An Efficient Spectral Method-based Algorithm for Solving a High-dimensional Chaotic Lorenz System

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Abstract. In this paper, we implement the multidomain spectral relaxation method to numerically study high dimensional chaos by considering the nine-dimensional Lorenz system. Chaotic systems are characterized by rapidly changing solutions, as well as sensitivity to small changes in initial data. Most of the existing numerical methods converge slowly for this kind of problems and this results in inaccurate approximations. Spectral methods are known for their high accuracy. However, they become less accurate for problems characterised by chaotic solutions, even with an increase in the number of grid points. As a result, in this work, we adopt the multidomain approach which assumes that the main interval can be decomposed into a finite number of subdomains and the solution obtained in each of the subdomains. This approach remarkably improves the results as well as the efficiency of the method.

Keywords: Spectral method, Multidomain, Chaotic systems.

1. Introduction

Chaos is a phenomenon that came into existence in 1963 when Lorenz [1] discovered chaotic behaviour in differential equations modelling weather phenomena. Since then, nonlinear chaotic and hyperchaotic systems have arisen from many science and engineering applications. The complexity of the chaotic behaviour is dependent on the number of positive Lyapunov exponents the system has. Basically, a chaotic system has only one positive Lyapunov exponent, whereas chaotic systems with more than one positive Lyapunov exponents are said to be hyperchaotic. Hyperchaos is a concept that was introduced by Rossler [2] in 1976 when he discovered hyperchaotic behaviour in differential equations for modeling chemical reactions. Since these two discoveries, several chaotic and hyperchaotic systems arising in different fields have been studied [3,4]. Hyperchaotic systems generally have more complex dynamical behaviours than the ordinary chaotic systems. Chaotic systems are characterized by strong sensitivity to initial conditions and swiftly changing solutions. These characteristics make chaotic dynamical systems difficult to analyze numerically.

Various direct numerical techniques have been employed to solve systems of ODEs exhibiting chaos. These methods include the differential quadrature method [5], power series method [6], differential transform method [7], Barycentric Lagrange Interpolation Collocation Method [8], to name a few. Unfortunately, most of the existing direct numerical methods converge slowly for these problems, and this leads to inaccurate approximations. Spectral methods are known for their high accuracy, however they also become less accurate for non-smooth solutions and large time-domain problems, even with an increase in the number of grid points. The use of many grid points lead to large memory requirements and can also lead to the approximations exhibiting spurious oscillations which can result in nonlinear instabilities. To overcome this limitation, the multidomain approach has been used by many researchers. The multidomain approach assumes that the main interval can be decomposed into a finite number of subintervals.

The idea of domain decomposition has predominantly been applied to semi-analytical methods when solving chaotic systems. These semi-analytical methods include the multistage Adomian decomposition method [9], multistage homotopy analysis method [10], multistage differential transformation method [11], multistage variational iteration method [12], and multistage homotopy perturbation methods [13]. The downside with the multidomain approach based on analytical approximations is that it becomes an arduous and time-consuming exercise to analytically integrate in each of the many subdomains.

Mota et al. [14, 15] presented a multidomain numerical method based on the spectral method. Their method differs from the previously listed multidomain methods in that it is completely numerical. Other numerical methods where domain decomposition has been used to solve chaotic systems include the Garlekin-Petrov time discretization method [16], piecewise successive linearization method [17], piecewise-spectral parametric iteration [18], piecewise spectral homotopy analysis method [19] and multidomain compact finite difference relaxation method [20].

In this paper, we present the multidomain spectral relaxation method (MSRM) to solve higher dimensional chaotic systems. The method uses the Chebyshev spectral method combined with the Gauss-Seidel iteration scheme to solve chaotic systems in a
2. Description of the method for 9D Lorenz

In this section, we give an outline of the multidomain spectral relaxation method (MSRM) to solve the 9D Lorenz system. The 9D Lorenz system was derived by applying a triple Fourier expansion to the Boussinesq-Oberbeck equations governing thermal convection in a 3D spatial domain by using an approach similar to the well known 3D Lorenz’s [21]. The system is given by

\[
\begin{align*}
\dot{x}_1 &= -\sigma x_1 - \sigma b_2 x_2 - x_2 x_4 + b_1 x_7^2 + b_3 x_3 x_5 \\
\dot{x}_2 &= -\sigma x_2 - 0.5\sigma x_5 + x_1 x_4 - x_2 x_5 + x_4 x_5 \\
\dot{x}_3 &= -\sigma x_3 + \sigma b_2 x_2 + x_3 x_4 - b_2 x_3 x_5 \\
\dot{x}_4 &= -\sigma x_4 + 0.5\sigma x_5 - x_2 x_4 - x_2 x_5 + x_4 x_5 \\
\dot{x}_5 &= -\sigma b_3 x_3 + 0.5x_2^2 - 0.5x_4^2 \\
\dot{x}_6 &= -b_4 x_6 + x_2 x_9 - x_4 x_9 \\
\dot{x}_7 &= -r x_1 - b_1 x_7 + 2x_5 x_7 - x_4 x_9 \\
\dot{x}_8 &= r x_2 - b_2 x_8 - 2x_5 x_7 + x_4 x_9 \\
\dot{x}_9 &= -r x_2 + r x_4 - x_9 - 2x_2 x_6 + 2x_4 x_6 + x_4 x_7 - x_2 x_8
\end{align*}
\]

where the constant parameters \( b_i \) are defined as

\[
\begin{align*}
b_1 &= 4 \frac{1 + a^2}{1 + 2a^2}, & b_2 &= \frac{1 + 2a^2}{2(1 + a^2)}, & b_3 &= \frac{1 - a^2}{1 + a^2} \\
b_4 &= \frac{a^2}{1 + a^2}, & b_5 &= \frac{8a^2}{1 + 2a^2}, & b_6 &= \frac{4}{1 + 2a^2}
\end{align*}
\]

without loss of generality, we express Eq. (1) in the form

\[
\dot{x} + Ax + F(x) = 0
\]

(2)

where \( x(t) = [x_1(t), x_2(t), ..., x_9(t)]^T \), \( A \) is a \( 9 \times 9 \) matrix with entries \( a_{ij}, \; i,j = 1,2,...,9 \), given by

\[
A = \begin{pmatrix}
0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and \( F(x) \) is a vector of nonlinear components of Eq. (2) given by

\[
F(x) = \begin{pmatrix}
x_2 x_4 - b_1 x_4^2 - b_2 x_3 x_5 \\
x_2 x_4 + x_2 x_5 - x_4 x_9 \\
x_2 x_4 + b_1 x_4^2 + b_2 x_3 x_5 \\
x_2 x_3 + x_2 x_5 - x_4 x_5 \\
x_2 x_3 + x_2 x_5 - 0.5x_2^2 + 0.5x_4^2 \\
x_4 x_9 + x_4 x_9 \\
-2x_2 x_4 + x_4 x_9 \\
2x_5 x_7 - x_2 x_8 \\
2x_5 x_7 - x_2 x_8
\end{pmatrix}
\]

we use the MSRM to solve Eq. (2). The MSRM is based on the use of the Chebyshev spectral method. The Chebyshev spectral method approximate functions by means of truncated series of Chebyshev orthogonal polynomials. The Chebyshev polynomials \( T_N(t) \) of order \( N \), are defined as

\[
T_N(t) = \cos(N \cos^{-1}(t)), \quad N \in \mathbb{N}
\]

(3)

The Chebyshev interpolation, \( u_N(t) \), of a function \( u(t) \) at \( t = t_i \), is defined by
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\[ u_N(t) = \sum_{i=0}^{N} u_i L_i(t). \] (4)

The collocation points \( t_i \) are chosen to be the extrema of \( T_N \):

\[ \{t_i\} = \left\{ \cos \left( \frac{i\pi}{N} \right) \right\}_{i=0}^{N} \] (5)

which are the Chebyshev-Gauss-Lobatto points. This choice is made from the simple reason that in Lagrangian interpolation, if the interpolation points are taken to be the zeros of the polynomial, the error is minimized. The polynomials \( L_i(t), \ i = 0, 1, ..., N \), are Lagrange polynomials of order \( N \) based on the Chebyshev-Gauss-Lobatto points, and is defined as

\[ L_i(t) = \frac{(-1)^{i+1}(1-t^2)T'_N(t)}{\hat{c}_N^2(t-t_i)}, \quad i = 0, 1, ..., N \] (6)

where \( \hat{c}_0 = \hat{c}_N = 2, \hat{c}_i = 1 \) for \( i = 1, 2, ..., N - 1 \). Therefore, the first order derivative of the approximate solution at the collocation points is computed as

\[ \frac{d}{dt} u(t) = \sum_{k=0}^{N} u(t_k) \frac{dL_k(t)}{dt} = \sum_{k=0}^{N} D_{ik} u(t_k) = DU, \quad i = 0, 1, ..., N \] (7)

where \( D_{ik} = \frac{dL_k(t)}{dt} \) is an \((N + 1) \times (N + 1)\) Chebyshev differentiation matrix for \( i, k = 0, 1, ..., N \). The first order Chebyshev differentiation matrix at the collocation points is given by [22-25]

\[ D_{ik} = \begin{cases} \frac{c_i}{2\hat{c}_N^2} (1 - t_i^2), & i = k = 0, N \\ \frac{2\hat{c}_k(1-t_i^2)}{2N^2 + 1}, & i = k \neq 0, N \\ \frac{-6}{2N^2 + 1}, & i = 0, k = 1, 2, ..., N \\ \frac{6}{6}, & i = k = N \\ \frac{c_i(1-t_i^2)}{2\hat{c}_N^2}, & i \neq k \end{cases} \] (8)

To solve Eq. (2), firstly, we decompose the time domain \( I = [0, T] \) into \( p \) subdomains of uniform length \( \frac{T}{p} \), with each subdomain \( I_j = [t_{j-1}, t_j] \) for \( j = 1, 2, ..., p \) and having the property that

\[ \bigcup_{j=1}^{p} [t_{j-1}, t_j] = [0, T]. \]

Each subdomain \( [t_{j-1}, t_j] \) is transformed to the domain \([-1,1] \), which is the domain of the Gauss-Lobatto points defined in Eq. (5), by using the linear transformation

\[ t = \frac{t_j - t_{j-1}}{2} + \frac{t_{j-1} + t_j}{2} = \frac{T}{2p} + \frac{t_{j-1} + t_j}{2}, \quad j = 1, 2, ..., p \]

where

\[ t_j - t_{j-1} = \frac{T}{p}. \]

We use the Chebyshev differentiation matrix to approximate the derivatives in Eq. (2), which results in a system of algebraic equations that we solve using the Gauss-Seidel technique. These equations are solved in each subdomain \( I_j \). We use \( t^c_j \) and \( X^c_j = [x_1(t^c_j), x_2(t^c_j), ..., x_9(t^c_j)]^T \) to represent the collocation points and the approximate solution in each subdomain, respectively. Using the Chebyshev differentiation matrix to approximate the first derivatives in Eq. (2), we get

\[ \left( \frac{2p}{T} 0 + a_r X \right) X^c_j + \sum_{i=1}^{9} a_{r,i} X^c_j + F^c_j(X) = 0, \quad r = 1, 2, ..., 9 \] (9)

where
To solve system (9), we use the Gauss-Seidel idea of solving algebraic equations to get the following iteration scheme,

\[
(2pD + a_{r,r}I)X^j_{r,s+1} = -\sum_{i=1}^{r-1} a_{r,i}X^j_{i,s+1} - \sum_{i=r+1}^{9} a_{r,i}X^j_{i,s} - F^j_{r,s+1}, \quad r = 1,2,\ldots,9
\]  

(10)

where \(X_{r,s+1}\) is the approximation of each \(x_{r}\) at the \((s+1)\)th iteration and

\(F_{r,s+1} = F_{r}(X_{1,s+1},X_{2,s+1},\ldots,X_{r-1,s+1},X_{r,s},\ldots,X_{9,s})\).

We remark that the nonlinear terms are evaluated using values from the previous iteration. Therefore, the approximate solution is obtained from

\[
X^j_{r,s+1} = A^{-1}B^j
\]  

(11)

where

\[A = \frac{2pD + a_{r,r}I}{T}, \quad B^j = \sum_{i=1}^{r-1} a_{r,i}X^j_{i,s+1} - \sum_{i=r+1}^{9} a_{r,i}X^j_{i,s} - F^j_{r,s+1}, \quad r = 1,2,\ldots,9
\]

The solution of the system (2) on \([0,T]\) is then given by

\[x_{r} = \bigcup_{j=1}^{p} X^j_{r}(t^j)
\]

3. Convergence and error analysis

In this section, we provide the effect of domain decomposition on the convergence and error estimates on the multidomain spectral relaxation method (MSRM). Eq. (9), can be written as the block matrix system

\[AX = B
\]  

(12)

where

\[
A = \begin{pmatrix}
\frac{2pD + a_{1,1}I}{T} & a_{2,1} & a_{3,1} & \cdots & a_{1,9} \\
a_{2,1} & \frac{2pD + a_{2,2}I}{T} & a_{3,2} & \cdots & a_{2,9} \\
a_{3,1} & a_{3,2} & \frac{2pD + a_{3,3}I}{T} & \cdots & a_{3,9} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{9,1} & a_{9,2} & a_{9,3} & \cdots & \frac{2pD + a_{9,9}I}{T}
\end{pmatrix}, \quad X^j = \begin{pmatrix}
X^j_1 \\
X^j_2 \\
X^j_3 \\
\vdots \\
X^j_9
\end{pmatrix} \quad \text{and} \quad B^j = \begin{pmatrix}
-F^j_1 \\
-F^j_2 \\
-F^j_3 \\
\vdots \\
-F^j_9
\end{pmatrix}
\]

**Theorem 1:** For any \(X^{(0)} \in \mathbb{R}^9\), the sequence \(\{x^{(k)}\}_{k=0}^{\infty}\) defined by \(X = A^{-1}B\) for each \(k \geq 1\), converge to the unique solution \(\hat{x}\) if the matrix \(A\) is strictly diagonally dominant.

**Proof:** To prove convergence, we show that matrix \(A\), above is diagonally dominant. Since we are dealing with the block matrix \(A\), we consider the norms of the matrices inside \(A\). For any \(m \times n\) matrix \(A\) with real entries, we have the norm equivalence

\[
\|A\|_{\max} \leq \|A\|_2 \leq \sqrt{mn} \|A\|_{\max}
\]  

(13)

where \(\|A\|_{\max} = \max_{i,j}|a_{i,j}|\) and \(\|A\|_2\) is the spectral norm. If \(p\), in Eq. (12), is sufficiently large, then

\[
2p\frac{D}{T} \gg a_{r,r}, \quad r = 1,2,\ldots,9
\]

This implies that

\[
\left\|\frac{2p}{T}D + a_{r,r}I\right\|_{\max} \approx \frac{2p}{T} \times C,
\]
where $C$ is a non-zero constant which is independent of $\frac{p}{T}$. Similarly,

$$
\sum_{j=1, j \neq r}^{q} \|a_{r,j}\|_{\max} = \sum_{j=1, j \neq r}^{q} |a_{r,j}|.
$$

Thus, for $p$ sufficiently large, we have

$$
\left\| \frac{2p}{T} D + a_{r,j} \right\|_{\max} \approx \frac{2p}{T} \times C \geq \sum_{j=1, j \neq r}^{q} |a_{r,j}| = \sum_{j=1, j \neq r}^{q} \|a_{r,j}\|
$$

Therefore, the block matrix $A$ is diagonally dominant in the sense of matrix norm. Hence, the method converges.

**Theorem 2:** Let $x_{r}(t) \in C^{N+1}[0,T]$, and let $X_{r}(t)$ be a polynomial of degree $\leq N$ that interpolates the function $x_{r}(t)$ at $N+1$ distinct points $t_{0}, t_{1}, \ldots, t_{N} \in [t_{0}, t_{N+1}]$, where

$$
\bigcup_{j=1}^{p} [t_{j-1}, t_{j}] = [0, T]
$$

If the nodes $t_{j}$ are chosen as the Gauss-Lobatto points $t_{i} = \cos \left( \frac{i \pi}{N} \right), (i = 0, 1, \ldots, N)$, then the error term for the polynomial interpolation in $[0,T]$, using the nodes $t_{j}$ in each subdomain, is

$$
E(t) = |x_{r}(t) - X_{r}(t)| \leq \left( \frac{T}{2} \right)^{N+1} \left( \frac{1}{p} \right)^{N} \frac{M}{2^{N+1}M + 1}
$$

where $M \neq 0$.

**Proof:** Since $x_{r}(t) \in C^{N+1}[0,T]$, it follows that its derivatives are bounded and thus there exists a constant $M$, such that

$$
\max |x^{(N+1)}(t)| \leq M
$$

The Chebsyhev polynomials are defined recursively via the formula

$$
T_{0}(t) = 1, \\
T_{1}(t) = t, \\
T_{N+1}(t) = 2tT_{N}(t) - T_{N-1}(t), \quad N = 1,2,3,4,\ldots
$$

Note that the leading term of the Chebyshev polynomial $T_{N}(t)$ is $2^{N-1}t^{N}$. For $t \in [-1,1]$, we have

$$
T_{N}(t) = \cos(N \cos^{-1}(t)),
$$

Hence

$$
|T_{N}(t)| \leq 1, \quad t \in [-1,1].
$$

Note that the expression of the form

$$
\prod_{i=0}^{N} (t - t_{i})
$$

are monic polynomials, as are the polynomials obtained from the Chebyshev polynomials by dividing through by the leading coefficient:

$$
Q_{N}(t) = \frac{1}{2^{N-1}} T_{N}(t).
$$

By construction, each of the $t_{i}$ is a distinct root of the monic polynomial $Q_{N}(t)$, which is of degree $N$. The fundamental theorem of algebra tells us that $Q_{N}(t)$ must therefore factorize as $Q_{N}(t) = (t - t_{1}) \cdots (t - t_{N})$. Therefore,

$$
(t - t_{1}) \cdots (t - t_{N}) = Q_{N}(t) \equiv 2^{1-N} T_{N}(t).
$$
We write

\[
\left| \prod_{i=0}^{N} (r - r_i) \right| = \frac{T_n(r)}{2^{N+1}} \leq \frac{1}{2^{N+1}},
\]

hence

\[
|x_\varepsilon(t) - X_\varepsilon(t)| = \left| 2^{1-N}T_n(r) \right| \frac{|x^{(N+1)}(\zeta(t))|}{(N+1)!} \leq \frac{M}{2^{N+1}(N+1)!}.
\]

Considering a general interval \( r \in [a, b] \), we can show that if \( [a, b] \neq [-1,1] \), use

\[
\tau_i = \frac{b + a}{2} - \frac{b - a}{2} \cos \left( \frac{\pi i}{N} \right), \quad i = 0, ..., N.
\]

Then

\[
|x_\varepsilon(r) - X_\varepsilon(r)| \leq \left( \frac{b - a}{2} \right)^{N+1} \frac{M}{2^{N+1}(N+1)!}.
\]

Therefore, in each interval \([t_{j-1}, t_j]\), we have

\[
E(r) = |x_\varepsilon(r) - X_\varepsilon(r)| \leq \left( \frac{T}{2p} \right)^{N+1} \frac{M}{2^{N+1}(N+1)!}.
\]

But

\[
t_j - t_{j-1} = \frac{T}{p},
\]

hence

\[
|x(r) - X(r)| \leq \left( \frac{T}{2p} \right)^{N+1} \frac{M}{2^{N+1}(N+1)!}.
\]

By adding the error in each subdomain given by Eq. (18), we get the total error bound across all the \( p \) subdomains

\[
E(t) = |x(t) - X(t)| \leq \left( \frac{T}{2p} \right)^{N+1} \frac{M}{2^{N+1}(N+1)!}.
\]

This finishes the proof.

It can be seen that, because of the factor \( \left( \frac{1}{p} \right)^N \ll 1 \) for large \( p \), the error in the multidomain case is much smaller than the single domain case, \( p = 1 \).

4. Results and Discussion

In this section, we present numerical results obtained by applying the MSRM to solve the nine-dimensional Lorenz system (2) and compare them with results of the Runge-Kutta (RK4) method. Reiterer et al. [19] observed that when the parameter value \( r \) is greater than 43.3, the system exhibit hyperchaotic behaviour, otherwise it remains chaotic. In this paper, we considered both chaotic and hyperchaotic cases by solving the system for values of \( r \). For the MSRM computations, \( N = 10 \) collocation points and \( p = 400 \) subdomains were sufficient to give accurate results. For the RK4, a step size of \( h = 10^{-4} \) is required to reach the same level of accuracy as the MSRM. A good agreement, to at least 10 decimal digits, between the two solutions is observed. In terms of computational time, the MSRM is much quicker than the RK4 as seen in Table 1. For the hyperchaotic case, \( r = 55 \), the time series solutions are depicted in Fig. 4. The results are also shown for selected values of the time in Table 2. For this case, \( N = 20 \) collocation points and \( p = 500 \) subdomains were used for the MSRM and \( h = 10^{-5} \) for the RK4. Again, a good agreement between the MSRM and the RK4 results is observed. The RK4 is very much slower than the MSRM. This can be attributed to the fact that a very small step size is required for the RK4 to acquire the same level of accuracy as the MSRM. As a result, the MSRM is much more efficient in terms of computation time. On the other hand, it has been shown, especially for large domain problems, that the accuracy of the spectral method is enhanced significantly if the problem is solved over multiple domains than using many grid points. This is because the multi-domain approaches allow the use of large step sizes and hence resulting in better-conditioned matrices.
Fig. 1. Phase portraits for the 9D attractor on the $x_0 - x_1$ plane for various values of $r$.

Fig. 2. Phase portraits for the 9D attractor on the $x_0 - x_1$ plane for various values of $r$. 
Table 1. Comparison of the MSRM and the RK4 solutions for $r = 14.1$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
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<td>-1.2023505190</td>
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<tr>
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</tr>
</tbody>
</table>

CPU time 1.777981 7.423930

Fig. 3. Time series solution for the 9D attractor for $r = 14.1$ (the chaotic case).
### Table 2. Comparison of the MSRM and the RK4 solutions for $r = 55$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
</tr>
</thead>
<tbody>
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<td>RK4</td>
<td>MSRM</td>
<td>RK4</td>
<td>MSRM</td>
<td>RK4</td>
<td>MSRM</td>
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<td>2.264802</td>
<td>5.374258</td>
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**CPU time**: 4.492820 s, 818.470979 s

**Fig. 4.** Time series solution for the 9D attractor for $r = 55$ (the hyperchaotic case).
5. Conclusion

In this paper, we have successfully computed solutions of a chaotic nine-dimensional Lorenz system, using a method based on blending the Gauss-Seidel relaxation method and the Chebyshev pseudo-spectral method. The method, called the multidomain spectral relaxation method MSRM, is a multidomain method which is adapted to solve complex dynamical systems like the hyperchaotic systems. The results presented in table and graphical forms are comparable to results obtained using the RK4 method, as well as other previously published results. The results also show that the proposed MSRM is accurate, computationally efficient, and a reliable method for solving complex dynamical systems with both chaotic and hyperchaotic behavior.

Author Contributions

The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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