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Research Paper

Meshfree Collocation Method for the Numerical Solution of Higher Order KdV Equation

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Abstract. In this paper, an efficient meshfree collocation scheme based on meshfree radial basis function is implemented for the numerical solution of 7th-order Korteweg-de Vires (KdV) equations. The demand of meshless techniques increment because of its meshless nature and simplicity of usage in higher dimensions. The proposed numerical scheme is tested on several test problems. The efficiency and accuracy of the suggested scheme is analyzed via $||L||_\infty$ and $||L||_2$ error norms.

Keywords: Meshfree method, Radial basis functions, 7th-order KdV equation, 7th-order Lax equation.

1. Introduction

The general 7th-order Kortweg-de Vires (KdV) equation [1, 2, 3] is written as:

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2u_{3x} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0\,, \tag{1}$$

where a, b, c, d, e, f and g are non-zero constants and $u_{nx} = \partial^n u / \partial x^n$. If a = 140, b = 70, c = 280, d = 70, e = 70, f = 42 and g = 14 then eq. (1):

$$u_t + 140u^3u_x + 70u_x^3 + 280uu_xu_{2x} + 70u^2u_{3x} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0\,. \tag{2}$$

is called Lax 7th-order KdV equation.
KdV equations have broad utilization for example in fission, cluster physics, super-deformed nuclei, thin films, radar rheology [4, 5] and fiber optics [6]. In plasma physics, such nonlinear equations give rise to the ion-acoustic soliton [7]. These kind of equations have been described as a shallow water wave in deep oceans and seas in fluid dynamics [8, 9]. However KdV equation physical conditions tend to be highly idealized due to the selection of the constant co-efficients. So for 7th oreder KdV equation has been discussed by some authors for its analytical solution. For example Ganji et. al. used Exp-Function method to find its analytical solution [10, 11]. While Cole-Hopf transformation method has been applied by Salas and Gomez [12, 13] and Soliman [14] applied Variational-Iteration method to investigate the exact solution. Darvishi, M. T., et al. has studied the approximate results of Lax 7th-order KdV equation by pseudospectral method [15]. Kashkari, B. S., Bakodah, H. O., applied Laplace decomposition method to discuss the approximate results of 7th-oreder KdV equation [3]. Alzaid, N. A., Alrayiqi, B. A., [16] discussed the approximate solution of 7th-order KdV equation by decomposition method while Joshi, N., Lustri, C. J., [17] used finite difference method. In the last few years VIM and its modification have been used accurately in solving complex PDE models (see [18, 19]).

In recent times the meshfree methods have got high attention from the researchers. Because these methods are very simple, easy to implement and highly accurate. One of the well known technique is the meshfree radial basis functions method. The method can be implemented on global interpolation and local interpolation. However global interpolation creates problems when the matrix turns into ill-conditioned form but local interpolation avoids these limitations and gives accurate and stable approximate solution [20-26].There are some authors who implemented RBF technique to some various problems. For example Ali, A., Islam, S., Haq, S., [27] tested RBFs on coupled burgers' equation, Mohyud-Din, S. T., Nagahdary, E., Usman, M., [28] on generalized fifth-order KdV equation, Dehaghan, M., Nikpour, A., [29] applied local RBFs based on differential quadrature on system of second-order boudary value problems. Dereli, Y., Dag, I., [30] discussed meshfree technique for the numerical results of Kawahara type equations. Recently Soleymani, F., Zaka-ullah, M., used multiquadric FBF-FD scheme for simulating the financial HHW equation utilizing exponential integrator [31]. Khan, M. N., Ahmad, I., Ahmad, H., used RBF collocation method for space-dependent inverse heat problems [32]. Hussain, M., Haq, S., used radial basis functions method to solve fractional advection diffusion models [33] and Shivanian, E., discussed numerical solution of 2D fractional time convection diffusion reaction equation via local meshless RBF method [34]. Srivastava, H. M., et.al. discussed numerical simulation of 3D fractional order convection diffusion PDEs by local meshless method [35]. Assari, P., Dehghan, M., used meshless RBFs method for the numerical solution of nonlinear weakly integral equations [36]. Golbabai, A., Nikam, O., Tousi, J. R., [37] discussed RBFs for solving nonlinear integral equations and Assari, P., [38] used RBFs to discuss the numerical solution of Fredholm-Hammerstein integral equations. In this article new meshfree numerical scheme has been developed for seventh-order KdV equation. Using local meshless radial basis function, the numerical results of seventh order Lax equation have been discussed. The Gaussian radial basis functions has been implemented. In section 4 Stability of the numerical scheme has been discussed while the numerical results of the scheme have been shown in section 5.

2. Numerical Scheme for KdV7 Equation

Consider eq. (1):

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{2x} + du^2u_{3x} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0\,,$$

where a, b, c, d, e, f and g are not all zero and $u_{nx} = \frac{\partial^n u}{\partial x^n}$, $x \in D = [\alpha,\beta] \subset \Re$, $t \ge 0$, with initial value and boundary value conditions

$$u(x,0) = u^{(0)}, \tag{3}$$

and

$$\begin{cases} u(\alpha,t) = \nu_1(t) \\ u(\beta,t) = \nu_2(t) \qquad t \ge 0, \end{cases} \tag{4}$$

here $u^{(0)}$, $\nu_1(t)$, $\nu_2(t)$ are known. In order to discretize the time derivatives of 7th-order KdV equation by applying finite difference method and ϑ–weighted $(0 \le \vartheta \le 1)$ scheme to δx (space step size) at consective time levels n and $n + 1$ as

$$\begin{aligned} &\frac{u^{n+1}-u^n}{\delta t} + \vartheta\Big\{a\left(u^3\right)^{n+1}u_x^{n+1} + b\left(u^3\right)^{n+1} + cu^{n+1}u_x^{n+1}u_{2x}^{n+1} + d\left(u^2\right)^{n+1}u_{3x}^{n+1} + eu_{2x}^{n+1}u_{3x}^{n+1} + fu_x^{n+1}u_{4x}^{n+1} \\ &+gu^{n+1}u_{5x}^{n+1} + u_{7x}^{n+1}\Big\} + (1-\vartheta)\Big\{a\left(u^3\right)^n u_x^n + b\left(u^3\right)^n + cu^nu_x^nu_{2x}^n + d\left(u^2\right)^n u_{3x}^n + eu_{2x}^nu_{3x}^n + fu_x^nu_{4x}^n \\ &+gu^nu_{5x}^n + u_{7x}^n\Big\} = 0, \end{aligned} \tag{5}$$

In eq. (5) the nonlinear terms can be estimated by linearization process in the following manner

$$\begin{cases} \left(u^3\right)^{n+1}u_x^{n+1} \approx \left(u^3\right)^n u_x^{n+1} + 3\left(u^2\right)^n u_x^n u^{n+1} - 3\left(u^3\right)^n u_x^n, \\ \left(u^3\right)^{n+1} \approx 3\left(u^2\right)^n u^{n+1} - 2\left(u^3\right)^n, \\ u^{n+1}u_x^{n+1}u_{2x}^{n+1} \approx u^{n+1}u_x^nu_{2x}^n + u^nu_x^{n+1}u_{2x}^n + u^nu_x^nu_{2x}^{n+1} - 2u^nu_x^nu_{2x}^n \\ \left(u^2\right)^{n+1}u_{3x}^{n+1} \approx \left(u^2\right)^n u_{3x}^{n+1} + 2u^nu_{3x}^nu^{n+1} - 2\left(u^2\right)^n u_{3x}^n \\ u_{2x}^{n+1}u_{3x}^{n+1} \approx u_{2x}^{n+1}u_{3x}^n + u_{2x}^nu_{3x}^{n+1} - u_{2x}^nu_{3x}^n \\ u_x^{n+1}u_{4x}^{n+1} \approx u_x^{n+1}u_{4x}^n + u_x^nu_{4x}^{n+1} - u_x^nu_{4x}^n \\ u^{n+1}u_{5x}^{n+1} \approx u^{n+1}u_{5x}^n + u^nu_{5x}^{n+1} - u^nu_{5x}^n \end{cases} \tag{6}$$

Setting up eq. (6) in eq. (5) and re-arrainging the terms, the result follows

$$\begin{aligned} &u^{n+1} + \vartheta\delta t\Big\{a\Big[\left(u^3\right)^n u_x^{n+1} + 3\left(u^2\right)^n u_x^nu^{n+1}\Big] + b\Big[3\left(u^2\right)^n u^{n+1}\Big] + c\left[u^{n+1}u_x^nu_{2x}^n + u^nu_x^{n+1}u_{2x}^n + u^nu_x^nu_{2x}^{n+1}\right] \\ &+d\Big[\left(u^2\right)^n u_{3x}^{n+1} + 2u^nu_{3x}^nu^{n+1}\Big] + e\left[u_{2x}^{n+1}u_{3x}^n + u_{2x}^nu_{3x}^{n+1}\right] + f\left[u_x^{n+1}u_{4x}^n + u_x^nu_{4x}^{n+1}\right] + g\left[u^{n+1}u_{5x}^n + u^nu_{5x}^{n+1}\right] + \\ &u_{7x}^{n+1}\Big\} = u^n - (1-\vartheta)\delta t\Big\{a\left(u^3\right)^n u_x^n + b\left(u^3\right)^n + cu^nu_x^nu_{2x}^n + d\left(u^2\right)^n u_{3x}^n + eu_{2x}^nu_{3x}^n + fu_x^nu_{4x}^n + gu^nu_{5x}^n + \\ &u_{7x}^n\Big\} + \vartheta\delta t\Big\{3a\left(u^3\right)^n u_x^n + 2b\left(u^3\right)^n + 2cu^nu_x^nu_{2x}^n + 2d\left(u^2\right)^n u_{3x}^n + eu_{2x}^nu_{3x}^n + fu_x^nu_{4x}^n + gu^nu_{5x}^n\Big\}. \end{aligned} \tag{7}$$

Now consider the collocation nodes $\{x_i \in D = [\alpha,\beta], 1 \le i \le N\}$ where $x_1 = \alpha$ and $x_N = \beta$. Then solution of eq. (7) can be calculated by

$$u^n(x) = \sum_{j=1}^{N} \lambda_j^n \psi\left(r_j\right), \tag{8}$$

Here ψ denotes RBF and $r_j(x) = \| x - x_j \|$ is the euclidean norm between x and x_j (are the centers of r). To calculate λ_j (unknown parameters) in eq. (8), therefore for each collocation node x_i eq. (8) can be written as

$$u^n(x_i) = \sum_{j=1}^{N} \lambda_j^n \psi\left(r_{ij}\right), \tag{9}$$

where $r_{ij} = \| x_i - x_j \|$. In eq. (9)

$$\psi(r_{ij}) = \begin{bmatrix} \psi(r_{11}) & \psi(r_{12}) & \cdots & \psi(r_{1N}) \\ \psi(r_{21}) & \psi(r_{22}) & \cdots & \psi(r_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi(r_{N1}) & \psi(r_{N2}) & \cdots & \psi(r_{NN}) \end{bmatrix}, \tag{10}$$

Let $A = \psi(r_{ij})$ and $\lambda^n = [\lambda_1^n, \lambda_2^n, \cdots\cdots, \lambda_N^n]^T$. Then eqs. (9) and (10) can be written as

$$u^n = A\lambda^n, \tag{11}$$

in eq. (10) matrix A order $N \times N$ has been separated into A_d and A_b corresponding to $N - 2$ interior and two boundary nodes in the subsequent arrangement. Then $A = A_d + A_b$, where $A_d = \{\psi(r_{ij}) : 1 \le j \le N, 2 \le i \le N-1 \text{ and } 0 \text{ elsewhere}\}$ and $A_b = \{\psi(r_{ij}) : 1 \le j \le N, i = 1, N \text{ and } 0 \text{ elsewhere}\}$.

The type of RBF has been used to solve eq. (7) is called Gaussain RBF which is

$$\text{GA} - \text{RBF} = \exp(-\eta r_j),$$

here η is the shape parameter of RBF which has a very important role to sort out the best numerical results. The value of η can be placed according to the best possible results in different cases as this can be seen in [39] that $\eta_{opt} = \arg\min_\eta \| e(\eta) \|$, where e is the error vecrtor between approximate solution and exact solution. It is also important to note that in Gaussian [40] $\eta = 1/2\sigma^2$ where σ is the varience between x and x_j. Now setting eq. (9) in eq. (7), then

$$\begin{aligned}
&\sum_{j=1}^{N} \lambda_j^{n+1}\psi(r_{ij}) + \vartheta\delta t\Bigg[a\Bigg\{\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^3 \sum_{j=1}^{N}\lambda_j^{n+1}\psi_x(r_{ij}) + 3\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^2 \sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij}) \\
&\sum_{j=1}^{N}\lambda_j^{n+1}\psi(r_{ij})\Bigg\} + 3b\left\{\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^2 \sum_{j=1}^{N}\lambda_j^{n+1}\psi(r_{ij})\right\} + c\Bigg\{\sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{2x}(r_{ij}) \\
&\sum_{j=1}^{N}\lambda_j^{n+1}\psi(r_{ij}) + \sum_{j=1}^{N}\lambda_j^{n+1}\psi_x(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{2x}(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij}) + \sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij})\sum_{j=1}^{N}\lambda_j^{n+1}\psi_{2x}(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\Bigg\} \\
&d\left\{\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^2 \sum_{j=1}^{N}\lambda_j^{n+1}\psi_{3x}(r_{ij}) + 2\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{3x}(r_{ij})\sum_{j=1}^{N}\lambda_j^{n+1}\psi(r_{ij})\right\} + e\Bigg\{\sum_{j=1}^{N}\lambda_j^{n+1}\psi_{2x}(r_{ij}) \\
&\sum_{j=1}^{N}\lambda_j^n\psi_{3x}(r_{ij}) + \sum_{j=1}^{N}\lambda_j^n\psi_{2x}(r_{ij})\sum_{j=1}^{N}\lambda_j^{n+1}\psi_{3x}(r_{ij})\Bigg\} + f\Bigg\{\sum_{j=1}^{N}\lambda_j^{n+1}\psi_x(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{4x}(r_{ij}) + \sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij}) \\
&\sum_{j=1}^{N}\lambda_j^{n+1}\psi_{4x}(r_{ij})\Bigg\} + g\left\{\sum_{j=1}^{N}\lambda_j^{n+1}\psi(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{5x}(r_{ij}) + \sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\sum_{j=1}^{N}\lambda_j^{n+1}\psi_{5x}(r_{ij})\right\} + \sum_{j=1}^{N}\lambda_j^{n+1}\psi_{7x}(r_{ij})\Bigg] \\
&= \sum_{j=1}^{N}\lambda_j^n\psi(r_{ij}) - \delta t(1-\vartheta)\Bigg[a\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^3 \sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij}) + b\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^3 + c\sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij}) \\
&\sum_{j=1}^{N}\lambda_j^n\psi_{2x}(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij}) + d\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^2 \sum_{j=1}^{N}\lambda_j^n\psi_{3x}(r_{ij}) + e\sum_{j=1}^{N}\lambda_j^n\psi_{2x}(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{3x}(r_{ij}) \\
&+ f\sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{4x}(r_{ij}) + g\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{5x}(r_{ij}) + \sum_{j=1}^{N}\lambda_j^n\psi_{7x}(r_{ij})\Bigg] + \\
&\delta t\vartheta\Bigg[3a\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^3 \sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij}) + 2b\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^3 + 2c\sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij}) \\
&\sum_{j=1}^{N}\lambda_j^n\psi_{2x}(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij}) + 2d\left(\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\right)^2 \sum_{j=1}^{N}\lambda_j^n\psi_{3x}(r_{ij}) + e\sum_{j=1}^{N}\lambda_j^n\psi_{2x}(r_{ij}) \\
&\sum_{j=1}^{N}\lambda_j^n\psi_{3x}(r_{ij}) + f\sum_{j=1}^{N}\lambda_j^n\psi_x(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{4x}(r_{ij}) + g\sum_{j=1}^{N}\lambda_j^n\psi(r_{ij})\sum_{j=1}^{N}\lambda_j^n\psi_{5x}(r_{ij}),
\end{aligned} \tag{12}$$

where $\psi_{kx}(r_j) = \psi(\frac{d_k}{dx^k} r_j) = \psi(\frac{d_k}{dx^k} \| x - x_j \|)$, $1 \le k \le 7$. Also from eqs. (4), (9) and (10)

$$\begin{cases} \sum_{j=1}^{N}\lambda_j^{n+1}\psi(r_{ij}) = \upsilon_m^{n+1}(t), \quad \text{where } m = 1,2 \text{ for } i = 1, N, \\ A = \left\{\psi(r_{ij}) : \ 1 \le j \le N, 1 \le i \le N\right\}, \\ A_{kx} = \left\{\psi_{kx}(r_{ij}) : 1 \le j \le N, 2 \le i \le N-1 \text{ and } 0 \text{ elsewhere}\right\}. \end{cases} \tag{13}$$

Now inserting eq. (13) in eq. (12) we get

$$
\begin{aligned}
&A\lambda_j^{n+1} + \delta t\vartheta\Big[a\Big\{\big(A\lambda_j^n\big)^3 A_x\lambda_j^{n+1} + 3\big(A\lambda_j^n\big)^2 A_x\lambda_j^n A\lambda_j^{n+1}\Big\} + 3b\Big\{\big(A\lambda_j^n\big)^2 A\lambda_j^{n+1}\Big\} + \\
&c\Big\{A_x\lambda_j^n A_{2x}\lambda_j^n A\lambda_j^{n+1} + A_x\lambda_j^{n+1}A_{2x}\lambda_j^n A\lambda_j^n + A_x\lambda_j^n A_{2x}\lambda_j^{n+1}A\lambda_j^n\Big\} + d\Big\{\big(A\lambda_j^n\big)^2 A_{3x}\lambda_j^{n+1} \\
&+2A\lambda_j^n A_{3x}\lambda_j^n A\lambda_j^{n+1}\Big\} + e\Big\{A_{2x}\lambda_j^{n+1}A_{3x}\lambda_j^n + A_{2x}\lambda_j^n A_{3x}\lambda_j^{n+1}\Big\} + f\Big\{A_x\lambda_j^{n+1}A_{4x}\lambda_j^n + \\
&A_x\lambda_j^n A_{4x}\lambda_j^{n+1}\Big\} + g\Big\{A\lambda_j^{n+1}A_{5x}\lambda_j^n + A\lambda_j^n A_{5x}\lambda_j^{n+1}\Big\} + A_{7x}\lambda_j^{n+1}\Big] \\
&= A\lambda_j^n - \delta t(1-\vartheta)\Big[a\big(A\lambda_j^n\big)^3 A_x\lambda_j^n + b\big(A\lambda_j^n\big)^3 + cA_x\lambda_j^n A_{2x}\lambda_j^n A\lambda_j^n + d\big(A\lambda_j^n\big)^2 A_{3x}\lambda_j^n \\
&eA_{2x}\lambda_j^n A_{3x}\lambda_j^n + fA_x\lambda_j^n A_{4x}\lambda_j^n + gA\lambda_j^n A_{5x}\lambda_j^n + A_{7x}\lambda_j^n\Big] + \delta t\vartheta\Big[3a\big(A\lambda_j^n\big)^3 A_x\lambda_j^n + \\
&2b\big(A\lambda_j^n\big)^3 + 2cA_x\lambda_j^n A_{2x}\lambda_j^n A\lambda_j^n + 2d\big(A\lambda_j^n\big)^2 A_{3x}\lambda_j^n + eA_{2x}\lambda_j^n A_{3x}\lambda_j^n + fA_x\lambda_j^n A_{4x}\lambda_j^n + gA\lambda_j^n A_{5x}\lambda_j^n\Big],
\end{aligned}
\tag{14}
$$

Since $u^n = A\lambda^n$ then $u_{kx}^n = A_{kx}\lambda^n$ where $1 \le k \le 7$. Let

$$
\begin{aligned}
&M_1 = \big(u^n\big)^3 * A_x + 3\big(u^n\big)^2 * u_x^n * A, \qquad M_2 = 3\big(u^n\big)^2 * A, \\
&M_3 = u_x^n * u_{2x}^n * A + u^n * u_{2x}^n * A_x + u^n * u_x^n * A_{2x}, \\
&M_4 = \big(u^n\big)^2 * A_{3x} + 2u^n * u_{3x}^n * A, \qquad M_5 = u_{2x}^n * A_{3x} + u_{3x}^n * A_{2x}, \\
&M_6 = u_x^n * A_{4x} + u_{4x}^n * A_x, \qquad M_7 = u^n * A_{5x} + u_{5x}^n * A, \\
&N_1 = a\big(u^n\big)^3 * A_x + b\big(u^n\big)^2 * A + cu^n * u_x^n * A_{2x} + d\big(u^n\big)^2 * A_{3x} + eu_x^n * A_{3x} + fu_x^n * A_{4x} + gu^n * A_{5x} + A_{7x}, \\
&N_2 = 3a\big(u^n\big)^3 * A_x + 2b\big(u^n\big)^2 * A + 2cu^n * u_x^n * A_{2x} + 2d\big(u^n\big)^2 * A_{3x} + eu_x^n * A_{3x} + fu_x^n * A_{4x} + gu^n * A_{5x},
\end{aligned}
$$

where " $*$ " show the product of j–component of coloum Vector to j–row of the Matrix. Again let

$$
\begin{aligned}
&M = A + \delta t\vartheta\big[aM_1 + bM_2 + cM_3 + dM_4 + eM_5 + fM_6 + gM_7 + A_{7x}\big], \\
&N = A - \delta t(1-\vartheta)N_1 + \delta t\vartheta N_2 \quad \text{and} \quad \Upsilon^{n+1} = \big[\upsilon_1(t^{n+1}), 0, \cdots\cdots, 0, \upsilon_2(t^{n+1})\big].
\end{aligned}
$$

Inserting these values in eq. (14) we get

$$M\lambda^{n+1} = N\lambda^n + \Upsilon^{n+1}, \tag{15}$$

$$\lambda^{n+1} = M^{-1}N\lambda^n + M^{-1}\Upsilon^{n+1}, \tag{16}$$

using eq. (11) we get

$$u^{n+1} = AM^{-1}NA^{-1}u^n + AM^{-1}\Upsilon^{n+1}. \tag{17}$$

3. Proposed Algorithm

1. Select time step δt and collocation nodes N from $D = [\alpha, \beta]$.
2. Choose the value of θ where ($0 \le \theta \le 1$).
3. From eq. (3) calculate the initial value of $u^{(0)}$ and then calculate $\lambda^n = A^{-1}u^n$ from eq. (11).
4. The parameters λ^{n+1} has to be found out from eq. (16).
5. The numerical solution u^{n+1} at the succeeding time level is calculated from step (4) and eq. (11).

4. Stability

In this part stability for eq. (16) has been put forward by using Matrix system. The nonlinear $u^m\left(\frac{\partial^n u}{\partial x^n}\right)^p$ where m,p are constants might be linearized. The error e^n at the n–th time level is given by

$$e^n = u_{exact}^n - u_{app}^n,$$

where u_{exact}^n and u_{app}^n are the respective analytical and numerical results at the n–th time level. The equation of error for linearized KdV7 may be as follows

$$Me^{n+1} = Ne^{n}, \tag{18}$$

here $M=A_d + A_b + \delta tS$ where $S=\vartheta[aM_1 + bM_2 + cM_3 + dM_4 + eM_5 + fM_6 + gM_7 + A_{7x}]$ and $N=A+\delta tT$ where $T=\vartheta N_2 - (1-\vartheta)N_1$. Now eq. (18) can be written as

$$e^{n+1} = Re^{n}, \tag{19}$$

here $R = M^{-1}N$ or $R=[A+\delta tT]^{-1}[A+\delta tS]$ or $R=[I+\delta tP]^{-1}[A+\delta tQ]$ where $P = A^{-1}T$ and $Q = A^{-1}S$.

The numerical scheme is stable if $\| R \|_2 \leq 1$, which is similar to $\rho(R) \leq 1$, where $\rho(R)$ shows the spectral radius of the matrix R. The stability of eq. (18) is ensured if the eigenvalues $M^{-1}N$ fulfill the following condition

$$\left|\frac{1+\delta t\lambda_Q}{1+\delta t\lambda_P}\right| \leq 1, \tag{20}$$

where λ_P and λ_Q are the eigenvalues of matrix P, Q respctively. It is clear from eq. (20) that the scheme is unconditionally stable if $\lambda_P \geq \lambda_Q$.

5. Numerical Results

In this section the numerical results of Lax KdV7 equation (eq. 2) have been discussed. The results have been found by gaussian radial basis functions method (GA-RBF). The initial and boundary value conditions for eq. (2) are

$$u(x,0) = 2k^2 \operatorname{sech}(kx) = u^0(x), \tag{21}$$

and

$$\begin{cases} u(\alpha,t) = 2k^2 \operatorname{sech}\left(k\left(\alpha - 64k^6 t\right)\right) = \upsilon_1(t), \\ u(\beta,t) = 2k^2 \operatorname{sech}\left(k\left(\beta - 64k^6 t\right)\right) = \upsilon_2(t). \end{cases} \tag{22}$$

The iterative numerical results have been computed from eq. (17) which are given in Table (1, 2 and 3). The errors between analytical solution and numerical solution has been computed by

$$\|L\|_\infty = \max_{1\leq j\leq N}\left\|u^n_{exact}(j) - u^n_{app}(j)\right\| \quad \text{and} \quad \|L\|_2 = \sqrt{\delta x \sum_{j=1}^{N}\left(u^n_{exact}(j) - u^n_{app}(j)\right)^2}. \tag{23}$$

The computed errors have been given in table 1 and table 2 according to different values of $\delta x$and collocation points N while in table 3, the computed errors have been given against different time intervals.

Table 1. Computed errors between u^n_{exact} and u^n_{app} at different δx where $-400 \leq x \leq 400$, $\delta t = 0.1$, $k = 0.01$, $\theta = 0.5$

		at t = 1		at t = 5	
N	η	L_∞–error	L_2–error	L_∞–error	L_2–error
80	0.45	$9.4976e^{-17}$	$1.2782e^{-15}$	$4.7480e^{-16}$	$6.3906e^{-15}$
100	0.7	$8.3890e^{-17}$	$1.1239e^{-15}$	$4.1918e^{-16}$	$5.6190e^{-15}$
160	1.9	$7.3075e^{-17}$	$9.7965e^{-16}$	$3.6524e^{-16}$	$4.8979e^{-15}$
400	13	$3.5074e^{-17}$	$4.6835e^{-16}$	$1.7551e^{-16}$	$2.3423e^{-15}$
800	58	$5.2042e^{-17}$	$6.9758e^{-16}$	$2.6007e^{-16}$	$3.4874e^{-15}$
1600	147	$4.1959e^{-17}$	$5.6208e^{-16}$	$2.0958e^{-16}$	$2.8100e^{-15}$

Table 2. Computed errors between u^n_{exact} and u^n_{app} at different δx where $-400 \leq x \leq 400$, $\delta t = 0.1$, $k = 0.001$, $\theta = 0.5$

		at t = 1		at t = 5	
N	η	L_∞–error	L_2–error	L_∞–error	L_2–error
80	0.48	$2.1176e^{-21}$	$1.3393e^{-20}$	$1.0588e^{-20}$	$6.6964e^{-20}$
100	0.75	$4.2352e^{-21}$	$3.1693e^{-20}$	$2.1176e^{-20}$	$1.5847e^{-19}$
160	1.97	$6.3527e^{-21}$	$9.5409e^{-20}$	$3.1764e^{-20}$	$4.7704e^{-19}$
400	13.95	$4.2021e^{-21}$	$8.5546e^{-20}$	$2.1191e^{-20}$	$4.2773e^{-19}$
800	60.5	$4.6587e^{-21}$	$7.7998e^{-20}$	$2.3293e^{-20}$	$3.8999e^{-19}$
1600	260	$5.2940e^{-21}$	$9.2540e^{-20}$	$2.6470e^{-20}$	$4.6270e^{-19}$

Table 3. Computed errors between u^n_{exact} and u^n_{app} at different δt where $-400 \le x \le 400$, N = 100, $\theta = 0.5$, t = 1

	$k = 0.01$		$k = 0.001$	
δt	L_∞–error	L_2–error	L_∞–error	L_2–error
0.5	$8.3782e^{-17}$	$1.1237e^{-15}$	$8.4703e^{-22}$	$1.7025e^{-20}$
0.1	$8.3890e^{-17}$	$1.1239e^{-15}$	$4.2352e^{-21}$	$3.1693e^{-20}$
0.05	$8.3890e^{-17}$	$1.1237e^{-15}$	$8.4703e^{-21}$	$5.8684e^{-20}$
0.02	$8.4324e^{-17}$	$1.1244e^{-15}$	$2.1176e^{-20}$	$1.4671e^{-19}$
0.01	$8.4974e^{-17}$	$1.1262e^{-15}$	$2.1176e^{-20}$	$1.1979e^{-19}$
0.001	$9.8527e^{-17}$	$1.1385e^{-15}$	$4.2352e^{-19}$	$4.7161e^{-18}$

Table 4. Convergence rate at space level at t=5, $\delta t = 0.1$

		$k = 0.001$		
N	L_∞–error	L_2–error	L_∞–error	L_2–error
80	$1.0588e^{-20}$	---	$6.6964e^{-20}$	---
100	$2.1176e^{-20}$	-3.1063	$1.5847e^{-19}$	1.9771
160	$3.1764e^{-20}$	-3.1736	$4.7704e^{-19}$	-2.3447
400	$2.1191e^{-20}$	1.6251	$4.2773e^{-19}$	0.4381
800	$2.3293e^{-20}$	-0.5020	$3.8999e^{-19}$	0.4903
1600	$2.6470e^{-20}$	-0.6786	$4.6270e^{-19}$	-0.9073
		$k = 0.01$		
N	L_∞–error	L_2–error	L_∞–error	L_2–error
80	$4.7480e^{-16}$	---	$6.3906e^{-15}$	---
100	$4.1918e^{-16}$	2.0540	$5.6190e^{-15}$	2.1214
160	$3.6524e^{-16}$	1.0782	$4.8979e^{-15}$	1.0750
400	$2.1191e^{-16}$	2.9423	$4.2773e^{-15}$	2.9617
800	$2.6007e^{-16}$	-2.0872	$3.4874e^{-15}$	-2.1125
1600	$2.0958e^{-16}$	1.1456	$2.8100e^{-15}$	1.1463

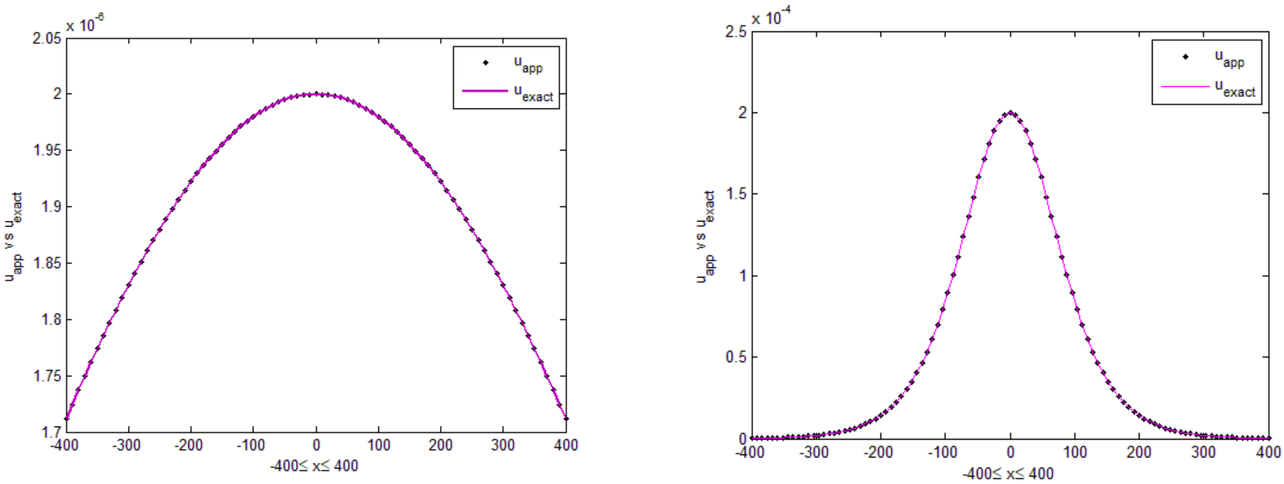

Fig. 1. Numerical u and Analytical u where k=0.001 (left) and k=0.01 (right)

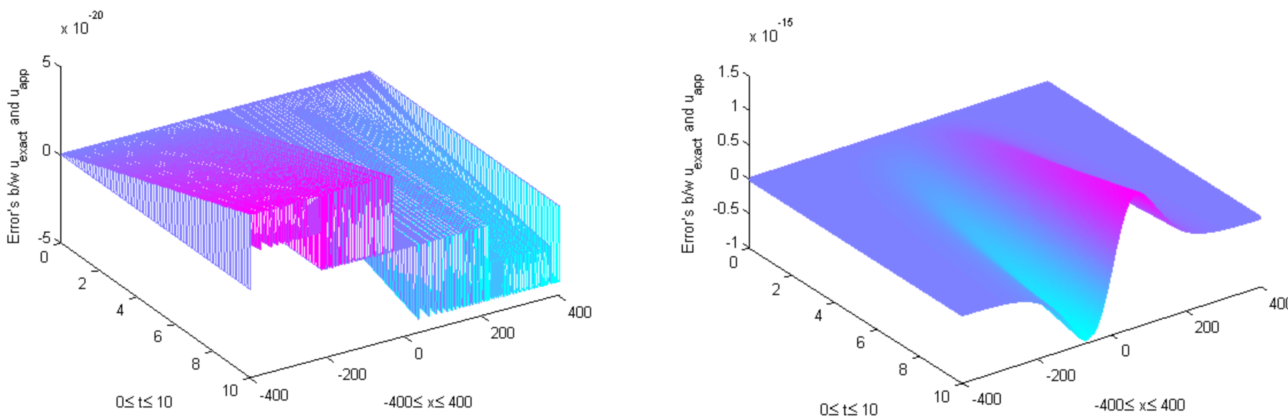

Fig. 2. Error's b/w u-exact and u-numerical where k=0.001 (left) and k=0.01 (right)

Table 5. Convergence rate w.r.t. time at t=5 and N=80

$k = 0.001$				
δt_i	L_∞–error	L_2–error	L_∞–error	L_2–error
1	4.2352e^{-21}	---	7.6819e^{-20}	---
0.5	4.2352e^{-21}	0.0000	8.5126e^{-20}	-0.5450
0.25	8.4703e^{-21}	-3.6787	1.0443e^{-19}	1.5844
0.125	1.6941e^{-20}	3.3234	1.3970e^{-19}	-1.5443
0.1	1.1176e^{-20}	-3.6786	1.5847e^{-19}	-2.0784
0.05	4.2352e^{-20}	-1.0000	2.9342e^{-19}	-3.2696
$k = 0.01$				
δt_i	L_∞–error	L_2–error	L_∞–error	L_2–error
1	4.1864e^{-16}	---	5.6184e^{-15}	---
0.5	4.1864e^{-16}	0.0000	5.6185e^{-15}	-0.0173
0.25	4.1864e^{-16}	0.0000	5.6189e^{-15}	-0.0188
0.125	4.1864e^{-16}	0.0000	5.6183e^{-15}	0.0282
0.1	4.1918e^{-16}	-0.2876	5.6190e^{-15}	-0.1023
0.05	4.1918e^{-16}	0.0000	5.6181e^{-15}	0.0423

The convergence rate w.r.t. space and time are calculated by the following formulae

$$\frac{\log_{10}\left(\left\|u_{exact}-u_{\delta x_i}\right\| / \left\|u_{exact}-u_{\delta x_{i+1}}\right\|\right)}{\log_{10}\left(\delta x_i/\delta x_{i+1}\right)} \quad \text{and} \quad \frac{\log_{10}\left(\left\|u_{exact}-u_{\delta t_i}\right\| / \left\|u_{exact}-u_{\delta t_{i+1}}\right\|\right)}{\log_{10}\left(\delta t_i/\delta t_{i+1}\right)}.$$

The results of convergence rate w.r.t space and time are shown in Table 4 and 5 respectively.

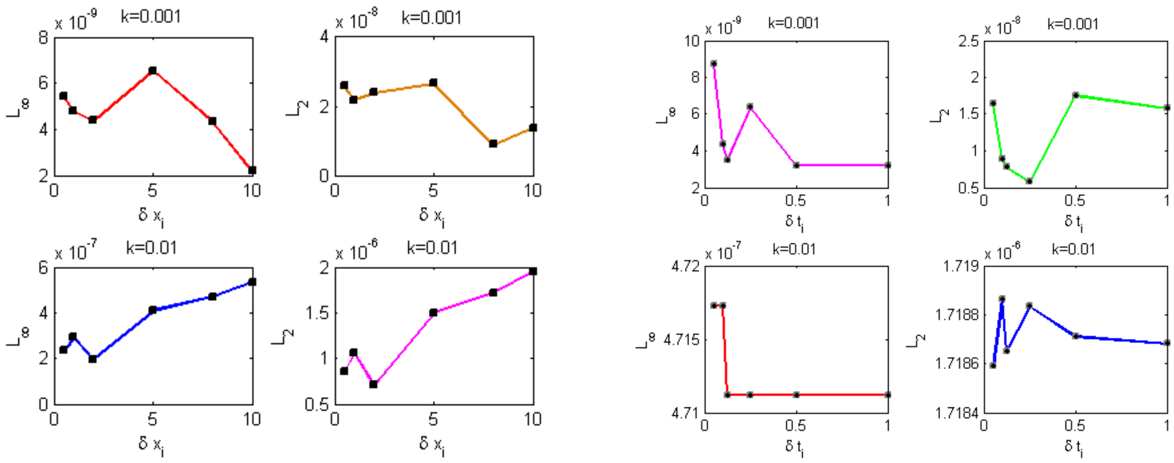

Fig. 3. Convergence rate w.r.t. space (left) and time (right)

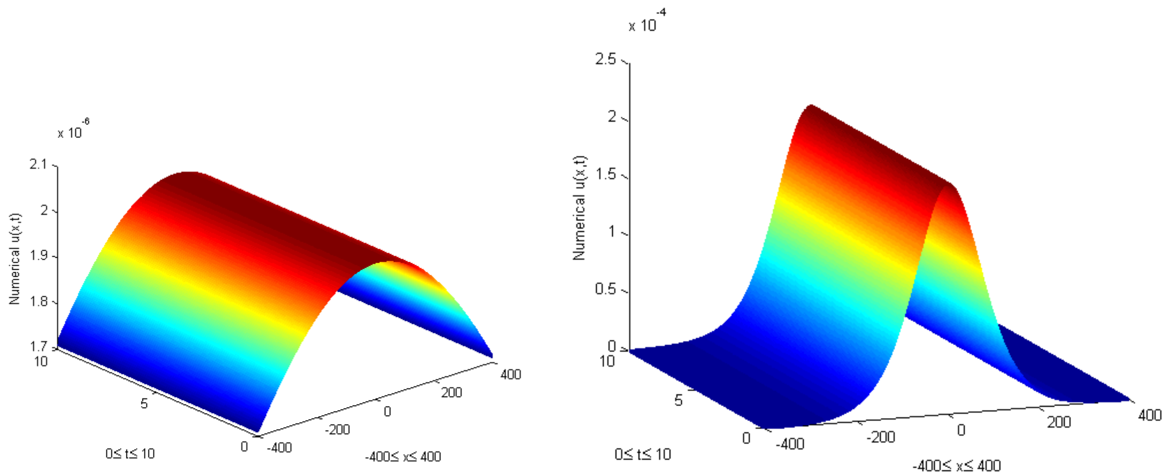

Fig. 4. 3D plot of Numerical u(x,t) k=0.001 (left) and k=0.01 (right)

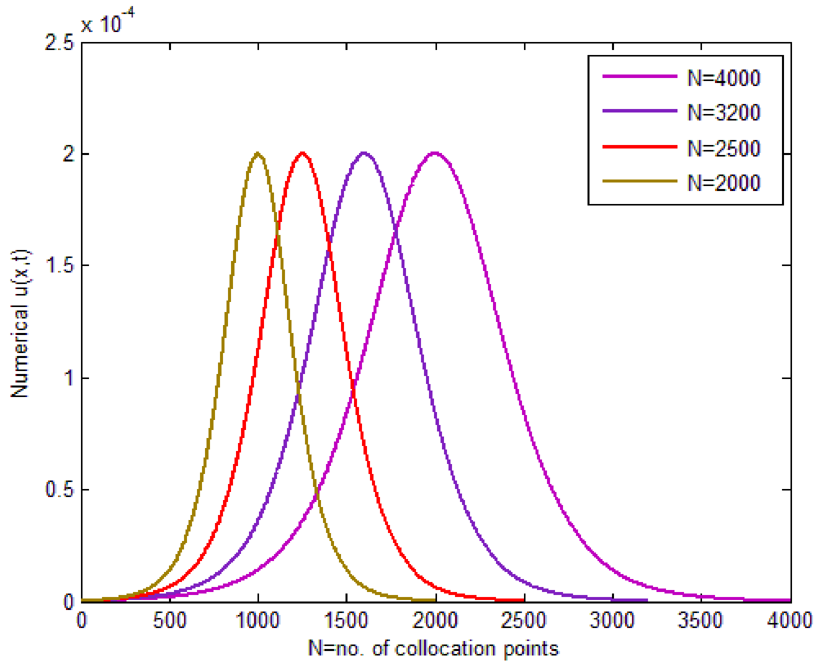

Fig. 5. Numerical u(x,t) at different collocation points

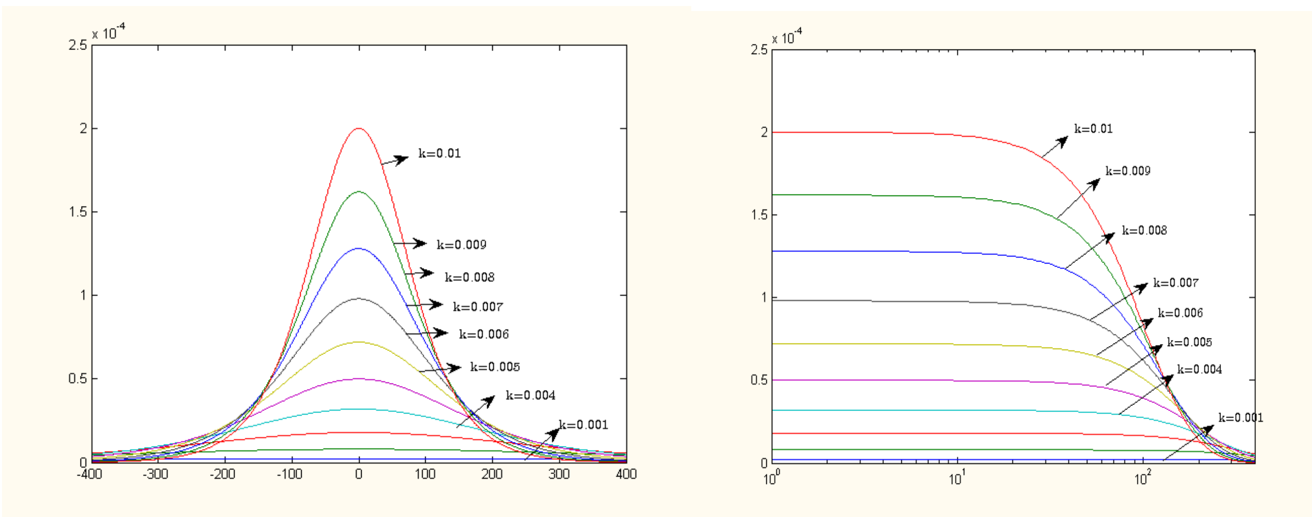

Fig. 6. Numerical u(x,t) over linear domain (left) and over logarithmic domain (right) where k=[0.001, 0.01]

6. Conclusion

In this Article the meshfree collocated numerical scheme has been developed for 7th-order Lax equation. The spital derivatives have been treated by GA-RBF while time derivatives have been treated by FDM. The method is accurate and less time consumable. The only problem is to adjust the value of shape parameter η. The efficiency and accuracy of the numerical scheme was analyzed by computing L_∞ and L_2. It is clear from Figure 6 that when the value of k is changed from 0.001 to 0.01, The height of the wave is increased and the thickness is decreased.

Author Contributions

All authors have equally contribution about publishing this research paper.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship,and publication of thisarticle.

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