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## Symmetry Reduction and Exact Solutions of a Class of Wave Equations

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**Abstract.** In this paper, the Lie symmetries and similarity reduction of a class of wave equations are investigated. First, Lie algorithm is used to get the determining equations of symmetry for the given equations which are complicated and difficult to be solved. Next, differential form of Wu's method is used to solve this problem. Moreover, a special case of differential invariant method is used to get similarity reduction of the given equations.

**Keywords:** Lie algorithm, Similarity reduction, Differential form Wu's method.

### 1. Introduction

The partial differential equations (PDEs) arises in many fields like the condense matter physics, fluid mechanics and optics, etc, which exhibit a rich variety of nonlinear phenomena. Finding the exact solutions of the PDEs is always one of the central interests in mathematics and physics. Over the past decades, various of methods have been developed trying to develop exact solutions for different types of PDEs. Few of the wellknown methods can be listed as: the homotopy perturbation method [1-3], the variational iteration method [4-6], the exp-function method [7-9], the semi-inverse variational method [10], the Taylor series method [11], the dynamical systems method [12], the Rayleigh-Ritz method [13], the method of separation of variables [14], etc. The Lie group method is a powerful and direct approach to construct exact solutions of nonlinear differential equations. Moreover, the Lie group method considers different types of exact solutions for PDEs can be considered, such as the traveling wave solutions, similarity solutions, and so on [15]. Tian et al. have performed some work on symmetries of PDEs; in [16] the Wu's method is used to complete symmetry classification of PDEs. The traditional Lie algorithm is used to complete symmetry classification of the diffusion-convection equation in [17]. The Wu's method, in [18], is used to simplify the symmetry computation of PDEs.

In this paper, it is focused on a class of wave equation:

$$\begin{pmatrix} 1 \\ u_t \end{pmatrix} = u_{xx} + u_x, \quad (1)$$

and its corresponding systems

$$v_t = \frac{1}{u}, v_x = u + u_x, \quad (2)$$

where, the physical interpretation is not known at present. In a special case of similarity reduction method, the analytical solution of Eqs. (1) and (2) are presented in this work. To the best of the author knowledge, there has been no report on this special case of similarity reduction method.

### 2. Lie Algorithm

Consider the following PDEs:

$$F_i[u] = F_i(x, u, u_{(1)}, \dots) = 0, i = 1, 2, \dots, \quad (3)$$

**Definition 1.** The one-parameter Lie group of point transformations



$$\begin{cases} x^* = X(x, u, \varepsilon), \\ u^* = U(x, u, \varepsilon), \end{cases} \quad (4)$$

leaves invariants the PDEs (3), i.e., is a point symmetry admitted by PDEs (3), if and only if its  $k$ th extension leaves Eq. (3) invariant.

**Theorem 1.** Let

$$X = \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta(x, u) \frac{\partial}{\partial u}, \quad (5)$$

be the infinitesimal generator of the Lie group of point transformations (4). Let

$$X^{(k)} = \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta(x, u) \frac{\partial}{\partial u} + \eta_i^{(1)}(x, u, \partial u) \frac{\partial}{\partial u_i} + \dots + \eta_{i_1 i_2 \dots i_k}^{(k)}(x, u, \partial u, \partial^2 u, \dots, \partial^k u) \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}}, \quad (6)$$

be the  $k$ th-extended infinitesimal generator of (5). Then one-parameter Lie group of point transformations given in (4) is admitted by Eq. (3), if and only if:

$$X^{(k)} F_i[u] = 0, \text{ when } F_i[u] = 0. \quad (7)$$

An over-determined system of PDEs

$$\Delta = \Delta(x, u, \xi_i, \eta), \quad (8)$$

Eq.s (8) are called determining equations (DTEs) for the infinitesimal  $\xi_i, \eta$ , which is obtained by equating the coefficients of the polynomials in Eq. (7) in  $\partial^j u$  to zero. Consequently, the problem of computing symmetry in (4) is reduced to find the corresponding (5), which is equivalent to solve the DTEs (8) for  $\xi_i$  and  $\eta$ . Generally it is difficult to solve Eq.(8). In this paper, the differential form Wu's method is used to solve it.

### 3. Differential Form of Wu's Method

Suppose DTEs and IP are differential polynomials system (dps), introduce the following notations:

Zero(DTEs)= All solutions of DTEs=0;

Zero(DTEs/IP)= All solutions of DTEs=0 such that IP  $\neq$  0;

Zero(DTEs,IP)= All solutions of DTEs=0 such that IP = 0;

In the following, we list the basic results of differential form Wu's theory [19].

**Theorem 2.** (characteristic set) Let DPS be a finite differential polynomials system, there is an algorithm (Wu's method) which allowing determining a differential chain DCS, called characteristic set of system DPS, such that

$$\text{Zero(DCS / IS)} \subset \text{Zero(DPS)} \subset \text{Zero(DCS)},$$

$$\text{Pr emd(DPS / DCS)} = 0,$$

$$\text{Zero(DPS)} = \text{Zero(DCS / IS)} \cup \text{Zero(DPS, IS)},$$

where, IS is a product of initials and separants of the characteristic set DCS.

**Theorem 3.** (zero decomposition theorem) For a given finite differential polynomials system DPS, then Wu's algorithmic yield finite number characteristic set DCS<sub>k</sub> of DPS with initials and separants products IS<sub>k</sub> such that the following zero decomposition

$$\text{Zero(DPS)} = \sum_k \text{Zero(DCS}_k \text{ / IS}_k),$$

is held.

The algorithm to determine characteristic set DCS is as follows:

**Input** A dps DPS.

**Output** A differential characteristic set DCS.

**Start** Let  $i = 0$ ;

**Step 1** Select a basic set DBS<sub>i</sub> from DPS<sub>i</sub>;

**Step 2** Compute all the nonzero coherent dps of DBS<sub>i</sub>, and put them in set IT<sub>i</sub>';

**Step 3**  $\forall IP$  (nonzero coherent differential polynomials)  $\in IT_i'$ , compute  $\text{Pr emd}(IP / \text{DBS}_i) \setminus \{0\}$ , and put the results in set IT<sub>i</sub>;



**Step 4** Compute  $R_i = \text{Pr emd}((DPS_i \setminus DBS_i) / DBS_i) \setminus \{0\}$ , and let  $RJ_i = IT_i \cup R_i$ ;

**Step 5** If  $RJ_i = \phi$  (Empty), then  $DCS = DBS_i$ , and stop, else  $i = i + 1$  and  $DPS_i = DPS_{i-1} \cup RJ_{i-1}$ , go to **Step 1**.

### 4. Symmetry and Exact Solutions of Equation (1)

Suppose the infinitesimal generator admitted by Eq.(1) is

$$X = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u \tag{9}$$

the determining equations derived from Eq. (7) are

$$DTE = \begin{cases} \xi_u = \xi_t = \eta_t = \tau_x = \tau_u = 0, \\ \eta_u - \xi_x = 0, \eta_{uu} = 0, \\ -\xi_x + \xi_{xx} - 2\eta_{xu} = 0, \\ \eta_{uu} + \eta_x + \eta_{xx} = 0, \tau_{tt} = 0, \end{cases} \tag{10}$$

taking left hand side for each equation, we have following corresponding differential polynomial system

$$DPS = \begin{cases} \xi_u, \xi_t, \eta_t, \tau_x, \tau_u, \\ \eta_u - \xi_x, \eta_{uu}, \\ -\xi_x + \xi_{xx} - 2\eta_{xu}, \\ \eta_{uu} + \eta_x + \eta_{xx}, \tau_{tt}, \end{cases} \tag{11}$$

using the differential form Wu’s method, we obtain characteristic set DCS of DPS

$$DCS = \{\xi_u, \xi_t, \xi_x + \xi_{xx}, \xi_x - \eta_u, \eta_t, \eta_x + \eta_{xx}, \tau_u, \tau_{tt}, \tau_x\} \tag{12}$$

solving the characteristic set corresponding equations  $DCS = 0$ , we have

$$\text{Zero}(DCS) = \begin{cases} \xi(x, t, u) = c_1 + c_2 e^{-x}, \\ \eta(x, t, u) = (c_3 - c_2 u) e^{-x} + c_4, \\ \tau(x, t, u) = c_5 + c_6 t, \end{cases} \tag{13}$$

where,  $c_1, c_2, \dots, c_6$  are arbitrary constants. Thus, Eq.(1) admits four parameter symmetry, the infinitesimal generator is

$$X = (c_1 + c_2 e^{-x})\partial_x + (c_5 + c_6 t)\partial_t + ((c_3 - c_2 u) e^{-x} + c_4)\partial_u, \tag{14}$$

From Eq.(14), the golbal symmetry is obtained

$$\begin{cases} x^* = -\ln\left(\frac{c_1}{(c_2 + c_1 e^x) e^{\varepsilon c_1} - c_2}\right) \\ t^* = \frac{-c_5 + (c_6 t + c_5) e^{\varepsilon c_6}}{c_6}, \\ u^* = \left(c_3 - \frac{c_4 c_2}{c_1}\right) \left(1 - \frac{1}{e^{\varepsilon c_1}}\right) + \varepsilon c_2 c_4 + (c_4 \varepsilon + u) c_1 e^x \left((c_2 + c_1 e^x) - c_2 e^{-\varepsilon c_1}\right)^{-1}, \end{cases} \tag{15}$$

where,  $c_1, c_2, \dots, c_6$  are arbitrary constants and  $c_1 c_6 \neq 0$ .

From Eq.(15), recurrence formula can be achieved. Once the recurrence formula is attained, a new exact solution  $u_n(x, t)$  of Eq. (1) can be obtained from already known  $u_{n-1}(x, t)$ ,

$$u_n(x, t) = \left( \left( c_3 - \frac{c_4 c_2}{c_1} \right) \left( 1 - \frac{1}{e^{-\varepsilon c_1}} \right) - c_4 \varepsilon (c_2 + c_1 x) + c_1 e^x u_{n-1}(x, t) \right) \left( (c_2 + c_1 e^x) - c_2 e^{\varepsilon c_1} \right)^{-1} \tag{16}$$

where

$$u_{n-1}(x, t) = u \left( -\ln\left(\frac{c_1}{(c_2 + c_1 e^x) e^{-\varepsilon c_1} - c_2}\right), \frac{-c_5 + (c_6 t + c_5) e^{-\varepsilon c_6}}{c_6} \right) \tag{17}$$

and  $n = 1, 2, \dots$ .



Next, Eq. (1) is solved; From (14), invariant solutions of Eq. (1) satisfies

$$\frac{du}{(c_3 - c_2 u)e^{-x} + c_4} = \frac{dt}{c_5 + c_6 t} = \frac{dx}{c_1 + c_2 e^{-x}} \quad (18)$$

solving this equation, we have

$$1. \quad c_1 c_2 c_6 \neq 0$$

Considering the following invariants

$$U = (c_6 t + c_5)(c_1 e^x + c_2)^{-c_6/c_1} c_6^{-1}$$

$$V = \frac{-c_2 c_4 (c_1 + c_2 e^{-x}) \ln(c_1 + c_2 e^{-x}) - c_4 x c_2^2 e^{-x} - ((-u c_1 + c_4(x-1))c_2 + c_3 c_1)c_1}{c_1^2 c_2 (c_1 + c_2 e^{-x})}$$

Therefore, Eq. (1) have the following form invariant solution

$$u = (c_1 + c_2 e^{-x})G(U) + \frac{c_2 c_4 (c_1 + c_2 e^{-x})(x + \ln(c_1 + c_2 e^{-x})) - c_4 c_1 c_2 + c_3 c_1^2}{c_1^2 c_2} \quad (19)$$

substitute of Eq. (19) into Eq. (1) results in an ODE where  $G(U)$  satisfies

$$c_4 + U c_6 (-c_1 + c_6) G'(U) + U^2 c_6^2 G''(U) + \frac{G''(U)}{G'(U)^2} = 0. \quad (20)$$

$$2. \quad c_1 = 0, c_2 c_6 \neq 0$$

Eq. (1) have the following form invariant solution

$$u = e^{-x} G(U) + \frac{2c_3 + e^x c_4}{2c_2}, \quad U = (c_6 t + c_5) e^{-c_6 e^x / c_3} c_6^{-1}, \quad (21)$$

in which  $G(U)$  is the solution of the following equation

$$\frac{c_2 c_4 G'(U)^2 + c_2^2 G''(U) + U c_6^2 G'(U)^2 (G'(U) + U G''(U))}{G'(U)^2} = 0. \quad (22)$$

$$3. \quad c_1 = c_6 = 0, c_2 \neq 0$$

Eq. (1) have the following form invariant solution

$$u = G(U) e^{-x} + \frac{2c_3 + e^x c_4}{2c_2}, \quad U = \frac{-c_5 e^x + t c_2}{c_2}, \quad (23)$$

in which  $G(U)$  is the solution of the following equation

$$\frac{c_4}{c_2} + \frac{c_5^2 G''(U)}{c_2^2} + \frac{G''(U)}{G'(U)^2} = 0. \quad (24)$$

$$4. \quad c_2 = 0, c_1 c_6 \neq 0$$

Eq. (1) have the following form invariant solution

$$u = G(U) + \frac{x c_4 - c_3 e^{-x}}{c_1}, \quad U = (c_6 t + c_5) e^{-c_6 e^x / c_1} c_6^{-1}, \quad (25)$$

in which  $G(U)$  is the solution of the following equation

$$\frac{c_1 G'(U)^2 (-c_4 + U c_6 G'(U)) - c_1^2 G''(U) - U c_6^2 G'(U)^2 (G'(U) + U G''(U))}{G'(U)^2} = 0. \quad (26)$$

$$5. \quad c_2 = c_6 = 0, c_1 \neq 0$$

Eq. (1) have the following form invariant solution



$$u = G(U) + \frac{xc_4 - c_3e^{-x}}{c_1}, \quad U = t - \frac{xc_5}{c_1}, \tag{27}$$

in which  $G(U)$  is the solution of the following equation

$$\frac{-c_4 + c_5G'(U)}{c_1} - \frac{c_5^2G''(U)}{c_1^2} - \frac{G'''(U)}{G'(U)^2} = 0. \tag{28}$$

6.  $c_6 = 0, c_1c_2 \neq 0$

Eq. (1) have the following form invariant solution

$$u = (c_1 + e^{-x}c_2)G(U) + \frac{c_2c_4(c_1 + e^{-x}c_2)(x + \ln(c_1 + e^{-x}c_2)) - c_4c_1c_2 + c_3c_1^2}{c_1^2c_2}, \tag{29}$$

where

$$U = t - \frac{c_5 \ln(c_1e^x + c_2)}{c_1}, \tag{30}$$

and  $G(U)$  is the solution of the following equation

$$\frac{c_4G'(U)^2 - c_1c_5G'(U)^3 + (1 + c_5^2G'(U)^2)G''(U)}{G'(U)^2} = 0. \tag{31}$$

### 5. Symmetry and Exact Solutions of Equation (2)

Suppose the infinitesimal generator admitted by Eq. (2) is

$$X = \xi(x, t, u, v)\partial_x + \tau(x, t, u, v)\partial_t + \eta(x, t, u, v)\partial_u + \varphi(x, t, u, v)\partial_v \tag{32}$$

the determining equations derived from (7) are

$$DTE = \begin{cases} \xi_v = \tau_u = 0, -u\tau_x + \varphi_x = 0, \\ \xi_t + \eta_v = 0, -\xi_t + \eta_v = 0 \\ u\eta_v + \eta_t = 0, \xi_u + \tau_v = 0, \xi_u - \tau_v = 0 \\ \xi_x - \eta_u - 2u\tau_v - \tau_t + \varphi_v = 0, \\ -\xi_x + \eta_u - 2u\tau_v - \tau_t + \varphi_v = 0, \\ -\tau_x + \varphi_u = 0, \tau_x + \varphi_u = 0, \\ -\xi_u - \eta - \eta_x + \tau_v - u^2\tau_v - u\tau_t + u\varphi_v + \varphi_t = 0, \end{cases} \tag{33}$$

taking left hand side of each equation, we have

$$DPS = \begin{cases} \xi_v, \tau_u, -u\tau_x + \varphi_x, \\ \xi_t + \eta_v, -\xi_t + \eta_v, \\ u\eta_v + \eta_t, \xi_u + \tau_v, \xi_u - \tau_v, \\ \xi_x - \eta_u - 2u\tau_v - \tau_t + \varphi_v, \\ -\xi_x + \eta_u - 2u\tau_v - \tau_t + \varphi_v, \\ -\tau_x + \varphi_u, \tau_x + \varphi_u, \\ -\xi_u - \eta - \eta_x + \tau_v - u^2\tau_v - u\tau_t + u\varphi_v + \varphi_t, \end{cases} \tag{34}$$

using the differential form Wu's method, characteristic set DCS of DPS are obtained as following

$$DCS = \begin{cases} \xi_v, \xi_u, \xi_t, \xi_x + \xi_{xx}, \\ \eta_v, \xi_x - \eta_u, \eta_t, \eta_x + \eta_{xx}, \\ \tau_v, \tau_u, \tau_{tt}, \tau_x, \tau_t - \varphi_v, \\ \varphi_u, \eta + \eta_x - \varphi_t, \varphi_x, \end{cases} \tag{35}$$

solving the characteristic set corresponding equations  $DCS = 0$ , we have



$$\text{Zero(DCS)} = \begin{cases} \xi(x, t, u, v) = c_1 + c_2 e^{-x}, \\ \tau(x, t, u, v) = c_4 + c_3 t, \\ \eta(x, t, u, v) = (c_6 - c_2 u) e^{-x} + c_5, \\ \varphi(x, t, u, v) = c_3 v + c_5 t + c_7, \end{cases} \quad (36)$$

where,  $c_1, c_2, \dots, c_7$  are arbitrary constants.

The above general solution contains seven arbitrary constants, hence, the symmetries of (2) form the seven-dimensional Lie algebra spanned by following linearly independent operators:

$$X_1 = \partial_x, X_2 = \partial_t, X_3 = \partial_v, X_4 = e^{-x} \partial_u, X_5 = \partial_u + t \partial_t,$$

$$X_6 = t \partial_t + v \partial_v, X_7 = e^{-x} \partial_x - u e^{-x} \partial_u,$$

in order to obtain symmetry reductions and exact solutions, the associated Lagrange equation has to be solved

$$\frac{dx}{\xi(x, t, u, v)} = \frac{dt}{\tau(x, t, u, v)} = \frac{du}{\eta(x, t, u, v)} = \frac{dv}{\varphi(x, t, u, v)}, \quad (37)$$

Following cases are considered in this paper:

**Case 1**  $c_1 c_2 c_3 \neq 0, c_4 = 0, c_5 = 0, c_6 = 0, c_7 = 0$ .

The symmetry  $c_1 X_1 + c_2 X_7 + c_3 X_6$  gives rise to the group-invariant solution

$$\begin{cases} u(U) = (c_1 + c_2 e^{-x}) G(U), \\ v(U) = (c_1 e^x + c_2)^{(c_3/c_1)} F(U), \end{cases} \quad (38)$$

where  $U = t(c_1 e^x + c_2)^{\frac{c_3}{c_1}}$ . Substitution of (38) into (2) results in the system of ODEs where  $G(U)$  and  $F(U)$  satisfy

$$\begin{cases} \frac{-1 + F(U) c_3 G'(U) - U c_3 F'(U) G'(U)}{G'(U)} = 0, \\ -G(U) c_1 + F'(U) + U c_3 G'(U) = 0. \end{cases} \quad (39)$$

**Case 2**  $c_1 = 0, c_2 c_3 \neq 0, c_4 = 0, c_5 = 0, c_6 = 0, c_7 = 0$ .

The symmetry  $c_2 X_7 + c_3 X_6$  gives rise to the group-invariant solution

$$\begin{cases} u(U) = e^{-x} G(U), \\ v(U) = e^{(c_3 e^x / c_2)} F(U), \end{cases} \quad (40)$$

where  $U = t e^{\frac{c_3 e^x}{c_2}}$ . Substitution of (40) into (2) results in the system of ODEs where  $G(U)$  and  $F(U)$  satisfy

$$\begin{cases} \frac{c_2 + c_3 (-F(U) + U F'(U)) G'(U)}{G'(U)} = 0, \\ c_2 F'(U) + U c_3 G'(U) = 0. \end{cases} \quad (41)$$

**Case 3**  $c_2 = 0, c_1 c_3 \neq 0, c_4 = 0, c_5 = 0, c_6 \neq 0, c_7 = 0$ .

The symmetry  $c_1 X_1 + c_6 X_4 + c_3 X_6$  gives rise to the group-invariant solution

$$\begin{cases} u(U) = G(U) - \frac{c_6 e^{-x}}{c_1}, \\ v(U) = e^{(c_3 x / c_1)} F(U), \end{cases} \quad (42)$$

where  $U = t e^{\frac{c_3 x}{c_1}}$ . Substitution of (42) into (2) results in the system of ODEs where  $G(U)$  and  $F(U)$  satisfy

$$\begin{cases} \frac{c_1 + c_3 (-F(U) + U F'(U)) G'(U)}{G'(U)} = 0, \\ -G(U) + F'(U) + \frac{U c_3 G'(U)}{c_1} = 0. \end{cases} \quad (43)$$



## 6. Conclusion

In this paper, we use Lie algorithm to determine the symmetry of a given PDEs. First, an over-determined PDEs (determining equations), for which an analytical solution is difficult to develop, is solve. Next, differential form Wu's method is used to decompose the determining equations into easier resolvable characteristic set. Finally, considering a special case of similarity reduction method, the given PDEs are reduced, the examples show the reliability of the proposed method.

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## Conflict of Interest

The author declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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## Nomenclature

|          |   |     |           |
|----------|---|-----|-----------|
| $u(x,t)$ | Speed of travelling wave [ $\text{ms}^{-1}$ ] | $x$ | Space [m] |
| $t$      | Time [s]                                      |     |           |

## References

- [1] He, J.H., Homotopy Perturbation Method with an Auxiliary Term, *Abstract and Applied Analysis*, 2012, 2012, 857612.
- [2] He, J.H., Homotopy perturbation method with two expanding parameters, *Indian Journal of Physics*, 88, 2014, 193-196.
- [3] Wu, Y., He, J.H., Homotopy perturbation method for nonlinear oscillators with coordinate dependent mass, *Results in Physics*, 10, 2018, 270-271.
- [4] Anjum, N., He, J.H., Laplace transform: Making the variational iteration method easier, *Applied Mathematics Letters*, 92, 2019, 134-138.
- [5] He, J.H., Some asymptotic methods for strongly nonlinear equations, *International Journal of Modern Physics B*, 20, 2006, 1141-1199.
- [6] He, J.H., Kong, H.Y., Chen, R.X., Variational iteration method for Bratu-like equation arising in electrospinning, *Carbohydrate Polymers*, 105, 2014, 229-230.
- [7] He, J.H., Exp-function Method for Fractional Differential Equations, *International Journal of Nonlinear Sciences and Numerical Simulation*, 14(6), 2013, 363-366.
- [8] He, J.H., Asymptotic Methods for Solitary Solutions and Compactons, *Abstract and Applied Analysis*, 2012, 2012, 916793.
- [9] He, J.H., Wu, X.H., Exp-function method for nonlinear wave equations, *Chaos Solitons & Fractals*, 30(3), 2006, 700-708.
- [10] He, J.H. A modified Li-He's variational principle for plasma, *International Journal of Numerical Methods for Heat and Fluid Flow*, 2019, DOI: 10.1108/HFF-06-2019-0523.
- [11] He, J.H., Ji, F.Y., Taylor series solution for Lane-Emden equation, *Journal of Mathematical Chemistry*, 57(8), 2019, 1932-1934.
- [12] Reetz, F., Kreilos, T., Schneider, T.M., Exact invariant solution reveals the origin of self-organized oblique turbulent-laminar stripes, *Nature Communications*, 10, 2019, 2277.
- [13] Shi, Z.Y., Yao, X.L., Pang, F. Z., Wang, Q.S., An exact solution for the free-vibration analysis of functionally graded carbon-nanotube-reinforced composite beams with arbitrary boundary conditions, *Scientific Reports*, 7, 2017, 12909.
- [14] Guvendi, A., Sahin, R., Sucu, Y., Exact solution of an exciton energy for a monolayer medium, *Scientific Reports*, 9, 2019, 8960.
- [15] Bluman, G.W., Kumei, S., *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989.
- [16] Tian, Y., Wang, K. L., Polynomial characterister method: An easy approach to lie symmetry, *Thermal Science*, 24(4), 2020, 2629-2635.
- [17] Tian, Y., Diffusion-convection equations and classical symmetry classification, *Thermal Science*, 23(4), 2019, 2151-2156.
- [18] Tian, Y., Symmetry reduction-A promising method for heat conduction equations, *Thermal Science*, 23(4), 2019, 2219-2227.
- [19] Wu, W.T., *Mathematics Mechanization*, Science Press, Beijing, 2000.

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