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Research Paper

Variational Principles for Two Compound Nonlinear Equations with Variable Coefficients

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Abstract. It is very important to seek explicit variational principles for nonlinear partial differential equations, which are theoretical bases for many methods to solve or analyze the nonlinear phenomena and problems. By designing the modified trial-Lagrange functional, different variational formulations are successfully and firstly established by the semi-inverse method for two kinds of compound nonlinear equation, i.e. the KdV-Burgers equation and the Burgers-BBM equation, respectively. Both of them contain the variable coefficients, which are time-dependent. Furthermore, the obtained variational principles are proved correct by minimizing the functionals with the calculus of variations.

Keywords: Variational principle, Calculus of variations, Compound KdV-Burgers equation, Compound Burgers-BBM equation.

1. Introduction

To solve the partial differential equations (PDEs) with integer or fractional orders is always an attractive and hot topic for many researchers in different scientific fields, because of their outstanding ability for modeling nonlinear phenomena. These phenomena have been investigated to study their physical properties by using the solutions of PDEs [1-5]. So, numerous methods have been developed to explore the approximate or exact solutions [4-25], of which variational-based methods have been very useful and effective, such as Ritz technique [12], variational iteration method [14-17], and variational approximation method [21-25] et al. When contrasted with other methods, variational ones show some advantages. For example, they can be used in investigating practical problems from a global perspective and provide physical insight into the nature of the solutions. And, the obtained solutions are the best among all possible trial-functions, and require much less strong local differentiability of variables than the ones that directly solve PDEs, such as finite difference method, finite volume method, et al. Because variational principles are so important for obtaining the approximate or exact solutions by variational methods, it is of great significance to seek explicit variational formulations for the nonlinear PDEs. The semi-inverse method [26-34] was firstly proposed in 1997 by Dr. Ji-Huan He, who is a famous Chinese mathematician. The semi-inverse method has been widely used to establish variational principles from the governing equations directly, and has become a significant and effective tool in the variational theory far beyond the well-known Lagrange multiplier method [14-16, 20, 26-34]. Because it is not necessary to introduce Lagrange multipliers, the Lagrange crisis frequently encountered in constructing variational principles can be avoided effectively and naturally [26-34]. Recently, many scientists have made a lot of attempts and great success for constructing variational principles in different kinds of fields such as fluid dynamics, meteorology, ocean, mathematical biology, solid state physics, optics, and plasma physics, and so forth [26-40]. Nonlinear PDEs with variable coefficients usually describe the physical phenomena more accurately and finely. In this paper, variational principles are established by the semi-inverse method [26-34], for the variable coefficient compound KdV-Burgers equation and the Burgers-BBM equation, respectively. Although the compound equations considered in this paper have been extensively studied for a long time by some scientists [41-55], but, up to now, variational principles for them have not been dealt with. Therefore, finding variational principles for them is of great value, and might find lots of applications in numerical simulations and scientific researches.

2. Variational Principles for the Variable Coefficient Compound KdV-Burgers Equation

The compound Korteweg-de Vries-Burgers (cKdVB) equation is a nonlinear partial differential equation, which can be thought as the combination of KdV and Burgers equation. Many researchers focus on the cKdVB equation, and a number of theoretical issues and solitary solutions have been obtained [41-49]. The exact traveling wave and soliton solutions in particular have been studied extensively [41,44-45,48-49]. Here, we consider the variable coefficient compound KdV-Burgers equation as following:

$$u_t + \alpha(t)uu_x - \beta(t)u^2u_x + \mu(t)u_{xx} + \delta(t)u_{xxx} = 0 \tag{1}$$

The model arises from plenty of physical applications and can be used as the control equation to a large number of nonlinear dynamical phenomena, such as weakly nonlinear plasma waves with dissipative effects, propagation of undular bores in shallow water waves, propagation of waves in elastic tube filled with a viscous fluid, and flow of liquids containing gas bubbles, etc [41-49]. In Eq.(1), $\alpha(t)$, $\beta(t)$, $\mu(t)$ and $\delta(t)$ are arbitrary functions of time t, which indicate the effects of nonlinearity, dispersion and dissipation at different time periods, respectively. In fact, if one takes different values for α, β, μ and δ, equation (1) will include quite a few equations as particular cases such as KdV, MKdV, CKdV, Burgers, and KdV-Burgers equation. When parameters $\beta = 0$, $\mu = 0$, Eq.(1) can be degenerated to the variable coefficient KdV equation. When parameters $\alpha = 0$, $\mu = 0$, Eq.(1) becomes the modified KdV equation. If we let $\beta = 0$, $\delta = 0$, Eq.(1) turns to the famous Burgers equation. When $\mu = 0$, Eq.(1) can be degenerated to the combined KdV equation, which is a compound of KdV and MKdV equation. If we let $\beta = 0$ only, Eq.(1) can be reduced to the KdV-Burgers equation.

In order to find its variational principles, Eq. (1) can be transformed into the following form

$$u_t + (\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 + \mu u_x + \delta u_{xx})_x = 0 \tag{2}$$

It is obvious that finding Lagrangian representations for the above compound KdV-Burgers equation is a nontrivial problem. Additionally, it is necessary to replace the physical field $u(x,t)$ by it's derivatives of potential fields. According to Eq. (2), a potential function Φ can be introduced, as following

$$\begin{aligned} &\Phi_x = u \\ &\Phi_t = -(\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 + \mu u_x + \delta u_{xx}) \end{aligned} \tag{3}$$

Thus, Eq. (2) will be automatically satisfied. We hope to construct different variational principles, according to Eq. (2) and the field equations (3).

For establishing the variational principles, whose Euler-Lagrange equations will be equivalent to the variable coefficient compound KdV-Burgers equation, we can firstly set a trial-functional in the following form:

$$J(u,\Phi) = \iint L(u,u_t,u_x,u_{xx},\Phi,\Phi_t,\Phi_x,\Phi_{xx})dxdt \tag{4}$$

where L is the trial-Lagrange functional. In view of Eq. (2) and (3), we design by the semi-inverse method [26-34], that the L can be written as

$$L = u\Phi_t + (\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 + \mu u_x + \delta u_{xx})\Phi_x + F \tag{5}$$

in which F is an unknown functional of only variable u and its derivatives, to be determined later. There are many alternative methods for constructing the trial-functional, see References. [26-34]. The great merit of the above trial-Lagrange functional (5) is whose stationary condition with respect to Φ leads to the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial x}(\frac{\partial L}{\partial \Phi_x}) - \frac{\partial}{\partial t}(\frac{\partial L}{\partial \Phi_t}) + \frac{\partial}{\partial x^2}(\frac{\partial L}{\partial \Phi_{xx}}) = 0 \tag{6}$$

After introducing Eq. (5), Eq. (6) is identical to the compound KdV-Burgers Equation (1). Subsequently, by calculating the stationary conditions of Eq. (5) with respect to u, we obtain:

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial L}{\partial u_x}) + \frac{\partial}{\partial x^2}(\frac{\partial L}{\partial u_{xx}}) + \frac{\delta F}{\delta u} = 0 \tag{7}$$

where $\delta F / \delta u$ is called the Frechet's variational derivative [6-39] of F. By using Eq. (5), Eq. (7) can be rewritten as:

$$\Phi_t + (\alpha u - \beta u^2)\Phi_x - \mu\Phi_{xx} + \delta\Phi_{xxx} + \frac{\delta F}{\delta u} = 0 \tag{8}$$

We hope to find such an F, so that Eq. (8) turns out to be the field equations (3). Accordingly, after substituting the Eq. (3) into Eq. (8), we get:

$$\frac{\delta F}{\delta u} = -\frac{\alpha}{2}u^2 + \frac{2\beta}{3}u^3 + 2\mu u_x \tag{9}$$

From Eq. (9), unfortunately, we cannot identify F through the calculus of variations, because of existing the term $2\mu u_x$, so we have to modify the trial-Lagrange function L into a new form [30] as,

$$L = Au\Phi_t + B\Phi_x\Phi_t + (\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 + \mu u_x + \delta u_{xx})\Phi_x + F \tag{10}$$

Again, by calculating the variational derivatives of L with respect to Φ and u, respectively, the new Euler-Lagrange equations can be obtained

$$\frac{\delta L}{\delta \Phi}: \quad -Au_t - 2B\Phi_{xt} - (\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 + \mu u_x + \delta u_{xx})_x + \frac{\delta F}{\delta \Phi} = 0 \tag{11}$$

$$\frac{\delta L}{\delta u}: \quad A\Phi_t + (\alpha u - \beta u^2)\Phi_x - \mu\Phi_{xx} + \delta\Phi_{xxx} + \frac{\delta F}{\delta u} = 0 \tag{12}$$

In view of Eq. (3) and $\delta F / \delta \Phi = 0$, Eq. (11) becomes:

$$(A + 2B)u_t + (\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 + \mu u_x + \delta u_{xx})_x = 0 \tag{13}$$

Because Eq. (13) should be identical to Eq. (2), we must set the coefficient of u_t to one. That is

$$A + 2B = 1 \tag{14}$$

After substituting Eq. (3) into Eq. (12), we obtain:

$$\frac{\delta F}{\delta u} = (\frac{A}{2} - 1)\alpha u^2 + (1 - \frac{A}{3})\beta u^3 + (A+1)\mu u_x + \delta(A-1)u_{xx} \tag{15}$$

In order to determine the unknown function F successfully, it is necessary to eliminate the term u_x, whose coefficient must be set to zero in Eq. (15). At the same time, according to the variational calculus and $\mu \neq 0$, we get

$$A+1 = 0 \tag{16}$$

From equations (14) and (16), we obtain $A = -1$ and $B = 1$. Furthermore,

$$\frac{\delta F}{\delta u} = -\frac{3\alpha}{2}u^2 + \frac{4\beta}{3}u^3 - 2\delta u_{xx} \tag{17}$$

From Eq. (17), F can be identified easily as

$$F = -\frac{\alpha}{2}u^3 + \frac{\beta}{3}u^4 + \delta u_x^2 \tag{18}$$

or

$$F = -\frac{\alpha}{2}u^3 + \frac{\beta}{3}u^4 - \delta u u_{xx} \tag{19}$$

Finally, we obtain the variational formulations for the variable coefficient compound KdV-Burgers equation (1), which read:

$$J(u,\Phi) = \iint \{\Phi_x\Phi_t - u\Phi_t + [\frac{\alpha(t)}{2}u^2 - \frac{\beta(t)}{3}u^3 + \mu(t)u_x + \delta(t)u_{xx}]\Phi_x - \frac{\alpha(t)}{2}u^3 + \frac{\beta(t)}{3}u^4 + \delta(t)u_x^2\}dxdt \tag{20}$$

and

$$J(u,\Phi) = \iint \{\Phi_x\Phi_t - u\Phi_t + [\frac{\alpha(t)}{2}u^2 - \frac{\beta(t)}{3}u^3 + \mu(t)u_x + \delta(t)u_{xx}]\Phi_x - \frac{\alpha(t)}{2}u^3 + \frac{\beta(t)}{3}u^4 - \delta(t)uu_{xx}\}dxdt \tag{21}$$

both of which are subject to the constraint equation $\Phi_x = u$. The established variational principles are firstly discovered by the semi-inverse method [26-34], and may find lots of applications in numerical simulations and researches of the compound KdV-Burgers. In the following, we will prove the obtained variational principles correct. By making anyone of the above functionals, Equations (20) and (21), stationary with respect to independent functions u and Φ severally, we can obtain two different Euler-Lagrange equations as:

$$\delta\Phi: \quad u_t - 2\Phi_{xt} - (\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 + \mu u_x + \delta u_{xx})_x = 0 \tag{22}$$

$$\delta u: \quad -\Phi_t + (\alpha u - \beta u^2)\Phi_x - \mu\Phi_{xx} + \delta\Phi_{xxx} - \frac{3\alpha}{2}u^2 + \frac{4\beta}{3}u^3 - 2\delta u_{xx} = 0 \tag{23}$$

in which $\delta\Phi$ and δu is the first-order variation for Φ and u. Substituting $\Phi_x = u$ into Eq. (22) leads to the original compound KdV-Burgers equation, obviously. After substituting $\Phi_x = u$ into Eq. (23), we can get that $\Phi_t + \mu u_x + \alpha u^2/2 - \beta u^3/3 + \delta u_{xx} = 0$, which is identical to the second one of equations (3). Hence, successfully, we proved the obtained variational principles (20) and (21) correct.

3. Variational Principles for the variable coefficient Burgers-BBM equation

The compound BBM-Burgers equation is a hydrodynamic model for propagation of small-amplitude long wave in nonlinear dispersive media, which is much better than the KdV equation. To get the approximate solution of the Burgers-BBM equation, some numerical and analytical methods have been proposed in recent years [50-55].

In this part, consider the following compound Burgers-BBM equation with variable coefficients

$$u_t + \alpha(t)u_x + \beta(t)uu_x - \gamma(t)u_{xx} - \mu u_{xxt} = 0 \tag{24}$$

The Eq. (24) is related to the well-known BBM equations which were advocated by Benjamin-Bona-Mahony as a refinement of the KdV equation [50]. In Eq. (24), $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are arbitrary functions of t, while μ is a constant parameter. Eq. (24) can be transformed into the following conservative form

$$(u - \mu u_{xx})_t + (\alpha u + \frac{\beta}{2}u^2 - \gamma u_x)_x = 0 \tag{25}$$

It is obvious that finding Lagrangian representations for the above equation (24) is not a trivial problem. Firstly, it is essential to replace original variable of wave height with the derivative of potential field. According to the equation (25), a potential function Π can be introduced as:

$$\begin{aligned} \Pi_x &= u - \mu u_{xx} \\ \Pi_t &= -(\alpha u + \frac{\beta}{2}u^2 - \gamma u_x) \end{aligned} \tag{26}$$

so that the Eq. (24) and (25) is automatically satisfied. We will construct different variational principles, directly from the original equation (24) and field equations (26).

Secondly, we can build a trial-functional in the following form by the semi-inverse method [26-34]:

$$J(u,\Pi) = \iint L(u,u_t,u_x,u_{xx},u_{xxt},\Pi,\Pi_t,\Pi_x)dxdt \tag{27}$$

where L is the trial-Lagrange functional. In view of Equation (27), It is designed that the L is written as following:

$$L = (u - \mu u_{xx})\Pi_t + (\alpha u + \frac{\beta}{2}u^2 - \gamma u_x)\Pi_x + G \tag{28}$$

Specially, G is an unknown functional of u and it's derivatives. The remarkable merit of the above trial-Lagrange functional (28) is whose stationary condition with respect to Π leads to the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial \Pi} - \frac{\partial}{\partial x}(\frac{\partial L}{\partial \Pi_x}) - \frac{\partial}{\partial t}(\frac{\partial L}{\partial \Pi_t}) = 0 \tag{29}$$

In view of Eq. (28), Eq. (29) is equivalent to the Burgers-BBM equation (25). Subsequently, by calculating the stationary conditions of Eq. (28) with respect to u, it leads to:

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial L}{\partial u_x}) - \frac{\partial}{\partial t}(\frac{\partial L}{\partial u_t}) + \frac{\partial}{\partial x^2}(\frac{\partial L}{\partial u_{xx}}) + \frac{\delta G}{\delta u} = 0 \tag{30}$$

where $\delta G / \delta u$ is called the Frechet's variational derivative [18-31] of G. By using Eq. (28), Eq. (30) can be rewritten, as follows:

$$\Pi_t - \mu\Pi_{xxt} + (\alpha + \beta u)\Pi_x + \gamma\Pi_{xx} + \frac{\delta G}{\delta u} = 0 \tag{31}$$

It is hoped to find such a G, so that Eq. (31) turns out to be the field Eq. (26). Accordingly, after substituting Eq. (26) into Eq. (31), we get:

$$\frac{\delta G}{\delta u} = -\frac{\beta}{2}u^2 - 2\gamma u_x + 2\mu\gamma u_{xxx} - \mu\beta(u_x)^2 \tag{32}$$

From Eq. (32), we cannot identify G successfully by the variational calculus, because of existing the terms $2\gamma u_x$ u_x and $2\mu\gamma u_{xxx}$. So, we have to modify the trial-Lagrange function L in the following new form [30],

$$L = A(u - \mu u_{xx})\Pi_t + B\Pi_x\Pi_t + (\alpha u + \frac{\beta}{2}u^2 - \gamma u_x)\Pi_x + G \tag{33}$$

where A, B are two free parameters to be setup. The variational derivatives of L with respect to Π and u are:

$$\frac{\delta L}{\delta \Pi}: \quad -(A+2B)u_t + \mu(A+2B)u_{xxt} - (\alpha u_x + \beta uu_x - \gamma u_{xx}) + \frac{\delta G}{\delta \Pi} = 0 \tag{34}$$

$$\frac{\delta L}{\delta u}: \quad A\Pi_t - \mu A\Pi_{xxt} + (\alpha + \beta u)\Pi_x + \gamma\Pi_{xx} + \frac{\delta G}{\delta u} = 0 \tag{35}$$

Because $\delta G / \delta \Pi = 0$, Eq. (34) can be rearranged as:

$$(A+2B)(u-\mu u_{xx})_t + (\alpha u_x + \beta u u_x - \gamma u_{xx}) = 0 \tag{36}$$

The equation (36) should be equal to Eq. (24), so we can setup:

$$A + 2B = 1 \tag{37}$$

After substituting Eq. (26) into (35), we get:

$$\begin{aligned}&(1-A)\alpha u + (A+1)\gamma u_x - (A+1)\mu\gamma u_{xxx} + (1-\frac{A}{2})\beta u^2 + \\ &(A-1)\alpha\mu u_{xx} + (A-1)\mu\beta u u_{xx} + \mu\beta A u_x^2 + \frac{\delta G}{\delta u} = 0\end{aligned} \tag{38}$$

In order to determine the unknown function G successfully through the variational calculuses, both the coefficient of terms u_x and u_{xxx} must be set to zero in Eq. (38)

$$A + 1 = 0 \tag{39}$$

By combining Eq. (37) and Eq. (39), it is easily obtain that $A = -1$ and $B = 1$, respectively. Furthermore, Eq. (38) becomes

$$\frac{\delta G}{\delta u} = -2\alpha u - \frac{3}{2}\beta u^2 + 2\alpha\mu u_{xx} + 2\mu\beta u u_{xx} + \mu\beta u_x^2 \tag{40}$$

It is not hard to verify that $\delta(u u_x^2)/\delta u = -u_x^2 - 2u u_{xx}$, so G can be identified from Eq. (40) as

$$G = -\alpha u^2 - \frac{\beta}{2}u^3 - \mu\beta u u_x^2 - \alpha\mu u_x^2 \tag{41}$$

or

$$G = -\alpha u^2 - \frac{\beta}{2}u^3 - \mu\beta u u_x^2 + \alpha\mu u u_{xx} \tag{42}$$

At last, we obtain the following variational principles for the Burgers-BBM equation with some variable coefficients, which read:

$$J(u,\Pi) = \iint \{\Pi_x\Pi_t - (u-\mu u_{xx})\Pi_t + [\alpha(t)u + \frac{\beta(t)}{2}u^2 - \gamma(t)u_x]\Pi_x - \alpha(t)u^2 - \frac{\beta(t)}{2}u^3 - \mu\beta(t)u u_x^2 + \mu\alpha(t)u u_{xx}\}dxdt \tag{43}$$

and

$$J(u,\Pi) = \iint \{\Pi_x\Pi_t - (u-\mu u_{xx})\Pi_t + [\alpha(t)u + \frac{\beta(t)}{2}u^2 - \gamma(t)u_x]\Pi_x - \alpha(t)u^2 - \frac{\beta(t)}{2}u^3 - \mu\beta(t)u u_x^2 - \mu\alpha(t)u_x^2\}dxdt \tag{44}$$

both of which are subject to the constraints of $\Pi_x = u - \mu u_{xx}$. The established variational principles by the semi-inverse method [26-34] provide conservation laws and may find lots of applications in numerical simulation and scientific analysis of Eq. (24). In the following, we will prove the obtained variational principles correct. By making anyone of two functionals, Eq. (43) and (44), stationary with respect to two independent functions u and Π severally, two Euler-Lagrange equations can be obtained as:

$$\delta\Pi:\ \ (u-\mu u_{xx})_t - (\alpha u + \frac{\beta}{2}u^2 - \gamma u_x)_x - 2\Pi_{xt} = 0 \tag{45}$$

$$\delta u:\ \ -(\Pi_t + \alpha u + \frac{\beta}{2}u^2 - \gamma u_x) + \mu(\Pi_t + \alpha u + \frac{\beta}{2}u^2 - \gamma u_x)_{xx} = 0 \tag{46}$$

in which $\delta\Pi$ and δu is the first-order variation of Π and u. Substituting $\Pi_x = u - \mu u_{xx}$ into Equation (45) leads to the original compound Burgers-BBM equation, obviously. Because Eq. (46) must be established in all definition domains, we conclude that $\Pi_t + \alpha u + \beta u^2/2 - \gamma u_x = 0$, which is identical to the second one in Eq. (26). Hence, successfully, we proved the obtained variational principles of the compound Burgers-BBM equation with variable coefficients correct.

4. Conclusion

In the second and third parts, different variational principles have been successfully constructed for the compound KdV-Burgers equation and Burgers-BBM equation with variable coefficients, respectively, by the semi-inverse method [26-34] and designing skillfully trial-Lagrange functionals. Subsequently, the obtained variational principles have proved correct by minimizing the corresponding functionals. From the results of analysis, it is concluded that the variational principle for the nonlinear PDEs studied in this paper have two different integral formulations, from which the same control equations can be derived. The procedure also reveals that the semi-inverse method [26-34] is effective and powerful. According to the obtained variational principles, on one hand, we can study possible solution structures for solitary waves. On the other hand, they also provide hints for numerical algorithms, the Equations (1) and (24) can be solved numerically by the variational-based methods. In the analytical analysis and numerical simulations, it is of great importance to choose an appropriate variational principle according to practical needs. Our work in the future will focus on the dynamics of soliton in the compound KdV-Burgers equation and Burgers-BBM equation, by the variational approximation method using the established variational principles in this paper.

Author Contributions

All authors have important contributions in this paper. The details are as following. Conceptualization, Xiao-qun Cao; Methodology, Xiao-qun Cao, Ke-Cheng Peng and Meng-Zhu Liu; Writing – original draft, Xiao-qun Cao; Writing – review & editing, Ke-Cheng Peng, Ya-Nan Guo and Cheng-Zhuo Zhang. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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