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Shahid Chamran
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An Efficient Spectral Method-based Algorithm for Solving a High-dimensional Chaotic Lorenz System

Phumlani Dlamini[1], Simphiwe Simelane[2]

[1] Department of Mathematics and Applied Mathematics, University of Johannesburg, P.O. Box 17011, Doornfontein, 2028, South Africa, Email: phgdlamini@gmail.com

[2] Department of Mathematics and Applied Mathematics, University of Johannesburg, P.O. Box 17011, Doornfontein, 2028, South Africa, Email: simphinho@gmail.com

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Corresponding author: P. Dlamini (phgdlamini@gmail.com)

Abstract. In this paper, we implement the multidomain spectral relaxation method to numerically study high dimensional chaos by considering the nine-dimensional Lorenz system. Chaotic systems are characterized by rapidly changing solutions, as well as sensitivity to small changes in initial data. Most of the existing numerical methods converge slowly for this kind of problems and this results in inaccurate approximations. Spectral methods are known for their high accuracy. However, they become less accurate for problems characterised by chaotic solutions, even with an increase in the number of grid points. As a result, in this work, we adopt the multidomain approach which assumes that the main interval can be decomposed into a finite number of subdomains and the solution obtained in each of the subdomains. This approach remarkably improves the results as well as the efficiency of the method.

Keywords: Spectral method, Multidomain, Chaotic systems.

1. Introduction

Chaos is a phenomenon that came into existence in 1963 when Lorenz [1] discovered chaotic behaviour in differential equations modelling weather phenomena. Since then, nonlinear chaotic and hyperchaotic systems have arisen from many science and engineering applications. The complexity of the chaotic behaviour is dependent on the number of positive Lyapunov exponents the system has. Basically, a chaotic system has only one positive Lyapunov exponent, whereas chaotic systems with more than one positive Lyapunov exponents are said to be hyperchaotic. Hyperchaos is a concept that was introduced by Rossler [2] in 1976 when he discovered hyperchaotic behaviour in diferential equations for modeling chemical reactions. Since these two discoveries, several chaotic and hyperchaotic sytems arising in different fields have been studied [3,4]. Hyperchaotic systems generally have more complex dynamical behaviours than the ordinary chaotic systems. Chaotic systems are characterized by strong sensitivity to initial conditions and swiftly changing solutions. These characteristics make chaotic dynamical systems difficult to analyze numerically.

Various direct numerical techniques have been employed to solve systems of ODEs exhibiting chaos. These methods include the differential quadrature method [5], power series method [6], differential transform method [7], Barycentric Lagrange Interpolation Collocation Method [8], to name a few. Unfortunately, most of the existing direct numerical methods converge slowly for these problems, and this leads to inaccurate approximations. Spectral methods are known for their high accuracy, however they also become less accurate for non-smooth solutions and large time-domain problems, even with an increase in the number of grid points. The use of many grid points lead to large memory requirements and can also lead to the approximations exhibiting spurious oscillations which can result in nonlinear instabilities. To overcome this limitation, the multidomain approach has been used by many researchers. The multidomain approach assumes that the main interval can be decomposed into a finite number of sub-intervals.

The idea of domain decomposition has predominantly been applied to semi-analytical methods when solving chaotic systems. These semi-analytical methods include the multistage Adomian decomposition method [9], multistage homotopy analysis method [10], multistage diferential transformation method [11], multistage variational iteration method [12], and multistage homotopy perturbation methods [13]. The downside with the multidomain approach based on analytical approximations is that it becomes an arduous and time-consuming exercise to analytically integrate in each of the many subdomains.

Motsa et al. [14, 15] presented a multidomain numerical method based on the spectral method. Their method difers from the previously listed multidomain methods in that it is completely numerical. Other numerical methods where domain decomposition has been used to solve chaotic systems include the Garlekin-Petrov time discretization method [16], piecewise successive linearization method [17], piecewise-spectral parametric iteration [18], piecewise spectral homotopy analysis method [19] and multidomain compact finite difference relaxation method [20].

In this paper, we present the multidomain spectral relaxation method (MSRM) to solve higher dimensional chaotic systems. The method uses the Chebyshev spectral method combined with the Gauss-Seidel iteration scheme to solve chaotic systems in a

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number of subdomains making up the entire domain of problem. We show that the advantage of the multidomain approach is that the accumulation of errors across the subdomains is significantly less than the error when a single domain is considered. We use the nine-dimensional Lorenz system derived by Reiterer et al. [21] as a test case. The paper is organized as follows: In section 2, we present the 9D Lorenz system and give a description of how the MSRM is used to solve it. In section 3, we discuss the convergence of the method. In section 4, we present the results and discussion of the numerical simulations and finally, the conclusion is given in section 5.

2. Description of the method for 9D Lorenz

In this section, we give an outline of the implementation of the multidomain spectral relaxation method (MSRM) to solve the 9D Lorenz system. The 9D Lorenz system was derived by applying a triple Fourier expansion to the Boussinesq-Oberbeck equations governing thermal convection in a 3D spacial domain by using an approach similar to the well known 3D Lorenz's [21]. The system is given by

$$\begin{cases} \dot{x}_1 = -\sigma b_1 x_1 - \sigma b_2 x_7 - x_2 x_4 + b_4 x_4^2 + b_3 x_3 x_5 \\ \dot{x}_2 = -\sigma x_2 - 0.5\sigma x_9 + x_1 x_4 - x_2 x_5 + x_4 x_5 \\ \dot{x}_3 = -\sigma b_1 x_3 + \sigma b_2 x_8 + x_2 x_4 - b_4 x_2^2 - b_3 x_1 x_5 \\ \dot{x}_4 = -\sigma x_4 + 0.5\sigma x_9 - x_2 x_3 - x_2 x_5 + x_4 x_5 \\ \dot{x}_5 = -\sigma b_5 x_5 + 0.5 x_2^2 - 0.5 x_4^2 \\ \dot{x}_6 = -b_6 x_6 + x_2 x_9 - x_4 x_9 \\ \dot{x}_7 = -r x_1 - b_1 x_7 + 2 x_5 x_8 - x_4 x_9 \\ \dot{x}_8 = r x_3 - b_1 x_8 - 2 x_5 x_7 + x_2 x_9 \\ \dot{x}_9 = -r x_2 + r x_4 - x_9 - 2 x_2 x_6 + 2 x_4 x_6 + x_4 x_7 - x_2 x_8 \end{cases} \tag{1}$$

where the constant parameters b_i are defined as

$$b_1 = 4\frac{1+a^2}{1+2a^2}, \quad b_2 = \frac{1+2a^2}{2(1+a^2)}, \quad b_3 = 2\frac{1-a^2}{1+a^2}$$

$$b_4 = \frac{a^2}{1+a^2}, \quad b_5 = \frac{8a^2}{1+2a^2}, \quad b_6 = \frac{4}{1+2a^2}.$$

without loss of generality, we express Eq. (1) in the form

$$\dot{x} + \Lambda x + F(x) = 0, \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \dots, x_9(t)]^T$, Λ is a 9×9 matrix with entries $\alpha_{i,j}$, $i,j = 1,2,\dots,9$, given by

$$\Lambda = \begin{pmatrix} \sigma b_1 & 0 & 0 & 0 & 0 & 0 & \sigma b_2 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0.5\sigma \\ 0 & 0 & \sigma b_1 & 0 & 0 & 0 & 0 & -\sigma b_2 & 0 \\ 0 & 0 & 0 & \sigma & 0 & 0 & 0 & 0 & -0.5\sigma \\ 0 & 0 & 0 & 0 & \sigma b_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_6 & 0 & 0 & 0 \\ r & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 \\ 0 & 0 & -r & 0 & 0 & 0 & 0 & b_1 & 0 \\ 0 & r & 0 & -r & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\mathbf{F}(x)$ is a vector of nonlinear components of Eq. (2) given by

$$F(x) = \begin{pmatrix} x_2 x_4 - b_4 x_4^2 - b_3 x_3 x_5 \\ -x_1 x_4 + x_2 x_5 - x_4 x_5 \\ -x_2 x_4 + b_4 x_2^2 + b_3 x_1 x_5 \\ x_2 x_3 + x_2 x_5 - x_4 x_5 \\ -0.5 x_2^2 + 0.5 x_4^2 \\ -x_2 x_9 + x_4 x_9 \\ -2 x_5 x_8 + x_4 x_9 \\ 2 x_5 x_7 - x_2 x_9 \\ 2 x_2 x_6 - 2 x_4 x_6 - x_4 x_7 + x_2 x_8 \end{pmatrix}$$

we use the MSRM to solve Eq. (2). The MSRM is based on the use of the Chebyshev spectral method. The Chebyshev spectral method approximate functions by means of truncated series of Chebyshev orthogonal polynomials. The Chebyshev polynomials $T_N(\tau)$ of order N, are defined as

$$T_N(\tau) = \cos(N \cos^{-1}(\tau)), \qquad N \in \mathbb{N} \tag{3}$$

The Chebyshev interpolation, $u_N(\tau)$, of a function $u(\tau)$ at $\tau = \tau_i$, is defined by

$$u_N(\tau) = \sum_{i=0}^{N} u_i L_i(\tau). \tag{4}$$

The collocation points τ_i are chosen to be the extrema of T_N:

$$\{\tau_i\} = \left\{\cos\left(\frac{\pi i}{N}\right)\right\}_{i=0}^{N} \tag{5}$$

which are the Chebyshev-Gauss-Lobatto points. This choice is made from the simple reason that in Lagrangian interpolation, if the interpolation points are taken to be the zeros of the polynomial, the error is minimized. The polynomials $L_i(\tau)$, $i = 0,1,\dots,N$, are Lagrange polynomials of order N based on the Chebyshev-Gauss-Lobatto points, and is defined as

$$L_i(\tau) = \frac{(-1)^{i+1}(1-\tau^2)T_N(\tau)}{\bar{c}_i N^2(\tau-\tau_i)}, \qquad i = 0,1,\dots,N \tag{6}$$

where $\bar{c}_0 = \bar{c}_N = 2$, $\bar{c}_i = 1$ for $i = 1,2,\dots,N-1$. Therefore, the first order derivative of the approximate solution at the collocation points is computed as

$$\frac{d}{dt}u(\tau) = \sum_{k=0}^{N} u(\tau_k)\frac{dL_k(\tau_i)}{d\tau} = \sum_{k=0}^{N} D_{ik}u(\tau_k) = DU_i, \qquad i = 0,1,\dots,N \tag{7}$$

where $D_{ik} = \frac{dL_k(\tau_i)}{d\tau}$ is an $(N+1)\times(N+1)$ Chebyshev differentiation matrix for $i,k = 0,1,\dots,N$. The first order Chebyshev differentiation matrix at the collocation points is given by [22-25]

$$D_{ik} = \begin{cases} \dfrac{c_i(-1)^{k+i}}{c_k(\tau_k-\tau_i)}, & i \neq k \\ -\dfrac{\tau_i}{2(1-\tau_i^2)}, & (i=k) \neq 0,N \\ \dfrac{2N^2+1}{6}, & i,k = 0 \\ -\dfrac{2N^2+1}{6}, & i,k = N \end{cases} \tag{8}$$

To solve Eq. (2), firstly, we decompose the time domain $I = [0,T]$ into p subdomains of uniform length $\frac{T}{p}$, with each subdomain $I_j = [t_{j-1}, t_j]$ for $j = 1,2,\dots,p$ and having the property that

$$\bigcup_{j=1}^{p}[t_{j-1}, t_j] = [0,T].$$

Each subdomain $[t_{j-1}, t_j]$ is transformed to the domain $[-1,1]$, which is the domain of the Gauss-Lobatto points defined in Eq. (5), by using the linear transformation

$$\hat{t} = \frac{t_j - t_{j-1}}{2}\tau + \frac{t_{j-1}+t_j}{2} = \frac{T}{2p}\tau + \frac{t_{j-1}+t_j}{2}, \qquad j = 1,2,\dots,p$$

where

$$t_j - t_{j-1} = \frac{T}{p}.$$

We use the Chebyshev differentiation matrix to approximate the derivatives in Eq. (2), which results in a system of algebraic equations that we solve using the Gauss-Seidel technique. These equations are solved in each subdmain I_j. We use τ_i^j and $X_r^j = \left[x_r(\tau_0^j), x_r(\tau_1^j), \cdots, x_r(\tau_N^j)\right]^T$ to represent the collocation points and the approximate solution in each subdomain, respectively. Using the Chebyshev differentiation matrix to approximate the first derivatives in Eq. (2), we get

$$\left(\frac{2p}{T}D + \alpha_{r,r}I\right)X_r^j + \sum_{\substack{i=1\\ i\neq r}}^{9}\alpha_{r,i}X_i^j + F_r^j(X) = 0, \qquad r = 1,2,\dots,9 \tag{9}$$

where

$$X = [X_1, X_2, X_3, \dots, X_9]^T,$$

$$F_r(X) = [F_r(X(\tau_0)), F_r(X(\tau_1)), \dots, F_r(X(\tau_N))]^T.$$

To solve system (9), we use the Gauss-Seidel idea of solving algebraic equations to get the following iteration scheme,

$$\left(\frac{2p}{T}D + \alpha_{r,r}I\right)X^j_{r,s+1} = -\sum_{i=1}^{r-1}\alpha_{r,i}X^j_{i,s+1} - \sum_{i=r+1}^{9}\alpha_{r,i}X^j_{i,s} - F^j_{r,s,s+1}, \qquad r = 1,2,\dots,9 \tag{10}$$

where $X_{r,s+1}$ is the approximation of each x_r at the $(s+1)^{th}$ iteration and

$$F_{r,s,s+1} = F_r(X_{1,s+1}, X_{2,s+1}, \dots, X_{r-1,s+1}, X_{r,s}, \dots, X_{9,s}).$$

We remark that the nonlinear terms are evaluated using values from the previous iteration. Therefore, the approximate solution is obtained from

$$X^j_{r,s+1} = A_r^{-1}B_r^j \tag{11}$$

where

$$A_r = \frac{2p}{T}D + \alpha_{r,r}I,$$

$$B^j_r = -\sum_{i=1}^{r-1}\alpha_{r,i}X^j_{i,s+1} - \sum_{i=r+1}^{9}\alpha_{r,i}X^j_{i,s} - F^j_{r,s,s+1}, \qquad r = 1,2,\dots,9$$

The solution of the system (2) on $[0, T]$ is then given by

$$x_r = \bigcup_{j=1}^{p} X^j_r(\tau^j)$$

3. Convergence and error analysis

In this section, we provide the effect of domain decomposition on the convergence and error estimates on the multidomain spectral relaxation method (MSRM). Eq. (9), can be written as the block matrix system

$$AX = B \tag{12}$$

where

$$A = \begin{pmatrix} \frac{2p}{T}D + \alpha_{1,1}I & \alpha_{1,2}I & \alpha_{1,3}I & \dots & \alpha_{1,9}I \\ \alpha_{2,1}I & \frac{2p}{T}D + \alpha_{2,2}I & \alpha_{2,3}I & \cdots & \alpha_{2,9}I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{9,1}I & \alpha_{9,2}I & \alpha_{9,3}I & \dots & \frac{2p}{T}D + \alpha_{9,9}I \end{pmatrix}, \quad X^j = \begin{pmatrix} X^j_1 \\ X^j_2 \\ \vdots \\ X^j_9 \end{pmatrix} \text{ and } B^j = \begin{pmatrix} -F^j_1 \\ -F^j_2 \\ \vdots \\ -F^j_9 \end{pmatrix}$$

Theorem 1: For any $X^{(0)} \in \mathbb{R}^n$, the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by $X = A^{-1}B$ for each $k \geq 1$, converge to the unique solution $\bar{X}$ if the matrix A is strictly diagonally dominant.

Proof: To prove convergence, we show that matrix A, above is diagonally dominant. Since we are dealing with the block matrix A, we consider the norms of the matrices inside A. For any $m \times n$ matrix A with real entries, we have the norm equivalence

$$\|A\|_{\max} \leq \|A\|_2 \leq \sqrt{mn}\,\|A\|_{\max} \tag{13}$$

where $\|A\|_{\max} = \max_{\{i,j\}}|a_{i,j}|$ and $\|\cdot\|_2$ is the spectral norm. If p, in Eq. (12), is sufficiently large, then

$$\frac{2p}{T}D \gg \alpha_{r,r}I, \quad r = 1,2,\dots,9$$

This implies that

$$\left\|\frac{2p}{T}D + \alpha_{r,r}I\right\|_{\max} \approx \left|\frac{2p}{T} \times C\right|,$$

where C is a non-zero constant which is independent of $\frac{p}{T}$. Similarly,

$$\sum_{j=1,j\neq r}^{9} \|\alpha_{r,j} I\|_{\max} = \sum_{j=1,j\neq r}^{9} |\alpha_{r,j}|.$$

Thus, for p sufficiently large, we have

$$\left\|\frac{2p}{T}D + \alpha_{r,r}I\right\|_{\max} \approx \left|\frac{2p}{T}\times C\right| \geq \sum_{j=1,j\neq r}^{9} |\alpha_{r,j}|_{\max} = \sum_{j=1,j\neq r}^{9} \|\alpha_{r,j} I\| \tag{14}$$

Therefore, the block matrix A is diagonally dominant in the sense of matrix norm. Hence, the method converges.

Theorem 2: Let $x_r(t) \in C^{N+1}[0,T]$, and let $X_r(t)$ be a polynomial of degree $\leq N$ that interpolates the function $x_r(t)$ at $N+1$ distinct points $\tau_0, \tau_1, \ldots, \tau_N \in [t_j, t_{j+1}]$, where

$$\bigcup_{j=1}^{p} [t_{j-1}, t_j] = [0,T]$$

If the nodes τ_i are chosen as the Gauss-Lobatto points $\tau_i = \cos\left(\frac{i\pi}{N}\right)$, $(i = 0,1,\cdots,N)$, then the error term for the polynomial interpolation in $[0,T]$, using the nodes τ_i in each subdomain, is

$$E(t) = |x_r(t) - X_r(t)| \leq \left(\frac{T}{2}\right)^{N+1} \left(\frac{1}{p}\right)^{N} \frac{M}{2^{N-1}(N+1)!} \tag{15}$$

where $M \neq 0$.

Proof: Since $x_r(t) \in C^{N+1}[0,T]$, it follows that its derivatives are bounded and thus there exists a constant M, such that

$$\max\left|x^{(N+1)}(t)\right| \leq M \tag{16}$$

The Chebyshev polynomials are defined recursively, via the formula

$$\begin{aligned} &T_0(\tau) = 1 \\ &T_1(\tau) = \tau \\ &T_{N+1}(\tau) = 2\tau T_N(\tau) - T_{N-1}(\tau), \qquad N = 1,2,3,4,\ldots \end{aligned}$$

Note that the leading term of the Chebyshev polynomial $T_N(\tau)$ is $2^{N-1}\tau^N$. For $\tau \in [-1,1]$, we have

$$T_N(\tau) = \cos(N\cos^{-1}(\tau)),$$

Hence

$$|T_N(\tau)| \leq 1, \quad \tau \in [-1,1].$$

Note that the expressions of the form

$$\prod_{i=0}^{N} (\tau - \tau_i)$$

are monic polynomials, as are the ploynomials obtained from the Chebyshev polynomials by dividing through by the leading coefficient:

$$Q_N(\tau) = \frac{1}{2^{N-1}} T_N(\tau).$$

By construction, each of the τ_i is a distinct root of the monic polynomial $Q_N(\tau)$, which is of degree N. The fundamental theorem of algebra tells us that $Q_N(\tau)$ must therefore factorize as $Q_N(\tau) = (\tau - \tau_1)\cdots(\tau - \tau_N)$. Therefore,

$$(\tau - \tau_1)\ldots(\tau - \tau_N) = Q_N(\tau) \equiv 2^{1-N} T_N(\tau).$$

We write

$$\left|\prod_{i=0}^{N}(\tau-\tau_i)\right| = \left|\frac{T_N(\tau)}{2^{N-1}}\right| \le \frac{1}{2^{N-1}},$$

hence

$$|x_r(t)-X_r(t)| = |2^{1-N}T_N(\tau)|\frac{\left|x^{(N+1)}(\xi(\tau))\right|}{(N+1)!} \le \frac{M}{2^{N-1}(N+1)!}. \tag{17}$$

Considering a general interval $\tau \in [a,b]$, we can show that if $[a,b] \neq [-1,1]$, use

$$\tau_i = \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{i\pi}{N}\right), \quad i = 0, \dots, N.$$

Then

$$|x_r(\tau)-X_r(\tau)| \le \left(\frac{b-a}{2}\right)^{N+1}\frac{M}{2^{N-1}(N+1)!}.$$

Therefore, in each interval $[t_{j-1}, t_j]$, we have

$$E(\tau) = |x_r(\tau)-X_r(\tau)| \le \left(\frac{t_j-t_{j-1}}{2}\right)^{N+1}\frac{M}{2^{N-1}(N+1)!}.$$

But

$$t_j - t_{j-1} = \frac{T}{p},$$

hence

$$|x(\tau)-X(\tau)| \le \left(\frac{T}{2p}\right)^{N+1}\frac{M}{2^{N-1}(N+1)!}. \tag{18}$$

By adding the error in each subdomain given by Eq. (18), we get the total error bound across all the p subdomains

$$E(t) = |x(t)-X(t)| \le \left(\frac{T}{2}\right)^{N+1}\left(\frac{1}{p}\right)^{N}\frac{M}{2^{N-1}(N+1)!} \tag{19}$$

This finishes the proof.

It can be seen that, because of the factor $\left(\frac{1}{p}\right)^N << 1$ for large p, the error in the multidomain case is much smaller than the single domain case, $p = 1$.

4. Results and Discussion

In this section, we present numerical results obtained by applying the MSRM to solve the nine-dimensional Lorenz system (2) and compare them with results of the Runge-Kutta (RK4) method. Reiterer et al. [19] observed that when the parameter value r is greater than 43.3, the system exhibit hyperchaotic behaviour, otherwise it remains chaotic. In this paper, we considered both chaotic and hyperchaotic cases by solving the system for values of r between 14.1 – 15.1 and $r = 55$, respectively. The initial condition was taken to be $x(0) = \{0.01, 0, 0.01, 0, 0, 0, 0, 0, 0.01\}$ and $\sigma = 0.5$. Phase projections on the $x_6 - x_7$ and $x_6 - x_9$ planes, for varying values of r, are depicted in Fig. 1 and Fig. 2, respectively. The phase portraits obtained are consistent with those of Reiterer et al. [21] and Kouagou et al.[26]. This shows that the MSRM is able to handle high dimensional chaotic systems.

The time series solutions for the chaotic case, $r = 14.1$, are shown in Fig. 3. The solid line represents the MSRM solution while the dotted line represent the RK4 solution. Table 1 shows the comparison between the two approaches at selected values of t. For the MSRM computations, $N = 10$ collocation points and $p = 400$ subdomains were sufficient to give accurate results. For the RK4, a step size of $h = 10^{-4}$ is required to reach the same level of accuracy as the MSRM. A good agreement, to at least 10 decimal digits, between the two solutions is observed. In terms of computational time, the MSRM is much quicker than the RK4 as seen in Table 1.

For the hyperchaotic case, $r = 55$, the time series solutions are depicted in Fig. 4. The results are also shown for selected values of the time in Table 2. For this case, $N = 20$ collocation points and $p = 500$ subdomains were used for the MSRM and $h = 10^{-5}$ for the RK4 . Again, a good agreement between the MSRM and the RK4 results is observed. The RK4 is very much slower than the MSRM. This can be attributed to the fact that a very small step size is required for the RK4 to acquire the same level of accuracy as the MSRM . As a result, the MSRM is much more efficient in terms of computation time. On the other hand, it has been shown, especially for large domain problems, that the accuracy of the spectral method is enhanced significantly if the problem is solved over multiple domains than using many grid points. This is because the multi-domain approaches allow the use of large step sizes and hence resulting in better-conditioned matrices.

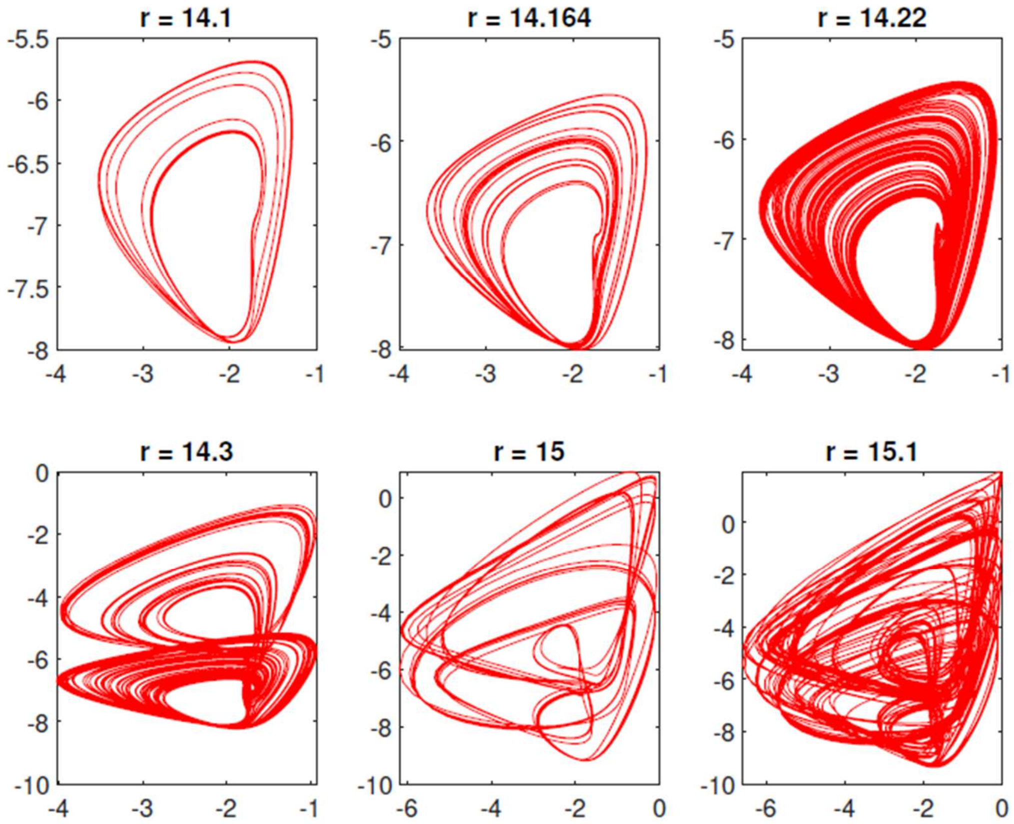

Fig. 1. Phase portraits for the 9D attractor on the $x_6 - x_7$ plane for various values of r.

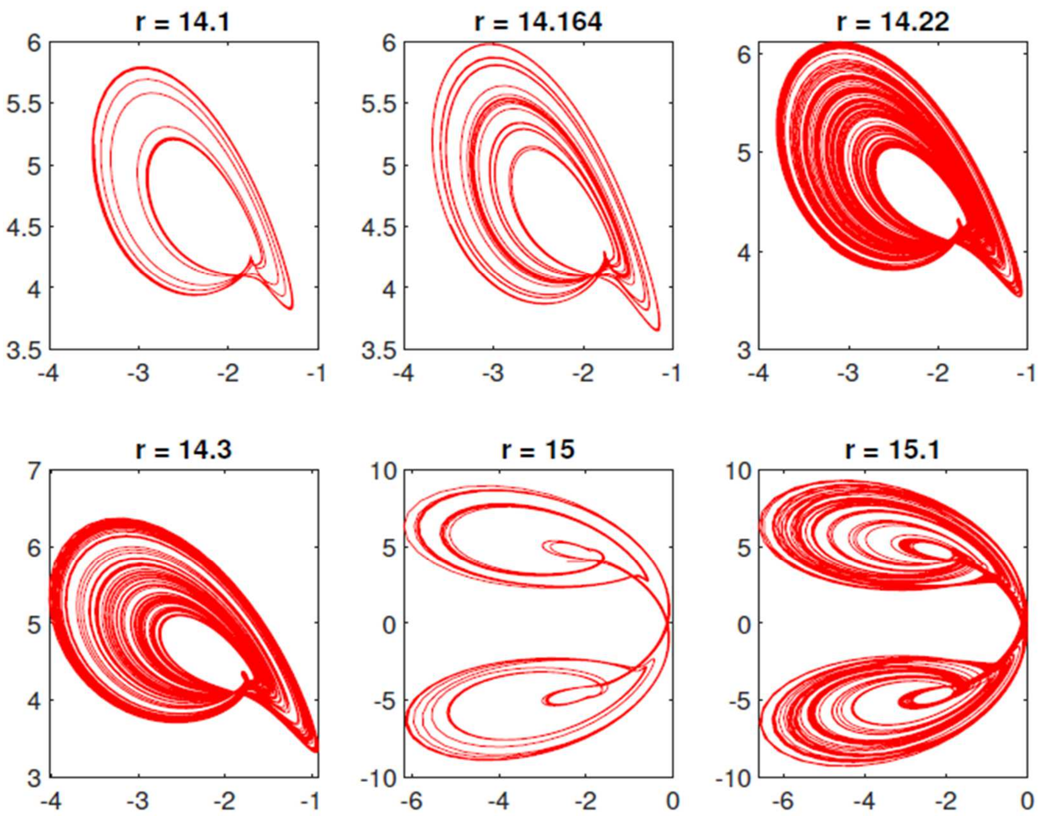

Fig. 2. Phase portraits for the 9D attractor on the $x_6 - x_9$ plane for various values of r

Table 1. Comparison of the MSRM and the RK4 solutions for $r = 14.1$

t	x_1		x_2		x_3	
	MSRM	RK4	MSRM	RK4	MSRM	RK4
20	1.7794616020	1.7794616020	-0.7204522789	-0.7204522789	-0.6362110296	-0.6362110296
40	1.5609534149	1.5609534149	-0.2347187741	-0.2347187741	-1.2023505190	-1.2023505190
60	1.4855822940	1.4855822940	-0.1307250241	-0.1307250241	-0.2401679408	-0.2401679408
80	1.7958197536	1.7958197536	-0.7477525490	-0.7477525490	-0.9536794241	-0.9536794241
100	1.6599843283	1.6599843283	-0.1433734338	-0.1433734338	-0.9914698728	-0.9914698728
t	x_4		x_5		x_6	
	MSRM	RK4	MSRM	RK4	MSRM	RK4
20	0.7245177009	0.7245177009	-0.3269195959	-0.3269195959	-2.6127239408	-2.6127239408
40	0.8327202872	0.8327202872	-0.1787710447	-0.1787710447	-1.6858331785	-1.6858331785
60	0.9777946806	0.9777946806	-0.6137459610	-0.6137459610	-1.7451784330	-1.7451784330
80	0.5724218362	0.5724218362	-0.1220058785	-0.1220058785	-2.3524652982	-2.3524652982
100	0.8524360371	0.8524360371	-0.2153136081	-0.2153136081	-1.5321979302	-1.5321979302
t	x_7		x_8		x_9	
	MSRM	RK4	MSRM	RK4	MSRM	RK4
20	-7.3469555719	-7.3469555719	-4.9185099929	-4.9185099929	4.5741481236	4.5741481236
40	-7.0111342856	-7.0111342856	-6.2312240135	-6.2312240135	4.1679032967	4.1679032967
60	-6.3019708209	-6.3019708209	-3.7653381446	-3.7653381446	4.5200856239	4.5200856239
80	-7.7542533122	-7.7542533122	-5.4472879550	-5.4472879550	4.1959347196	4.1959347196
100	-7.4421079409	-7.4421079409	-5.4831875435	-5.4831875435	3.9713031504	3.9713031504
CPU time	1.777981	7.423930				

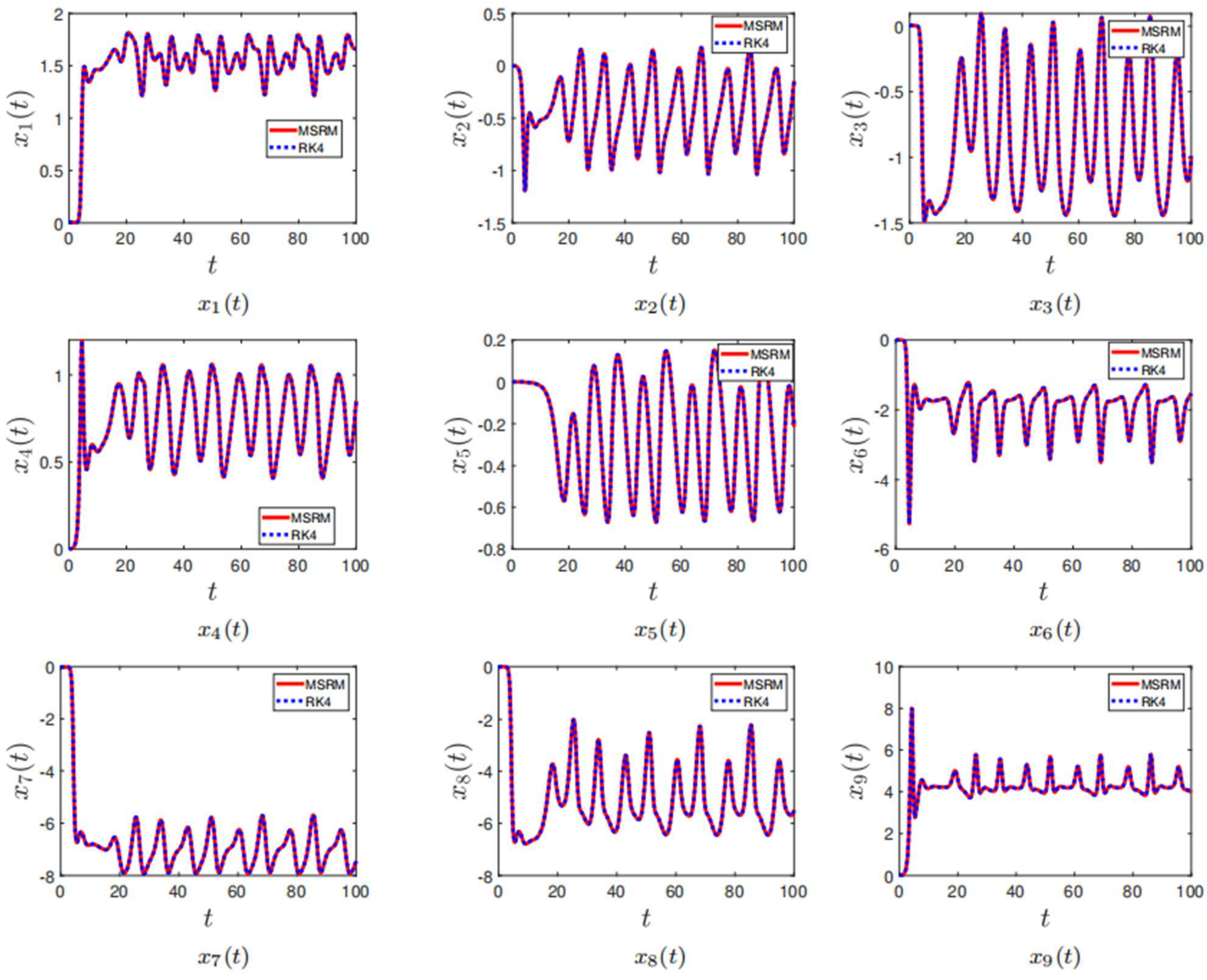

Fig. 3. Time series solution for the 9D attractor for $r = 14.1$ (the chaotic case).

Table 2. Comparison of the MSRM and the RK4 solutions for $r = 55$

t	x_1		x_2		x_3	
	MSRM	RK4	MSRM	RK4	MSRM	RK4
4	2.264802	2.264802	5.374258	5.374258	1.663534	1.663534
8	0.254814	-0.254814	-4.806631	-4.806631	5.133758	5.133758
12	4.177500	-4.177500	3.236132	3.236132	4.609824	4.609824
16	5.971705	5.971705	-3.474095	-3.474095	-2.599292	-2.599292
20	0.458099	-0.458099	1.194282	1.194282	1.946396	1.946396
t	x_4		x_5		x_6	
	MSRM	RK4	MSRM	RK4	MSRM	RK4
4	-5.680465	-5.680465	3.295077	3.295077	-44.575344	-44.575344
8	-6.166781	-6.166781	2.900768	2.900768	-23.072529	-23.072529
12	1.230726	1.230726	-0.997081	-0.997081	-8.179131	-8.179131
16	-3.455944	-3.455944	0.276657	0.276657	-8.746022	-8.746022
20	-4.536999	-4.536999	-0.265394	-0.265394	-23.120070	-23.120070
t	x_7		x_8		x_9	
	MSRM	RK4	MSRM	RK4	MSRM	RK4
4	1.558516	1.558516	-0.874860	-0.874860	-8.160863	-8.160863
8	19.065008	19.065008	9.809716	9.809716	3.606984	3.606984
12	25.129832	25.129832	36.175991	36.175991	-26.497018	-26.497018
16	-55.935017	-55.935017	-17.578250	-17.578250	35.292195	35.292195
20	-2.946511	-2.946511	17.728235	17.728235	-39.708822	-39.708822
CPU time	4.492820	818.470979				

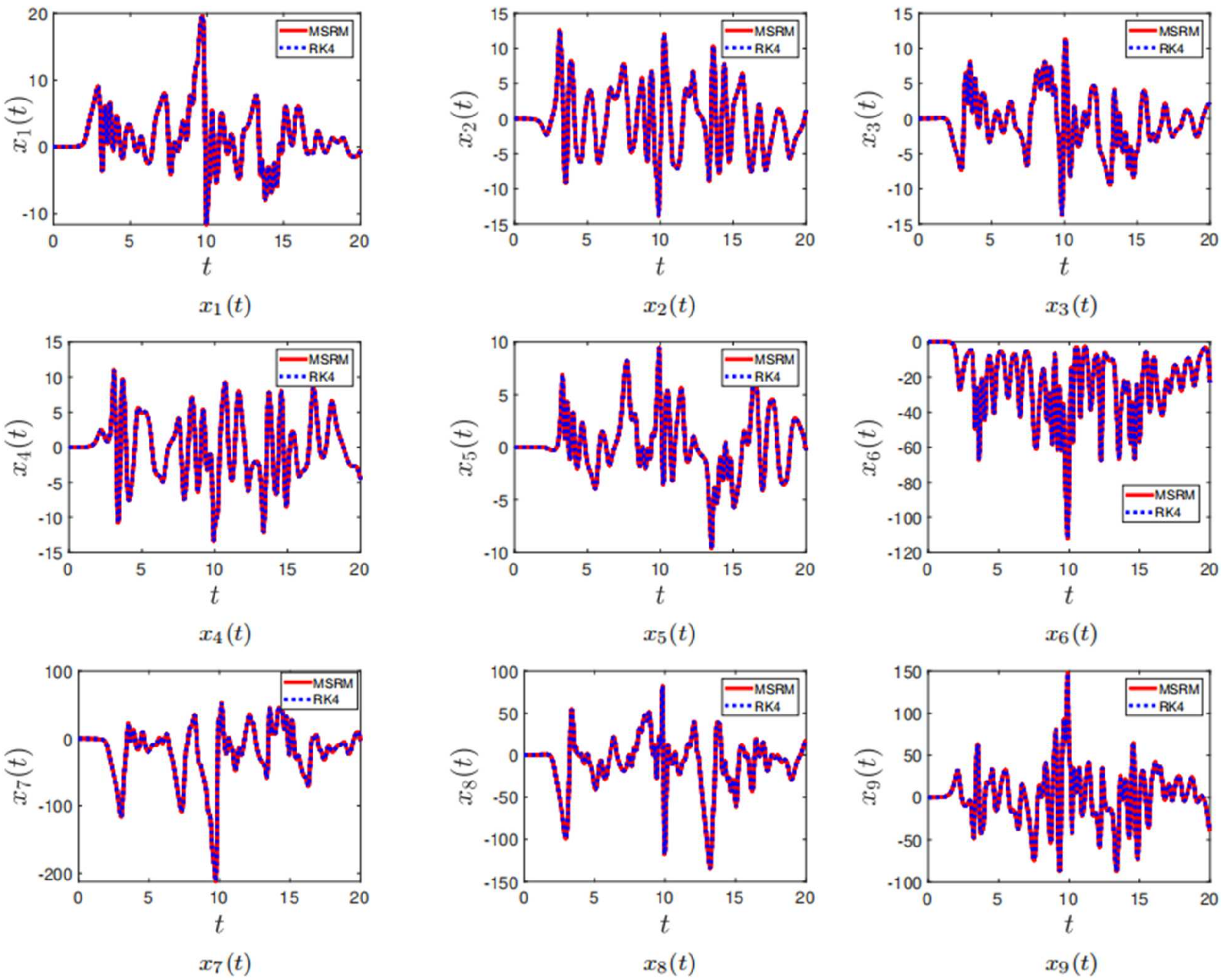

Fig. 4. Time series solution for the 9D attractor for $r = 55$ (the hyperchaotic case).

5. Conclusion

In this paper, we have successfully computed solutions of a chaotic nine-dimensional Lorenz system, using a method based on blending the Gauss-Seidel relaxation method and the Chebyshev pseudo-spectral method. The method, called the multidomain spectral relaxation method MSRM, is a multidomain method which is adapted to solve complex dynamical systems like the hyperchaotic systems. The results presented in table and graphical forms are comparable to results obtained using the RK4 method, as well as other previously published results. The results also show that the proposed MSRM is accurate, computationally efficient, and a reliable method for solving complex dynamical systems with both chaotic and hyperchaotic behavior.

Author Contributions

The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

References

[1] Lorenz, E., Deterministic nonperiodic flow, *J. Atmospheric Sci.*, 20, 1963, 130-141.
[2] R¨ossler, O.E., An equation for continuous chaos, *Phys. Lett. A*, 57(5), 1976, 397-398.
[3] Ispolatov, I., Madhok, V., Allende, S., Doebeli, M., Chaos in high-dimensional dissipative dynamical systems, *Scientific Reports*, 5, 2015, 12506.
[4] Eilersen, A., Jensen, M.H., Sneppen, K., Chaos in disease outbreaks among prey, *Scientific Reports*, 10, 2020, Article 3907.
[5] Eftekhari, S.A., Jafari, A.A., Numerical simulation of chaotic dynamical systems by the method of differential quadrature, *Scientia Iranica*, 19(5), 2012, 1299–1315.
[6] Lozi, R., Pogonin, V.A., Pchelintsev, A.N., A new accurate numerical method of approximation of chaotic solutions of dynamical model equations with quadratic nonlinearities, *Chaos, Solitons and Fractals*, 91, 2016, 108–114 .
[7] Odibat, Z.M., Bertelle, C., Aziz-Alaoui, M.A., Duchamp, G.H.E., A multi-step differential transform method and ap- plication to non-chaotic or chaotic systems, *Computers and Mathematics with Applications*, 59(4), 2010, 1462–1472.
[8] Zhou, X., Li, J., Wang, Y., Zhang, W., Numerical Simulation of a Class of Hyperchaotic System Using Barycentric Lagrange Interpolation Collocation Method, *Complexity*, 2019(1), 1–13.
[9] Abdulaziz, O., Noor, N.F.M., Hashim, I., Noorani M.S.M., Further accuracy tests on Adomian decomposition method for chaotic systems, *Chaos Solitons Fractals*, 36, 2008, 1405-1411.
[10] Alomari, A.K., Noorani, M.S.M., Nazar, R., Adaptation of homotopy analysis method for the numeric analytic solution of Chen system, *Commun. Nonlinear Sci. Numer. Simul.*, 14, 2009, 2336-2346.
[11] Do, Y., Jang B., Enhanced multistage differential transform method: application to the population models, *Abstr. Appl. Anal.*, 2012, 253890.
[12] Batiha, B:, Noorani, M.S.M., Hashim, I., Ismail, E.S., The multistage variational iteration method for a class of nonlinear system of ODEs, Phys. Scr., 76, 2007, 388-392.
[13] Chowdhury, M.S.H., Hashim, I., Momani, S., The multistage homotopy-perturbation method: a powerful scheme for handling the Lorenz system, *Chaos Solitons Fractals*, 40, 2009, 1929-1937.
[14] Motsa, S.S., Dlamini, P., Khumalo, M., A new multistage spectral relaxation method for solving chaotic initial value systems, *Nonlinear Dynam.*, 72, 2013, 265–283.
[15] Motsa, S.S., Dlamini, P.G., Khumalo, M., Solving hyperchaotic systems using the spectral relaxation method, *Abstr. Appl. Anal.*, 2012, 1–18.
[16] Khan, M.S., Khan, M.I., A novel numerical algorithm based on Galerkin–Petrov time-discretization method for solving chaotic nonlinear dynamical systems, *Nonlinear Dynam.*, 91(3), 2018, 1555–1569 .
[17] Motsa, S.S., A new piecewise-quasilinearization method for solving chaotic systems of initial value problems, *Cent. Eur. J. Phys.*, 10(4), 2012, 936–946.
[18] Ghorbani, A., Saberi-Nadjafi, J., A piecewise-spectral parametric iteration method for solving the nonlinear chaotic Genesio system, *Math. Comput. Modelling*, 54(1–2), 2011, 131–139.
[19] Karimi, M., Saberi Nik, H., A piecewise spectral method for solving the chaotic control problems of hyperchaotic finance system, *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, 31(3), 2018, 1–14.
[20] Mathale, D., Dlamini, P. G., Khumalo, M., Compact finite difference relaxation method for chaotic and hyperchaotic initial value systems, *Computational and Applied Mathematics*, 37(4), 2018, 5187–5202.
[21] Reiterer, P., Lainscsek, C., Sch, F., Maquet, J., A nine-dimensional Lorenz system to study high-dimensional chaos, *J. Phys. A: Math. Gen.*, 31, 1998, 7121-7139.
[22] Canuto, C., Hussaini, M.Y., Quarteroni, A., Zang, T.A., *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York , 1988.
[23] Trefethen, L. N., *Spectral Methods in MATLAB*, SIAM, Philadelphia, Pa, USA, 2000.
[24] Trivedi, M., Otegbeye, O, Ansari, Md. S., Motsa S.S., A Paired Quasi-linearization on Magnetohydrodynamic Flow and Heat Transfer of Casson Nanofluid with Hall Effects, *Journal of Applied and Computational Mechanics*, 5(5), 2019, 849-860.
[25] Mondal, H., Bharti, S., Spectral Quasi-linearization for MHD Nanofluid Stagnation Boundary Layer Flow due to a Stretching/Shrinking Surface, *Journal of Applied and Computational Mechanics*, 6(4), 2020, 1058-1068.
[26] Kouagou J.N., Dlamini P.G., Simelane S.M., On the multi-domain compact finite difference relaxation method for high dimensional chaos: The nine-dimensional Lorenz system, *Alexandria Engineering Journal*, 59, 2020, 2617-2625.

ORCID iD

Phumlani Dlamini https://orcid.org/0000-0001-9261-308X
Simphiwe Simelane https://orcid.org/0000-0002-0067-2996

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