array} \tag{34II}$$

The inspection of the equations (34I), and (34II) indicates that these equations are coupled algebraic equations in the artificial frequencies. Actually, they are rather difficult to be split. For simplicity, one may consider the following transformation:

$$\upsilon = \delta\lambda, \tag{35}$$

where, δ is some constant. It must be of non-unity value. In view of eq. (35), using the HPM again, one may obtain a real value of the artificial frequency υ as follows:

$$\upsilon^4 = a_0 - \rho\left(a_1\upsilon^6 + a_2\upsilon^8\right),\ \rho \in [0,1] \tag{36I}$$

where

$$a_0 = \frac{-24\alpha\gamma\delta^4 - 21\alpha^2(\delta^2-1) + 4\gamma\delta^4(4\gamma + (\gamma+3\beta)\delta^2 - 3\beta\delta^4)}{32(\delta^2-1)(3\alpha+2\gamma\delta^2)},\ a_1 = \frac{-4\omega^2}{3\alpha+2\gamma\delta^2} \text{ and } a_2 = \frac{4}{3\alpha+2\gamma\delta^2}. \tag{36II}$$

In a similar manner, one gets

$$\lambda^4 = b_0 - \rho\left(b_1\lambda^6 + b_2\lambda^8\right),\ \rho \in [0,1] \tag{36III}$$

where

$$b_0 = \frac{\delta^4\left(24\beta\gamma - 8\gamma^2 - 21\beta^2\delta^2(\delta^2-1)\right) - 12\alpha\gamma(\delta^2-1)}{32\delta^4(\delta^2-1)(2\gamma+3\beta\delta^2)},\quad b_1 = \frac{4\delta^2\sigma^2}{(2\gamma+3\beta\delta^2)} \text{ and } b_2 = \frac{4\delta^2}{(2\gamma+3\beta\delta^2)}. \tag{36IV}$$

Following a similar procedure as given above, an approximate solution, up to the second-order, of the artificial frequency, may be written as:

$$\upsilon = \lim_{\rho\to 1}\left(\upsilon_0 + \rho\upsilon_1 + \rho^2\upsilon_2\right) \tag{37I}$$

where

$$\upsilon_0 = \left(\frac{-24\alpha\gamma\delta^4 - 21\alpha^2(\delta^2-1) + 4\gamma\delta^4(4\gamma + (\gamma+3\beta)\delta^2 - 3\beta\delta^4)}{32(\delta^2-1)(3\alpha+2\gamma\delta^2)}\right)^{1/4},\ \upsilon_1 = \frac{-\omega^2\upsilon_0^3 + \upsilon_0^5}{3\alpha+2\gamma\delta^2} \text{ and } \upsilon_2 = \frac{-12\omega^2\upsilon_0^3\upsilon_1 + 16\upsilon_0^5\upsilon_1 - 9\alpha\upsilon_1^2 - 6\alpha\delta^2\upsilon_1^2}{2(3\alpha+2\gamma\delta^2)\upsilon_0}. \tag{37II}$$

Following a similar procedure as given above, an approximate solution, up to the second-order, of the artificial frequency, may be written as:

$$\lambda = \lim_{\rho\to 1}\left(\lambda_0 + \rho\lambda_1 + \rho^2\lambda_2\right) \tag{37III}$$

In light of the real value of the artificial frequencies, the approximate bounded solutions require the following criteria:

$$a_0 > 0 \text{ and } b_0 > 0 \tag{38}$$

It is worthy to note that the inequalities that are given in (38) include all the characteristics of the considered system (8) and (9), in addition to the virtual parameter ($\delta \neq 1$). Indeed, these inequalities guaranteed the bounded solutions of the problem at hand. For more convenience, the bounded solutions of the coupled equations (12I) and (12II) must be pictured. These solutions are given by equations (33I) and (33II). Simultaneously, they depend mainly on the values of the nonlinear artificial frequencies υ, and λ, which are presented by equations (37I) and (37II). Therefore, in what follows, some graphs are plotted, for some chosen sample systems, to indicate the bounded solutions of equations (33I) and (33II), as shown in the following figures:
The figures 3I, and 3II are plotted, simultaneously, in case of $\delta < 1$. Whereas; this not completely verified as in Fig. 4II, where, the calculations showed that $\delta > 1$. Therefore, the condition of the bounded solution of $x(t)$ is automatically satisfied. In contrast, as $b_0 < 0$, one cannot obtain the bounded solution. Consequently, the finiteness the solution of $y(t)$ is not satisfied. As a conclusion, in case of $\delta < 1$, the function $x(t)$ is only pictured. On the other hand, in case of $\delta > 1$, the function $y(t)$ is only displayed Fig. 4II. In all these figures the periodic solutions of both $x(t)$, and $y(t)$ is observed.

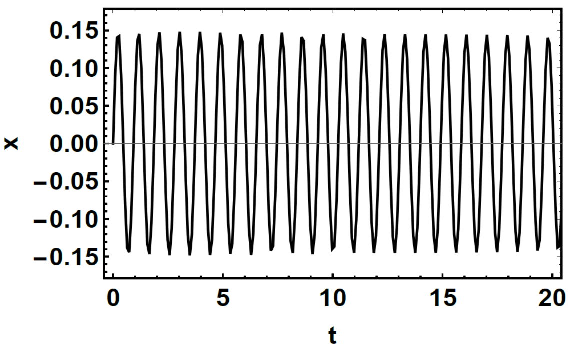

Fig. 3I. Plots the approximate solution that is given in eq. (33I), for a system having the particulars:
$\delta = 0.99, \alpha = 16, \beta = 1, \gamma = 6, \omega = \sqrt{2}, \sigma = 4$.

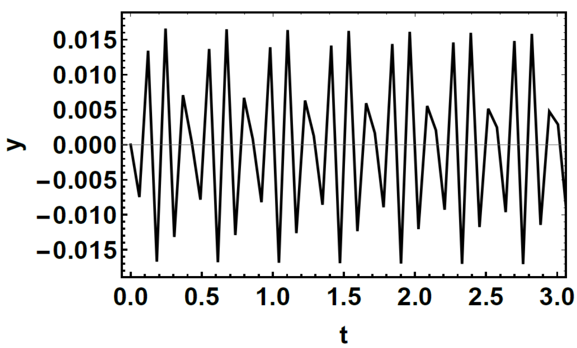

Fig. 3II. Plots the approximate solution that is given in eq. (33II), for a system having the particulars as given in Fig. (2I).

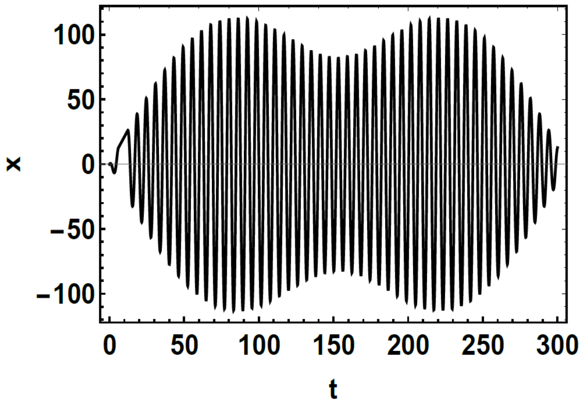

Fig. 4I. Plots the approximate solution that is given in eq. (33I), for a system having the particulars: $\delta = 0.99, \alpha = 1, \gamma = 1, \beta = 1, \omega = 1, \sigma = 1$.

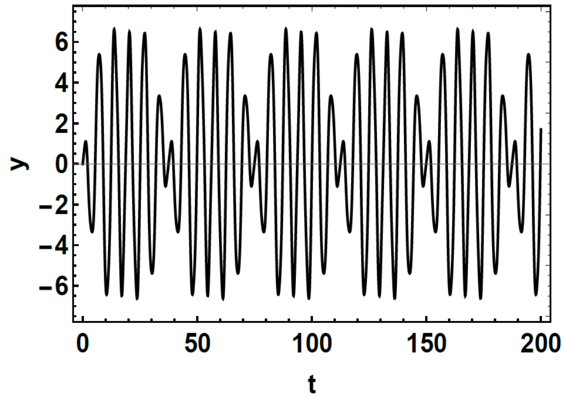

Fig. 4II. Plots the approximate solution that is given in eq. (33II), for a system having the particulars as given in Fig. (3I), but in case of $\delta = 1.1$

4. Multiple-time Scale Technique

To analyze the stability nature, of the coupled nonlinear oscillations as presented in equations (8) and (9), the multiple time-scales technique is adopted; for instance, see Nayfeh [40]. Recently, Ren et al. [41] made a couple of the HPM along with the multiple time scales. The new method has been proved to be promising, effective and powerful mathematical tool in analyzing several nonlinear equations. Additionally, it is extremely effective for the forced nonlinear oscillators. Throughout the following analysis, a uniform valid expansion of the governed equation of motion will be given as a function in terms of three independent variables rather a single parameter. In light of the HPM, one may consider the dependent variable as a function of $t, \rho t$ and $\rho^2 t$. Therefore, the expansion, that gives the response, is addressed as a function of multiple independent variables, or scales, instead of a single one. The technique of multiple-time scales is considered as a more general method in the perturbation theory.
For this purpose, one begins by introducing the new independent variables according to

$$T_n = \rho^n t, \ n = 0,1,2,.... \tag{39}$$

Therefore, the ordinary derivatives with respect to the time-dependent variable will expanded in terms of the partial derivatives with respect to the time-scales $T_0, T_1, T_2....$ as follows:

$$\frac{d}{dt} \equiv \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \frac{dT_2}{dt}\frac{\partial}{\partial T_2} + ... = D_0 + \rho D_1 + \rho^2 D_2 + ... \tag{40}$$

and

$$\frac{d^2}{dt^2} \equiv D_0^2 + 2\rho D_0 D_1 + \rho^2\left(D_1^2 + 2D_0 D_2\right) + ... \tag{41}$$

where $D_n \equiv \partial / \partial T_n$. Assuming that the solutions of equations (8) and (9) may be represented as an expansion as follows:

$$x(t;\rho) = x_0(T_0,T_1,T_2,....) + \rho\, x_1(T_0,T_1,T_2,....) + \rho^2 x_2(T_0,T_1,T_2,....) + ... \tag{42I}$$

and

$$y(t;\rho) = y_0(T_0,T_1,T_2,....) + \rho\, y_1(T_0,T_1,T_2,....) + \rho^2 y_2(T_0,T_1,T_2,....) + ... \tag{42II}$$

It should be noted that the number of the independent time scales equals the order at which the expansion is carried out. Namely, if the expansion is carried out up to $O(\rho^2)$, then T_0 and T_1 are only needed.

For more convenience, to obtain an accurate expansion, the expansion is carried up to $O(\rho^3)$. In this case, three time scales, T_0, T_1 and T_2 are required.
Substituting from equations (41), (42I) and (42II) into the equations (8) and (9), then equating the coefficient of like powers of ρ to zero, one finds the following equations:

$$\rho^0: \quad \left(D_0^2 + \omega^2\right)x_0 = 0, \tag{43I}$$

$$\rho^0: \quad \left(D_0^2 + \sigma^2\right)y_0 = 0, \tag{43II}$$

$$\rho: \quad \left(D_0^2 + \omega^2\right)x_1 = -2D_0 D_1 x_0 - \alpha x_0^3 - \gamma x_0 y_0^2, \tag{44I}$$

$$\rho: \quad \left(D_0^2 + \sigma^2\right)y_1 = -2D_0 D_1 y_0 - \beta y_0^3 - \gamma y_0 x_0^2, \tag{44II}$$

$$\rho^2: \quad \left(D_0^2 + \omega^2\right)x_2 = -\left(2D_0 D_2 x_0 + 2D_0 D_1 x_1 + D_1^2 x_0 + 3\alpha x_0^2 x_1 + \gamma x_1 y_0^2 + 2\gamma x_0 y_0 y_1\right) \tag{45I}$$

and

$$\rho^2: \quad \left(D_0^2 + \sigma^2\right)y_2 = -\left(2D_0 D_2 y_0 + 2D_0 D_1 y_1 + D_1^2 y_0 + 3\beta y_0^2 y_1 + \gamma y_1 x_0^2 + 2\gamma y_0 x_0 x_1\right). \tag{45II}$$

With this approach, it is convenient to write the solutions of equations (43I) and (43II) in the following form:

$$x_0(T_0,T_1,T_2)=A(T_1,T_2)e^{i\omega T_0}+c.c. \tag{46I}$$

$$y_0(T_0,T_1,T_2)=B(T_1,T_2)e^{i\sigma T_0}+c.c. \tag{46II}$$

where $c.c.$ represents the complex conjugate of the preceding terms. Substituting from equations (46I) and (46II) into equations (44I) and (44II), one finds

$$\left(D_0^2+\omega^2\right)x_1=-\left(2i\omega D_1A+3\alpha A^2\overline{A}+2\gamma AB\overline{B}\right)e^{i\omega T_0}-\alpha A^3e^{3i\omega T_0}-\gamma AB^2e^{i(\omega+2\sigma)T_0}-\gamma A\overline{B}^2e^{i(\omega-2\sigma)T_0}+c.c. \tag{47I}$$

$$\left(D_0^2+\sigma^2\right)y_1=-\left(2i\sigma D_1B+3\beta B^2\overline{B}+2\gamma BA\overline{A}\right)e^{i\sigma T_0}-\beta B^3e^{3i\sigma T_0}-\gamma BA^2e^{i(\sigma+2\omega)T_0}-\gamma B\overline{A}^2e^{i(\sigma-2\omega)T_0}+c.c. \tag{47II}$$

The required uniform expansions of the functions x_1 and y_1 need the elimination of the secular terms. Therefore, to obtain a uniform valid expansion, the secular terms must be excluded. The sources of these secular terms come from the coefficients of the exponentials $e^{\pm i\omega T_0}$, and $e^{\pm i\sigma T_0}$. Therefore, the uniform valid expansions require

$$2i\omega D_1A+2\gamma AB\overline{B}+3\alpha A^2\overline{A}=0 \tag{48I}$$

$$2i\sigma D_1B+2\gamma BA\overline{A}+3\beta B^2\overline{B}=0. \tag{48II}$$

Equations (48I) and (48II) are well-known as the solvability conditions. Sometimes, they may called the amplitude equations. It follows that uniform solution of the particular solutions of equations (47I) and (47II) may be written as

$$x_1(T_0,T_1,T_2)=\frac{\alpha A^3}{8\omega^2}e^{3i\omega T_0}-\frac{\gamma AB^2}{\omega^2-(\omega+2\sigma)^2}e^{i(\omega+2\sigma)T_0}-\frac{\gamma A\overline{B}^2}{\omega^2-(\omega-2\sigma)^2}e^{i(\omega-2\sigma)T_0}+c.c. \tag{49I}$$

$$y_1(T_0,T_1,T_2)=\frac{\beta B^3}{8\sigma^2}e^{3i\sigma T_0}-\frac{\gamma BA^2}{\sigma^2-(\sigma+2\omega)^2}e^{i(\sigma+2\omega)T_0}-\frac{\gamma B\overline{A}^2}{\sigma^2-(\sigma-2\omega)^2}e^{i(\sigma-2\omega)T_0}+c.c.. \tag{49II}$$

Substituting from equations (46I) (46II), (49I) and (49II) into equations (45I) and (45II), after lengthy, but straightforward calculations, one finds the following equations:

$$\begin{aligned}
(D_0^2+\omega^2)x_2=&\left(-2i\omega D_2A+\frac{15\alpha^2A^3\overline{A}^2}{8\omega^2}+\frac{\gamma\left(3\alpha\sigma^2+(\gamma-3\alpha)\omega^2\right)A^2\overline{A}B\overline{B}}{\omega^2\left(\omega^2-\sigma^2\right)}+\frac{\gamma^2\left(3\omega^2-2\sigma^2\right)AB^2\overline{B}^2}{2\omega^2\left(\omega^2-\sigma^2\right)}\right)e^{i\omega T_0}+\\
&\left(\frac{21\alpha^2A^4\overline{A}}{8\omega^2}+\frac{\gamma\left(2\alpha\sigma^2+(\gamma-2\alpha)\omega^2\right)A^3B\overline{B}}{\omega^2\left(\omega^2-\sigma^2\right)}\right)e^{3i\omega T_0}-\frac{3\alpha^2A^5}{8\omega^2}e^{5i\omega T_0}+\left(\frac{\gamma\left(4\gamma\sigma^2+\beta\omega(11\sigma+5\omega)\right)AB^3\overline{B}}{4\omega\sigma^2(\omega+\sigma)}-\right.\\
&\left.\frac{\gamma\left(\gamma(\sigma-\omega)(\sigma^2-4\sigma\omega-2\omega^2)-3\alpha\sigma^2(\sigma-2\omega)\right)A^2\overline{A}B^2}{2\omega\sigma^2\left(\sigma^2-\omega^2\right)}\right)e^{i(\omega+2\sigma)T_0}+\left(\frac{\gamma\left(4\gamma\sigma^2+\beta\omega(5\omega-11\sigma)\right)AB\overline{B}^3}{4\sigma^2(\omega-\sigma)}+\right.\\
&\left.\frac{\gamma\left(\gamma(\sigma+\omega)(\sigma^2+4\sigma\omega-2\omega^2)-3\alpha\sigma^2(\sigma+2\omega)\right)A^2\overline{A}\overline{B}^2}{2\omega\sigma^2\left(\sigma^2-\omega^2\right)}\right)e^{i(\omega-2\sigma)T_0}-\frac{\gamma\left(\gamma\sigma+\beta(\omega+\sigma)\right)AB^4}{4\sigma^2(\omega+\sigma)}e^{i(\omega+4\sigma)T_0}+\\
&\frac{\gamma\left(\gamma\sigma-\beta(\omega-\sigma)\right)A\overline{B}^4}{4\sigma^2(\omega-\sigma)}e^{i(\omega-4\sigma)T_0}-\frac{\gamma\left(4\gamma\sigma\omega+\alpha(\sigma^2+\sigma\omega+6\omega^2)\right)A^3B^2}{8\sigma\omega^2(\omega+\sigma)}e^{i(2\omega+3\sigma)T_0}-\\
&\frac{\gamma\left(4\gamma\sigma\omega-\alpha(\sigma^2+-\sigma\omega+6\omega^2)\right)A^3B^2}{8\sigma\omega^2(\omega-\sigma)}e^{i(2\omega-3\sigma)T_0}+c.c.
\end{aligned} \tag{50I}$$

and

$$\begin{aligned}
(D_0^2+\sigma^2)y_2=&\left(-2i\sigma D_2B+\frac{15\beta^2B^3\overline{B}^2}{8\sigma^2}+\frac{\gamma\left(3\beta\omega^2+(\gamma-3\beta)\sigma^2\right)B^2\overline{B}A\overline{A}}{\sigma^2\left(\omega^2-\sigma^2\right)}+\frac{\gamma^2\left(3\sigma^2-2\omega^2\right)BA^2\overline{A}^2}{2\sigma^2\left(\sigma^2-\omega^2\right)}\right)e^{i\sigma T_0}+\\
&\left(\frac{21\beta^2B^4\overline{B}}{8\sigma^2}+\frac{\gamma\left(2\beta\omega^2+(\gamma-2\beta)\sigma^2\right)B^3A\overline{A}}{\sigma^2\left(\omega^2-\sigma^2\right)}\right)e^{3i\sigma T_0}-\frac{3\beta^2B^5}{8\sigma^2}e^{5i\sigma T_0}+\left(\frac{\gamma\left(4\gamma\omega^2+\alpha\sigma(11\omega+5\sigma)\right)BA^3\overline{A}}{4\sigma\omega^2(\omega+\sigma)}+\right.\\
&\left.\frac{\gamma\left(\gamma(\sigma-\omega)(-\omega^2+4\sigma\omega+2\sigma^2)+3\beta\omega^2(2\sigma-\omega)\right)A^2\overline{B}B^2}{2\sigma\omega^2\left(\sigma^2-\omega^2\right)}\right)e^{i(\sigma+2\omega)T_0}+\left(\frac{\gamma\left(4\gamma\omega^2+\alpha\sigma(5\sigma-11\omega)\right)AB\overline{A}^3}{4\omega^2(\sigma-\omega)}+\right.\\
&\left.\frac{\gamma\left(\gamma(\sigma+\omega)(2\sigma^2-4\sigma\omega-\omega^2)-3\beta\omega^2(2\sigma+\omega)\right)B^2\overline{A}^2\overline{B}}{2\sigma\omega^2\left(\sigma^2-\omega^2\right)}\right)e^{i(\sigma-2\omega)T_0}-\frac{\gamma\left(\gamma\omega+\alpha(\omega+\sigma)\right)BA^4}{4\omega^2(\omega+\sigma)}e^{i(\sigma+4\omega)T_0}+\\
&\frac{\gamma\left(\gamma\omega-\alpha(\omega-\sigma)\right)B\overline{A}^4}{4\omega^2(\sigma-\omega)}e^{i(\sigma-4\omega)T_0}-\frac{\gamma\left(4\gamma\sigma\omega+\beta(6\sigma^2+\sigma\omega+\omega^2)\right)A^2B^3}{8\omega\sigma^2(\omega+\sigma)}e^{i(2\sigma+3\omega)T_0}+\\
&\frac{\gamma\left(4\gamma\sigma\omega-\beta(6\sigma^2-\sigma\omega+\omega^2)\right)A^3B^2}{8\omega\sigma^2(\omega-\sigma)}e^{i(2\sigma-3\omega)T_0}+c.c.
\end{aligned} \tag{50II}$$

To eliminate the secular terms of equations (50I) and (50II), one finds

$$-2i\omega D_2A+\frac{15\alpha^2A^3\bar{A}^2}{8\omega^2}+\frac{\gamma\left(3\alpha\sigma^2+(\gamma-3\alpha)\omega^2\right)A^2\bar{A}B\bar{B}}{\omega^2\left(\omega^2-\sigma^2\right)}+\frac{\gamma^2\left(3\omega^2-2\sigma^2\right)AB^2\bar{B}^2}{2\omega^2\left(\omega^2-\sigma^2\right)}=0 \tag{51I}$$

and

$$-2i\sigma D_2B+\frac{15\beta^2B^3\bar{B}^2}{8\sigma^2}+\frac{\gamma\left(3\beta\omega^2+(\gamma-3\beta)\sigma^2\right)B^2\bar{B}A\bar{A}}{\sigma^2\left(\omega^2-\sigma^2\right)}+\frac{\gamma^2\left(3\sigma^2-2\omega^2\right)BA^2\bar{A}^2}{2\sigma^2\left(\sigma^2-\omega^2\right)}=0. \tag{51II}$$

The solution of the solvability conditions will be presented in two categories; the first category is concerned with the non-resonance case; whereas, the second one concerns with the resonance one. These approaches will be given in the following Subsections:

4.1. Stability Analysis in the non-Resonance Case

In order to investigate the stability profile, throughout the non-resonance case, one may return back to the amplitude equations that are given by equations (48I), (48II), (51I) and (51II). Actually, these equations may be used to determine the unknown complex functions A and B in terms of the time-independent variables T_1 and T_2. Furthermore, the stability behavior depends mainly on the behavior of these functions. For this purpose, one may integrate equations (48I) and (48II) partially with respect to the variable T_1. Then integrate equations (51I) and (51II) partially with respect to the variable T_2. In another procedure, simply, multiply equations (48I), (48II), (51I) and (51II) by ρ and ρ^2, respectively, then adding them. It follows that the partial differentiation in these amplitude equations may be transformed into dA/dt and dB/dt. Finally, one may find the following amplitude equations:

$$2i\omega\frac{dA}{dt}+3\alpha A^2\bar{A}-\frac{15\alpha^2}{8\omega^2}A^3\bar{A}^2+2\gamma AB\bar{B}-\frac{\gamma\left(3\alpha\sigma^2+(\gamma-3\alpha)\omega^2\right)A^2\bar{A}B\bar{B}}{\omega^2\left(\omega^2-\sigma^2\right)}-\frac{\gamma^2\left(3\omega^2-2\sigma^2\right)AB^2\bar{B}^2}{2\omega^2\left(\omega^2-\sigma^2\right)}=0 \tag{52I}$$

and

$$2i\sigma\frac{dB}{dt}+3\beta B^2\bar{B}-\frac{15\beta^2}{8\sigma^2}B^3\bar{B}^2+2\gamma BA\bar{A}-\frac{\gamma\left(3\beta\omega^2+(\gamma-3\beta)\sigma^2\right)B^2\bar{B}A\bar{A}}{\sigma^2\left(\omega^2-\sigma^2\right)}-\frac{\gamma^2\left(3\sigma^2-2\omega^2\right)BA^2\bar{A}^2}{2\sigma^2\left(\sigma^2-\omega^2\right)}=0. \tag{52II}$$

Equations (52I) and (52II) are first-order nonlinear differential equations with complex coefficients. The amplitude equations will govern the stability criteria of the problem in the non-resonance case. The solutions of these equations may be obtained by utilizing the polar form formula as given below:

$$A(t)=\Gamma_1(t)e^{i\lambda_1(t)} \tag{53I}$$

and

$$B(t)=\Gamma_2(t)e^{i\lambda_2(t)}, \tag{53II}$$

where the functions $\Gamma_1(t)$, $\Gamma_2(t)$, $\lambda_1(t)$ and $\lambda_2(t)$ are real functions on time.

Substituting from equations (53I) and (53II) into equations (52I) and (52II), then equating the real and imaginary parts on both sides. Simultaneously, two differential equations and two algebraic equations will be obtained. The solution of these equations may be written as: $\Gamma_1(t)=c_1$, $\Gamma_2(t)=c_2$, $\lambda_1(t)=s_1t$ and $\lambda_2(t)=s_2t$, where, c_1 and c_2 are two real constants. The coefficients s_1 and s_2 are given in the Appendix. One may choose appropriate boundary conditions, so that they may be considered as unities. On the other hand, the quantities s_1 and s_2 are real ones. They are composed of the given parameters of the problem at hand. Therefore, in view of the non-resonance case, the considered system has, always, a periodic solution during all time. Finally, the constants s_1 and s_2 are given as:
Along with the solutions in the non-resonance case, one gets:

$$s_1=\frac{4\gamma^2c_2^4(2\sigma^2-3\omega^2)+3\alpha c_1^2(\omega^2-\sigma^2)(8\omega^2-5\alpha c_1^2)+8\gamma c_2^2\left(2\omega^2(\omega^2-\sigma^2)\right)+8\gamma c_1^2c_2^2\left(3\alpha\sigma^2+\omega^2(\gamma-3\alpha)\right)}{16\omega^3(\omega^2-\sigma^2)},$$

$$s_2=\frac{4\gamma^2c_1^4(2\omega^2-3\sigma^2)+3\beta c_2^2(\sigma^2-\omega^2)(8\sigma^2-5\beta c_2^2)+8\gamma c_1^2\left(2\sigma^2(\sigma^2-\omega^2)\right)+8\gamma c_1^2c_2^2\left(3\beta\omega^2+\sigma^2(\gamma-3\beta)\right)}{16\sigma^3(\sigma^2-\omega^2)}$$

As a final result in the case of non-resonant, the approximate periodic solutions of the equation that is given in equations (8) and (9) may be written as follows:

$$x(t)=\lim_{\rho\to1}\left(x_0(t)+\rho x_1(t)+\rho^2x_2(t)\right) \tag{54I}$$

$$y(t)=\lim_{\rho\to1}\left(y_0(t)+\rho y_1(t)+\rho^2y_2(t)\right). \tag{54II}$$

4.2. Stability Analysis of the Resonant Case

As seen in the previous Subsection, the non-resonant case fails to achieve the stability criteria. Subsequently, the following discussion is devoted to introducing a resonant case. It should be noted that resonance occurs when the natural frequency ω of eq. (12I) is nearer to the natural frequency σ of eq. (12II).

Actually, there are several powers in the different stages as shown in equations (47I), (47II), (50I) and (50II). It follows that there are two super-harmonic resonance: as $\sigma \approx \omega/2$ and $\sigma \approx \omega/3$ and one sub-harmonic resonance: as $\sigma \approx \omega$. To avoid the lengthy of the calculations, the present analysis is only concerned with one of these resonance cases. As an example, one may consider the sub-hamonic resonance. Therefore, if the natural frequency σ is nearer to the natural frequency ω, one may introduce a detaining parameter Ω, this advantage helps us in introducing an additional secular term. Accordingly, one may write

$$\sigma = \omega + \rho\Omega \tag{55}$$

In this case, one finds

$$-i(\omega - 2\sigma)T_0 = i\omega T_0 + 2i\Omega T_1 \tag{56}$$

Therefore, the secular terms will be rising again. At this stage, the solvability conditions given by equations (48I) and (48II) will be modified to become:

$$2i\omega D_1 A + 2\gamma AB\overline{B} + 3\alpha A^2\overline{A} + \gamma B^2\overline{A}e^{2i\Omega T_1} = 0 \tag{57I}$$

$$2i\sigma D_1 B + 2\gamma AB\overline{A} + 3\beta B^2\overline{B} + \gamma A^2\overline{B}e^{-2i\Omega T_1} = 0. \tag{57II}$$

The uniform valid solutions of equations (47I) and (47II), in this case, may be expressed as:

$$x_1(T_0,T_1,T_2) = \frac{\alpha A^3}{8\omega^2}e^{3i\omega T_0} - \frac{\gamma AB^2}{\omega^2 - (\omega+2\sigma)^2}e^{i(\omega+2\sigma)T_0} + c.c. \tag{58I}$$

$$y_1(T_0,T_1,T_2) = \frac{\beta B^3}{8\sigma^2}e^{3i\sigma T_0} - \frac{\gamma BA^2}{\sigma^2 - (\sigma+2\omega)^2}e^{i(\sigma+2\omega)T_0} + c.c. \tag{58II}$$

Substituting from the transformation (56), into the right-hand side of equations (50I) and (50II), the elimination of these secular terms requires

$$\begin{aligned}&2i\omega D_2 A - \frac{15\alpha^2 A^3\overline{A}^2}{8\omega^2} + \frac{\gamma\left(3\alpha\sigma^2 + (\gamma-3\alpha)\omega^2\right)A^2\overline{A}B\overline{B}}{\omega^2\left(\omega^2-\sigma^2\right)} + \frac{\gamma^2\left(3\omega^2-2\sigma^2\right)AB^2\overline{B}^2}{2\omega^2\left(\omega^2-\sigma^2\right)} + \\ &e^{2i\Omega T_1}\left(\frac{\gamma\left(4\gamma\sigma^2+\beta\omega(5\omega-11\sigma)\right)B^3\overline{A}\overline{B}}{4\omega\sigma^2(\sigma-\omega)} + \frac{\gamma\left(\gamma(\sigma+\omega)(\sigma^2+4\sigma\omega-2\omega^2)-3\alpha\sigma^2(\sigma+2\omega)\right)AB^2\overline{A}^2}{2\omega\sigma^2\left(\sigma^2-\omega^2\right)}\right) = 0\end{aligned} \tag{59I}$$

and

$$\begin{aligned}&2i\sigma D_2 B - \frac{15\beta^2 B^3\overline{B}^2}{8\sigma^2} + \frac{\gamma\left(3\beta\omega^2 + (\gamma-3\beta)\sigma^2\right)AB^2\overline{A}\overline{B}}{\sigma^2\left(\sigma^2-\omega^2\right)} + \frac{\gamma^2\left(3\sigma^2-2\omega^2\right)BA^2\overline{A}^2}{2\sigma^2\left(\omega^2-\sigma^2\right)} + \\ &e^{-2i\Omega T_1}\left(\frac{\gamma\left(4\gamma\omega^2+\alpha\sigma(5\sigma-11\omega)\right)A^3\overline{A}\overline{B}}{4\sigma\omega^2(\sigma-\omega)} + \frac{\gamma\left(\gamma(\sigma+\omega)(2\sigma^2-4\sigma\omega-\omega^2)-3\beta\omega^2(2\sigma+\omega)\right)BA^2\overline{B}^2}{2\sigma\omega^2\left(\sigma^2-\omega^2\right)}\right) = 0.\end{aligned} \tag{59II}$$

Using similar arguments as given in the non-resonance case to obtain equations (52I) and (52II), one finds

$$\frac{dA}{dt} + ia_1 A^2\overline{A} + ia_2 A^3\overline{A}^2 + ia_3 AB\overline{B} + ia_4 A^2\overline{A}B\overline{B} + ia_5 AB^2\overline{B}^2 + ie^{2i\Omega t}\left(a_6 B^2\overline{A} + a_7 B^3\overline{A}\overline{B} + a_8 AB^2\overline{A}^2\right) = 0 \tag{60I}$$

and

$$\frac{dB}{dt} + ib_1 B^2\overline{B} + ib_2 B^3\overline{B}^2 + ib_3 AB\overline{A} + ib_4 AB^2\overline{A}\overline{B} + ib_5 BA^2\overline{A}^2 + ie^{-2i\Omega t}\left(b_6 A^2\overline{B} + b_7 A^3\overline{A}\overline{B} + b_8 BA^2\overline{B}^2\right) = 0, \tag{60II}$$

where the coefficients a_i and b_i are given as follows:

$$\begin{aligned}&a_1 = -\frac{3\alpha}{2\omega},\ a_2 = \frac{15\alpha^2}{16\omega^3},\ a_3 = -\frac{\gamma}{\omega},\ a_4 = -\frac{\gamma\left(3\alpha\sigma^2+(\gamma-3\alpha)\omega^2\right)}{2\omega^3\left(\omega^2-\sigma^2\right)},\ a_5 = \frac{\gamma^2\left(3\omega^2-2\sigma^2\right)}{4\omega^3\left(\omega^2-\sigma^2\right)},\ a_6 = -\frac{\gamma}{2\omega},\\ &a_7 = \frac{\gamma\left(-3\alpha\sigma^2(\sigma-2\omega)+\gamma(\sigma-\omega)(\sigma^2+4\omega\sigma-2\omega^2)\right)}{4\omega^2\sigma^2\left(\sigma^2-\omega^2\right)} \text{ and } a_8 = -\frac{\gamma\left(4\gamma\sigma^2+\beta\omega(5\omega-11\sigma)\right)}{8\omega^2\sigma^2(\sigma-\omega)}.\end{aligned}$$

$$\begin{aligned}&b_1 = -\frac{3\beta}{2\sigma},\ b_2 = \frac{15\beta^2}{16\sigma^3},\ b_3 = -\frac{\gamma}{\sigma},\ b_4 = \frac{\gamma\left(3\beta\omega^2+(\gamma-3\beta)\sigma^2\right)}{2\sigma^3\left(\omega^2-\sigma^2\right)},\ b_5 = \frac{\gamma^2\left(3\sigma^2-2\omega^2\right)}{4\sigma^3\left(\sigma^2-\omega^2\right)},\ b_6 = -\frac{\gamma}{2\sigma},\\ &b_7 = \frac{\gamma\left(3\beta\omega^2(\omega+2\sigma)+\gamma(\sigma+\omega)(2\sigma^2-4\omega\sigma-\omega^2)\right)}{4\omega^2\sigma^2\left(\sigma^2-\omega^2\right)} \text{ and } b_8 = -\frac{\gamma\left(4\gamma\omega^2+\alpha\sigma(5\sigma-11\omega)\right)}{8\omega^2\sigma^2(\sigma-\omega)}.\end{aligned}$$

Equations (60I) and (60II) are first-order nonlinear differential equations with complex and variable coefficients. Their solutions may be obtained by the following transformation:

$$A(t) = \xi_1(t)e^{2i\Omega t}, \tag{61I}$$

$$B(t) = \xi_2(t) e^{i\Omega t}, \tag{61II}$$

where, $\xi_1(t)$ and $\xi_2(t)$ are two real functions on time. Substituting from equations (61I) and (61II) into equations (60I) and (60II), then equating the real and imaginary terms, the resulting differential equations showed that the two functions $\xi_1(t)$ and $\xi_2(t)$ are of constant values. These constant values may be written as follows:

$$a_2\xi_1^4 + (a_5 + a_7)\xi_2^4 + (a_4 + a_8)\xi_1^2\xi_2^2 + a_1\xi_1^2 + (a_3 + a_6)\xi_2^2 + 2(\sigma - \omega) = 0 \tag{62I}$$

and

$$b_2\xi_2^4 + (b_5 + b_7)\xi_1^4 + (b_4 + b_8)\xi_1^2\xi_2^2 + b_1\xi_2^2 + (b_3 + b_6)\xi_1^2 + \sigma - \omega = 0 \tag{62II}$$

Equations (62I), and (62II) are coupled equations for ξ_1, and ξ_2. Actually, they are rather difficult to be divided. As previously shown, throughout the expanded frequency analysis, one may consider a virtual parameter ε, such that

$$\xi_1 = \varepsilon \xi_2, \tag{63}$$

where, ε is some constant. It must be of non-unity value. Equations (62I) and (62II), are then become

$$\xi_1^4 + q_1\xi_1^2 + q_2 = 0 \tag{64I}$$

$$\xi_2^4 + q_3\xi_2^2 + q_4 = 0 \tag{64II}$$

where, $q_i, (i = 1,2,3,4)$ are given as follows:

$$q_1 = \frac{\varepsilon^2(a_1\varepsilon^2 + a_3 + a_6)}{a_2\varepsilon^4 + (a_4 + a_8)\varepsilon^2 + a_5 + a_7}, \quad q_2 = \frac{2\Omega\varepsilon^4}{a_2\varepsilon^4 + (a_4 + a_8)\varepsilon^2 + a_5 + a_7},$$
$$q_3 = \frac{\varepsilon^2(b_3 + b_6) + b_1}{(b_5 + b_7)\varepsilon^4 + (b_4 + b_8)\varepsilon^2 + b_2} \quad \text{and} \quad q_4 = \frac{\Omega}{(b_5 + b_7)\varepsilon^4 + (b_4 + b_8)\varepsilon^2 + b_2}.$$

As shown in equations (61I), and (61II) the functions $\xi_1(t)$ and $\xi_2(t)$ are two real functions on time, and consequently, the constants ξ_1, and must have a real nature. From the elementary algebra: the criteria of eq. (64I), require

$$q_1 < 0, \quad q_2 > 0, \quad \text{and} \quad q_1^2 - 4q_2 > 0, \tag{65I}$$

whereas, for eq. (64II), one finds

$$q_3 < 0, \quad q_4 < 0, \quad \text{and} \quad q_3^2 - 4q_4 > 0. \tag{65II}$$

For more convenience, the criteria of the periodic approximate solutions will be plotted. Therefore, in what follows, the figures 5I, and 5II are pictured to illustrate the stability criteria (65I), and (65II). In these figures the nonlinear natural frequencies (σ and ω) are plotted versus the parameter γ, for sample chosen systems. Evidently, the ordered pairs in the shaded region indicate that the solution as it is a periodic one.

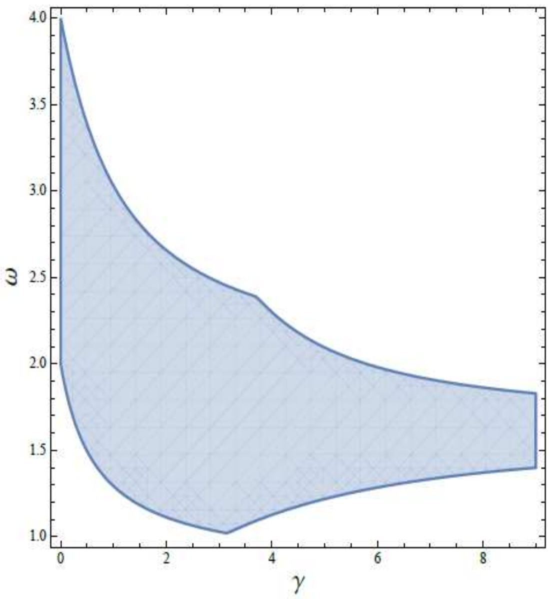

Fig. 5I. Plots the phase diagram to show the natural frequency ω versus the parameter γ, for a dynamical system having the particulars: $\beta = 1.0$, $\Omega = 0.25$, $\alpha = 16$, $\sigma = 4.0$, and $\varepsilon = 0.6$

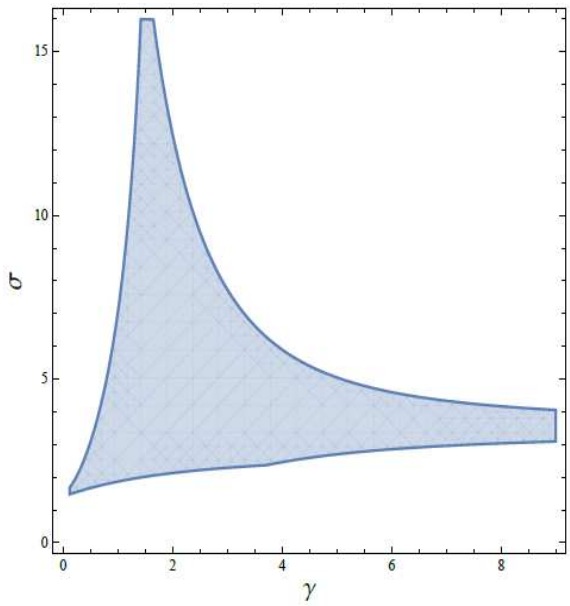

Fig. 5II. Plots the phase diagram to show the natural frequency σ versus the parameter γ, for a dynamical system having the particulars: $\beta = 1.0$, $\Omega = 0.25$, $\alpha = 16$, $\omega = \sqrt{2}$, and $\varepsilon = 0.6$

5. Concluding Remarks

The current paper is concerned with the analysis of governing equations of coupled nonlinear oscillations. This system is comprehensively examined throughout different approaches. The first one deals with a traditional analytic approximate solution. The He-Laplace approach is utilized to produce this solution. Unfortunately, because of the presence of the secular terms, this solution is physically un-preferred. Simultaneously, no one has any authority to ignore these terms. Therefore, in order to eliminate these secular terms, a modification to the classical HPM is made. This approach needs an inclusion of an artificial frequency, which sometimes called the nonlinear frequency. Therefore, the nonlinear frequency of each differential equation is achieved. Indeed, the significance of the expanded frequency is an early concept. It was first proposed by Lindstedt-Poincaré technique throughout the perturbation theory. This approach resulted in periodic approximate solutions of the considered system. During the latter approach, the criteria in finding the periodic approximate solutions are achieved. For more convenience, the solutions of the previous two cases are plotted. Furthermore, the multiple time scale technique is utilized. For more accuracy, three-time scales are considered. Unluckily, the non-resonance case does not produce any criteria to the periodic solution. Subsequently, the analysis proceeds to include the resonance case. Several criteria, for achieving the periodic solutions, are accomplished. Furthermore, the obtained criteria are plotted. It is concluded that the multiple time scales method is the most powerful mathematical tool in analyzing the nonlinear oscillation system that arises in physical application and practical engineering. The concluding remarks of the present work may be summarized as follows:

- The classical approximate solutions, involving secular terms, are given in equations (23I) and (23II).
- The periodic approximate solutions via an expanded parameter are given in equations (33I) and (33II).
- The non-resonance case concludes that the system is always stable.
- The resonance case reveals the stability criteria as given in (65I) and (65II).
- Figures (5I) and (5II) result all the ordered pairs that yield the periodic solutions of the considered coupled nonlinear oscillations.

Author Contributions

All authors planned the scheme, initiated the project, developed the mathematical modeling and examined the theory validation. The manuscript was written through the contribution by all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript

Conflict of Interest

The authors declared that no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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