J. Appl. Comput. Mech., 7(1) (2021) 177-181
DOI: 10.22055/JACM.2020.34857.2486

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

A Simple Approach for the Fractal Riccati Differential Equation

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Received August 31 2020; Revised September 21 2020; Accepted for publication September 27 2020.
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Abstract. In this paper, a fractal modification of the Riccati differential equation is presented, and the two-scale transform method combined with Taylor series is used to solve the equation. Two examples are given to verify the correctness and effectiveness of the proposed method.

Keywords: Two-scale transform method; Taylor series method; Fractal Riccati differential equation.

1. Introduction

Riccati differential equation is a kind of typical nonlinear differential equation, which has been widely used in the field of applied science such as random processes, optimal control, diffusion problems and so on [1-3].The expression of RDE takes the form as:

$$\Theta'(\tau) = m(\tau)\Theta(\tau)^2 + n(\tau)\Theta(\tau) + r(\tau), \quad 0 \le \tau \le k, \tag{1.1}$$

with the initial conditions as:

$$\Theta(0) = c.$$

where $\Theta(\tau)$ is the unknown function, $m(\tau)$, $n(\tau)$ and $r(\tau)$ are known continuous functions defined in $[0,k]$ and $c \in N$.

In the past few decades, as many researchers have pointed out, compared with the classical integer derivative, the fractal derivative has obvious advantages in modeling complex phenomenon arising in circuit [4], cold plasma [5], vibration [6], filter [7-9], microgravity space [10] so on [10-14]. So we present the FRDE as the expansion of the classical Riccati differential equation, which is used to describe many complex science phenomenon, it takes the form as:

$$\frac{D_{\Im}^{H}\Theta(\tau)}{D\tau^{\Im}} = m(\tau)\Theta(\tau)^2 + n(\tau)\Theta(\tau) + r(\tau), \quad 0 \le \tau \le k, \tag{1.2}$$

with the following initial conditions:

$$\Theta(0) = c$$

where $D_{\Im}^{H}\Theta(\tau) / D\tau^{\Im}$ is the fractal derivative that defined as [15-16]:

$$\frac{D_{\Im}^{H}\Theta}{D\tau^{\Im}}(\tau_0) = \Gamma(1+\Im) \lim_{\substack{\tau-\tau_0=\Delta\tau \\ \Delta\tau\neq 0}} \frac{\Theta(\tau)-\Theta(\tau_0)}{(\tau-\tau_0)^{\Im}}. \tag{1.3}$$

2. Solution for the FRDE

The two-scale transform is considered as an extension of complex transformation [17], which was first proposed by He in 2019. It can convert the fractal equation into the ordinary equation [12] and has been used widely to solve many fractal problems.

Applying the two-scale transform [18, 19] as:

Published online: September 28 2020

$$T = \tau^{\vartheta}, \tag{2.1}$$

Eq. (1.2) can be converted into the following one:

$$\frac{d\Theta}{dT} = m(T)\Theta(T)^2 + n(T)\Theta(T) + r(T), \quad 0 \le T \le k \tag{2.2}$$

with the initial conditions as:

$$\Theta(0) = c,$$

Now, we apply Taylor series [20-22] to solve the Eq. (2.2). The Taylor series is named after Sir Brook Taylor, an English mathematician who published the Taylor formula in 1715, and is accessible to all students and engineers. To illustrate the solution process, we take $T = 0$ into Eq. (2.2), it obtains:

$$\Theta'(0) = m(0)\Theta^2(0) + n(0)\Theta(0) + r(0), \tag{2.3}$$

With use of the initial condition of Eq. (2.2) for the above equation, we have

$$\Theta'(0) = m(0)a^2 + cn(0) + r(0), \tag{2.4}$$

Then, we take a derivation on T for Eq. (2.2), which yields:

$$\Theta''(T) = m'(T)\Theta(T)^2 + 2m(T)\Theta(T)\Theta'(T) + n'(T)\mu(T) + n(T)\Theta'(T) + r'(T), \tag{2.5}$$

which leads to:

$$\Theta''(0) = m'(0)c^2 + 2m(0)c\Theta'(0) + cn'(0) + n(0)\Theta'(0) + r'(0), \tag{2.6}$$

where $\Theta'(0)$ is a given known parameter that can be determined by Eq. (2.4).

Use a similar approach, differentiating Eq. (2.5) with regard to T produces the following results:

$$\Theta'''(T) = m''(T)\Theta(T)^2 + 4m'(T)\Theta'(T)\Theta(T) + 2m(T)\left(\Theta'(T)\right)^2 + 2m(T)\Theta(T)\Theta''(T) + n''(T)\Theta(T) + 2n'(T)\Theta'(T) + n(T)\Theta''(T) + r''(T), \tag{2.7}$$

There is

$$\Theta'''(0) = c^2m''(0) + 4cm'(0)\Theta'(t) + 2m(0)\left(\Theta'(0)\right)^2 + 2cm(0)\Theta''(0) + cn''(0) + 2n'(0)\Theta'(0) + n(0)\Theta''(0) + r''(0), \tag{2.8}$$

where $\Theta'(0)$ and $\Theta''(0)$ can be obtained from Eq. (2.4) and Eq. (2.6) respectively.

Similarly, we can get the values of $\Theta^{(4)}(0), \Theta^{(5)}(0), \Theta^{(6)}(0)$ …., and so on.

Hence the approximate solution of $\Theta(T)$ with Taylor series can be written as:

$$\Theta(T) = \Theta(0) + \Theta'(0)T + \frac{1}{2}\Theta''(0)T^2 + \frac{1}{3!}\Theta'''(0)T^3 + \frac{1}{4!}\Theta^{(4)}(0)T^4 + \frac{1}{5!}\Theta^{(5)}(0)T^5 + \frac{1}{6!}\Theta^{(6)}(0)T^6 + \frac{1}{7!}\Theta^{(7)}(0)T^7 + \ldots \tag{2.9}$$

Accordingly, the solution of Eq. (1.2) can be expressed as

$$\Theta(\tau) = \Theta(0) + \Theta'(0)\tau^{\vartheta} + \frac{1}{2}\Theta''(0)\tau^{2\vartheta} + \frac{1}{3!}\Theta'''(0)\tau^{3\vartheta} + \frac{1}{4!}\Theta^{(4)}(0)\tau^{4\vartheta} + \frac{1}{5!}\Theta^{(5)}(0)\tau^{5\vartheta} + \frac{1}{6!}\Theta^{(6)}(0)\tau^{6\vartheta} + \frac{1}{7!}\Theta^{(7)}(0)\tau^{7\vartheta} + \ldots \tag{2.10}$$

Next we will use two examples to show the efficiency and accuracy of the proposed method.

3. Results Verification

Example 1. Considering the FRDE reads as

$$\begin{aligned} &\frac{D_{\vartheta}^{H}\Theta(\tau)}{D\tau^{\vartheta}} = \Theta^2(\tau) - \Theta(\tau), \quad 0 \le \tau \le 1, \\ &\Theta(0) = \frac{1}{2}, \end{aligned} \tag{3.1}$$

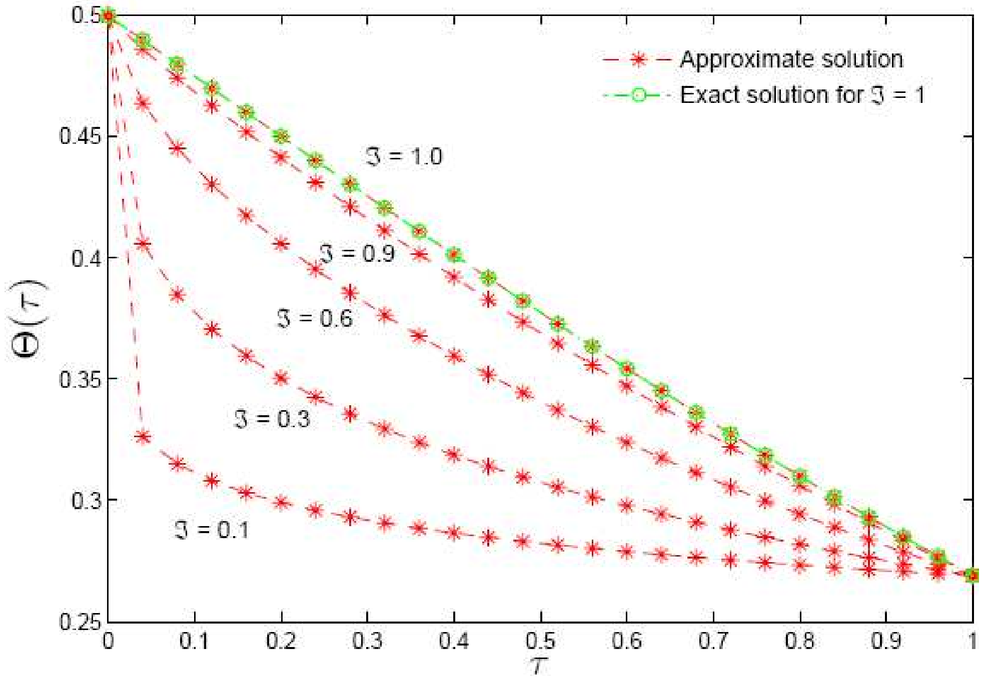

Fig. 1. The solution property of Eq. (3.1) ($\Im \le 1$)

The above equation can be converted into the following one via two-scale transform:

$$\begin{aligned} &\frac{d\Theta(T)}{dT} = \Theta^2(T) - \Theta(T), \qquad 0 \le T \le 1, \\ &\Theta(0) = \frac{1}{2}, \end{aligned} \tag{3.2}$$

Using the solution method we mentioned before for Eq. (3.2), we can get:
$\Theta(0) = 1/2, \Theta'(0) = -1/4, \Theta''(0) = 0, \Theta'''(0) = 1/8, \Theta^{(4)}(0) = 0, \Theta^{(5)}(0) = -1/4, \Theta^{(6)}(0) = 0, \Theta^{(7)}(0) = 17/16$. Then the solution of Eq. (3.2) can be approximated by 7-*th* Taylor series as:

$$\Theta(T) = \frac{1}{2} - \frac{1}{4}T + \frac{1}{48}T^3 - \frac{1}{480}T^5 + \frac{17}{80640}T^7 \tag{3.3}$$

The solution of Eq. (3.1) is obtained as:

$$\Theta(\tau) = \frac{1}{2} - \frac{1}{4}\tau^{\Im} + \frac{1}{48}\tau^{3\Im} - \frac{1}{480}\tau^{5\Im} + \frac{17}{80640}\tau^{7\Im}. \tag{3.4}$$

Fig. 1 depicts the solution property of Eq. (3.4) with different $\Im$. Obviously, the value of fractal order has a significant impact on the solution property. With the decrease of fractal order, the value of function also decreases. For the special case $\Im = 1$, the FRDE Eq. (3.1) becomes into the ordinary RDE, which has the exact solution as [3]:

$$\Theta(\tau) = \frac{e^{-\tau}}{1 + e^{-\tau}},$$

when $\Im = 1$, the comparison between the approximated solution and exact solution can be seen from Fig.1. Obviously, the result computed by our method and the exact solution have a good agreement. The detailed comparison results are shown in Table 1.

Example 2. Considering the FRDE reads as

$$\begin{aligned} &\frac{D^H_{\Im}\Theta(\tau)}{D\tau^{\Im}} = -e^{\tau}\mu^2(\tau) + 2e^{2\tau}\mu(\tau) - e^{3\tau} + e^{\tau}, \qquad 0 \le t \le 1, \\ &\Theta(0) = 1, \end{aligned} \tag{3.5}$$

Applying two-scale transform for Eq. (3.5), yields its partner as:

$$\begin{aligned}&\frac{d\Theta(T)}{dT}=-e^{T}\mu^{2}(T)+2e^{2T}\mu(T)-e^{3T}+e^{T}, \quad 0\le T\le 1,\\&\Theta(0)=1,\end{aligned} \tag{3.6}$$

Then we get the following results about Eq. (3.6)

$$\Theta(0)=1,\Theta'(0)=1,\Theta''(0)=1,\Theta'''(0)=1,\Theta^{(4)}(0)=1,\Theta^{(5)}(0)=1$$

Then the solution of Eq. (3.5) with *7-th* Taylor series can be approximately written as

$$\begin{aligned}\Theta(T)&=\Theta(0)+\Theta'(0)T+\frac{1}{2!}\Theta''(0)T^{2}+\frac{1}{3!}\Theta'''(0)T^{3}+\frac{1}{4!}\Theta^{(4)}(0)T^{4}+\frac{1}{5!}\Theta^{(5)}(0)T^{5}+\frac{1}{6!}\Theta^{(6)}(0)T^{6}+\frac{1}{7!}\Theta^{(7)}T^{7}\\&=1+T+\frac{1}{2!}T^{2}+\frac{1}{3!}T^{3}+\frac{1}{5!}T^{5}+\frac{1}{6!}T^{6}+\frac{1}{7!}T^{7}.\end{aligned} \tag{3.7}$$

By the above formula, we can easily get the approximate solution of equation (3.5), which reads

$$\begin{aligned}\Theta(\tau)&=\Theta(0)+\Theta'(0)\tau^{\Im}+\frac{1}{2!}\Theta''(0)\tau^{2\Im}+\frac{1}{3!}\Theta'''(0)\tau^{3\Im}+\frac{1}{4!}\Theta^{(4)}(0)\tau^{4\Im}+\frac{1}{5!}\Theta^{(5)}(0)\tau^{5\Im}+\frac{1}{6!}\Theta^{(6)}(0)\tau^{6\Im}+\frac{1}{7!}\Theta^{(7)}\tau^{7\Im}\\&=1+\tau^{\Im}+\frac{1}{2!}\tau^{2\Im}+\frac{1}{3!}\tau^{3\Im}+\frac{1}{5!}\tau^{5\Im}+\frac{1}{6!}\tau^{6\Im}+\frac{1}{7!}\tau^{7\Im}.\end{aligned} \tag{3.8}$$

We present the behavior of Eq. (3.8) for various fractal orders in Fig. 2. In this example, it can be found that the value of the function decreases with the increase of fractal order.

The exact solution for equation (3.5) in the case $\Im=1$ is given by [3] as:

$$\Theta(\tau)=e^{\tau}.$$

As expected, the results obtained by our method and the exact solution agree very well for $\Im=1$. The detailed compared results can be obtained in Table 1.

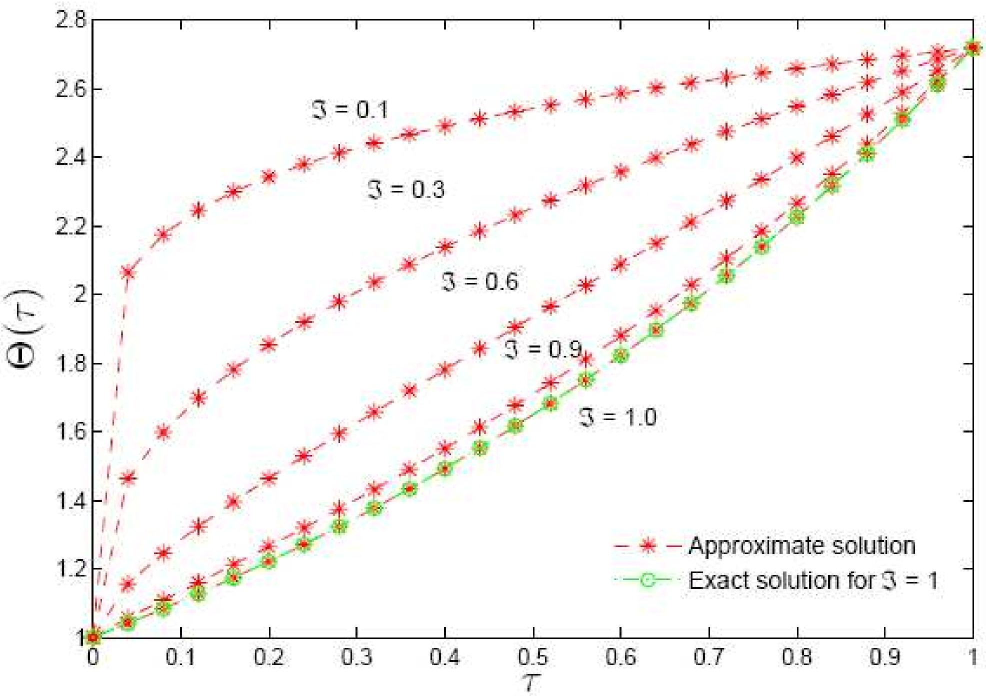

Fig. 2. The solution property of Eq. (3.8) ($\Im\le 1$)

Table 1. Comparison between our method and the exact solution for $\Im = 1$

τ	Example one		Example two	
	Our method	Exact solution	Our method	Exact solution
0	0.5	0.5	1	1
0.1	0.4750208	0.4750208	1.1051709	1.1051709
0.2	0.450166	0.450166	1.2214028	1.2214028
0.3	0.4255575	0.4255575	1.3498588	1.3498588
0.4	0.4013123	0.4013123	1.4918247	1.4918247
0.5	0.3775407	0.3775407	1.6487212	1.6487213
0.6	0.3543439	0.3543437	1.8221184	1.8221188
0.7	0.331813	0.3318122	2.0137512	2.0137527
0.8	0.3100282	0.3100255	2.2255364	2.2255409
0.9	0.2890581	0.2890505	2.4595913	2.4596031
1	0.2689608	0.2689414	2.718254	2.7182818

4. Conclusion

In this paper, we present a simple but effective approximate method to solve the FRDE using Two-scale transform and Taylor series, the whole solving process is easy and clear. The results obtained in this paper are expected to shed a new light on the study of fractal calculus in fractal nonlinear differential equation.

Acknowledgments

This work is supported by Program of Henan Polytechnic University (No.B2018-40).

Conflict of Interest

The author declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

References

[1] Reid, W.T., *Riccati Differential Equations*, Academic Press, New York, 1972.
[2] Dehghan, M., Taleei, A., A compact split-step finite difference method for solving the nonlinear Schrödinger equations with constant and variable coefficients, *Computer Physics Communications*, 181, 2010, 43–51.
[3] Mukherjee, S., Roy, B., Solution of Riccati equation with variable co-efficient by differential transform method, *International Journal of Nonlinear Science*, 14(2), 2012, 251–256.
[4] Wang, K.J. *et al.*, The transient analysis for zero-input response of fractal RC circuit based on local fractional derivative, *Alexandria Engineering Journal*, 2020, https://doi.org/10.1016/j.aej.2020.08.024.
[5] Goswami, A., Singh, J., Kumar, D., *et al.*, An efficient analytical approach for fractional equal width equations describing hydro-magnetic waves in cold plasma, *Physica A*, 524, 2019, 563-575.
[6] Kumar, D., Singh, J., Baleanu, D., On the analysis of vibration equation involving a fractional derivative with Mittag-Leffler law, *Mathematical Methods in the Applied Sciences*, 43(1), 2019, 443-457.
[7] Wang, K.J., On a High-pass filter described by local fractional derivative, *Fractals*, 2020, 28(3), 2050031.
[8] Wang, K.J., Sun, H.C., Cui, Q.C., The fractional Sallen-Key filter described by local fractional derivative, *IEEE Access*, 8, 2020, 166377-166383.
[9] Wang, K.J., *et al.*, A a-order R-L high-pass filter modeled by local fractional derivative, *Alexandria Engineering Journal*, 59(5), 2020, 3244-3248.
[10] He, J.H., A fractal variational theory for one-dimensional compressible flow in a microgravity space, *Fractals*, 28(2), 2020, 2050024.
[11] He, J.H., Ain, Q.T., New promises and future challenges of fractal calculus: from two-scale thermodynamics to fractal variational principle, *Thermal Science*, 24(2A), 2020, 659-681.
[12] He, J.H., Variational Principle for the Generalized KdV-Burgers Equation with Fractal Derivatives for Shallow Water Waves, *Journal of Applied and Computational Mechanics*, 6(4), 2020, 735-740.
[13] Wang, K.L., Wang, K.J., He, C.H., Physical insight of local fractional calculus and its application to fractional KdV-Burgers-Kuramoto equation, *Fractals*, 27(7), 2019, 1950122.
[14] Wang, K.L., Wang, K.J., A Modification of the Reduced Differential Transform Method for Fractional Calculus, *Thermal Science*, 22(4), 2018, 1871-1875.
[15] He, J.H., A tutorial review on fractal spacetime and fractional calculus, *International Journal of Theoretical Physics*, 53(11), 2014, 3698-3718.
[16] He, J.H., Fractal calculus and its geometrical explanation, *Results in Physics*, 10, 2018, 272–276.
[17] He, J.H., Li, Z.-B., Converting Fractional Differential Equations into Partial Differential Equations, *Thermal Science*, 16(2), 2012, 331-334.
[18] Ain, Q.T., He, J.H., On two-scale dimension and its applications, *Thermal Science*, 23(3B), 2019, 1707-1712.
[19] He, J.H., Ji, F.Y., Two-scale mathematics and fractional calculus for thermodynamics, *Thermal Science*, 23(4), 2019, 2131-2133.
[20] He, C.H., Shen, Y., Ji, F.Y., He, J.H., Taylor series solution for fractal Bratu-type equation arising in electrospinning process, *Fractals*, 28(1), 2020, 2050011.
[21] He, J.H., Taylor series solution for a third order boundary value problem arising in architectural engineering, Ain Shams Engineering Journal, 2020, https://doi.org/10.1016/j.asej.2020.01.016.
[22] He, J.H., A simple approach to one-dimensional convection-diffusion equation and its fractional modification for E reaction arising in rotating disk electrodes, *Journal of Electroanalytical Chemistry*, 2019, 854, 113565.

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How to cite this article: Wang K.-J. A Simple Approach for the Fractal Riccati Differential Equation, *J. Appl. Comput. Mech.*, 7(1), 2021, 177–181. https://doi.org/10.22055/JACM.2020.34857.2486