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Research Paper

Applications of Higher-Order Derivatives to the Subclasses of Meromorphic Starlike Functions

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Abstract. In this paper, we introduce and study some new classes of multivalent (p-valent) meromorphically starlike functions involving Higher-Order derivatives. For these multivalent classes of functions, we derive several interesting properties including sharp coefficient bounds, neighborhoods, partial sums and inclusion relationships. For validity of our results relevant connections with those in earlier works are also pointed out.

Keywords: Meromoprhic functions, Meromorphically starlike functions, Functions with positive real parts, Higher-order derivatives.

1. Introduction

Let $\Sigma(p)$ denotes the class of functions f of the form

$$f(z)=\frac{1}{z^p}+\sum_{k=p}^{\infty}a_k z^k, \qquad \left(p\in \mathrm{N}=\{1,2,3,\ldots\}\right) \tag{1}$$

which are regular and p-valent in punctured open unit disk $E^* =: E\setminus\{0\}$, where

$$E=\{z\ :\ z\in \mathrm{C}\quad \text{and}\quad |z|<1\} \tag{2}$$

and having a pole of order p at $z=0$.

A function $f\in\Sigma(p)$ is said to be in the class $\mathbf{MS}_p^*(\alpha)$ (the class of p-valent meromorphically starlike of order α $(0\le\alpha\le p)$) in E^*, if and only if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z\in E). \tag{3}$$

Let $\mathbf{P}$ denotes the class of functions p given by

$$p(z)=1+\sum_{k=1}^{\infty}c_k z^k, \qquad (z\in E) \tag{4}$$

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which are analytic in E and satisfy the condition $\Re(p(z))>0$.

The class $\mathbf{P}$ plays a central role in the theory of analytic and multivalent functions because almost all the elementary subclassses of analytic functions were defined using this class of functions (see, for example [1-4]).

Given function f of the form (1) and another function g, given by

$$g(z)=\frac{1}{z^p}+\sum_{k=p}^{\infty}b_k z^k,$$

the Hadamard product (or convolution) denoted by $f*g$ and is defined as

$$(f*g)(z)=\frac{1}{z^p}+\sum_{k=p}^{\infty}a_k b_k z^k=(g*f)(z).$$

Many important properties and characteristics of various interesting subclasses of the class $\Sigma(p)$ of meromorphically p-valent functions were investigated extensively by many authors. For example Aouf *et al.* found new criteria for meromorphically starlike functions and also investigated partial sums for these functions, (see [5, 6]). The class of α-convex functions, subordination and superordination conditions was investigated by Ali *et al.* [7, 8], Darus *et al.* [9, 10], studied the coefficient estimates. Various inclusion and other properties of a certain class of meromorphicaly p-valent functions were defined and investigated by Liu and Srivastava [11] (see also [12-14] and Frasin [15]). For some recent investigations, see [16-19].

In recent times mathematical techniques and theories are highly recommended for application in different field of sciences for example mesh free radial basis function has been implemented by many authors to treat several kind of partial differential equation problems and variational iteration method has been used to treat some complex partial differential equations see [20-31]. Similarly there are some applications of quantum calculus and higher-order derivatives to study certain subclasses of analytic and starlike functions from deferent aspects see for example [32-37].

In this paper, we propose to implement certain higher-order derivatives and generalize the work of Wang *et al.* [38, 39], we first define some new subclasses $\mathbf{H}_p^m(\beta,\lambda)$ and $\mathbf{H}_p^{m+}(\beta,\lambda)$ of $\Sigma(p)$ of multivalent meromorphic starlike functions. We then derive several interesting properties including sharp coefficient bounds, neighborhoods, partial sums and inclusion relationships for the general function class which we introduce here. We also indicate a number of other related works on this subject.

Definition 1. For $\beta\geq 0$ and $1-\frac{p}{2}\leq\lambda<p$. A function $f\in\Sigma(p)$ is said to be in the class $\mathbf{H}_p^m(\beta,\lambda)$ if it satisfies the following condition

$$\Re\left(\mathbf{F}(f,m,\beta)\right)<\beta\lambda\left(\lambda-\frac{p}{2}+1\right)+\frac{1}{2}p\beta-\lambda \qquad \left(z\in E,\ m,p\in \mathrm{N}=\{1,2,3,..\}\right) \tag{5}$$

where

$$\mathbf{F}(f,m,\beta)=\frac{zf^{(m)}(z)}{f^{(m-1)}(z)}+\beta\frac{z^2 f^{(m+1)}(z)}{f^{(m-1)}(z)}.$$

Here $f^{(m)}$ denotes the mth derivative of f with respect to z.

Throughout in this paper, we assume that $\beta\geq 0$ and $1-\frac{p}{2}\leq\lambda<p$ unless otherwise mentioned.

It is worth mentioning that

$$\mathbf{H}_p^1(0,\lambda)=\mathbf{MS}_p^*(\lambda) \quad (0\leq\lambda\leq p)$$

and

$$\mathbf{H}_1^1(\beta,\lambda)=\mathbf{H}(\beta,\lambda),$$

where $\mathbf{MS}_p^*(\lambda)$ and $\mathbf{H}(\beta,\lambda)$ are the functions classes, which were studied by Wang *et al.* [39,40]. Also Wang *et al.* [40] proved that $\mathbf{H}(\beta,\lambda)$ is a subclass of $\mathbf{MS}^*(\lambda)$.

Let $\mathbf{H}_p^{m+}(\beta,\lambda)$ denotes the subset of $\mathbf{H}_p^m(\beta,\lambda)$ such that all functions f belonging to $\mathbf{H}_p^m(\beta,\lambda)$ with

$$f^{(m-1)}(z)=\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^p}-\sum_{k=p}^{\infty}\frac{k!}{(k-m+1)!}a_k z^{k-m+1} \qquad (a_k\geq 0). \tag{6}$$

In the present investigation, we will investigate some coefficient inequalities, neighborhoods, partial sums and inclusion relationships for the functions classes $\mathbf{H}_p^m(\beta,\lambda)$ and $\mathbf{H}_p^{m+}(\beta,\lambda)$.

2. Preliminary Results

Each of the following lemma will be needed in our present investigation.

Lemma 1. (see [41]) If the function $p \in \mathbf{P}$ is given by (4), then

$$|c_k| \le 2.$$

Lemma 2. Let $\beta > 0$ and $(p+m-1)(1-\beta(p+m)) - \gamma_p > 0$. Suppose that the sequence $\{A_k\}_{k=p}^{\infty}$ is defined by

$$A_p = \frac{(p+m-2)!(p-m+1)!(2p+m-1)!}{(2p)!p!(p-1)!}\left(\frac{(p+m-1)(1-\beta(p+m))-\gamma_p}{(1+\beta(1-2m))p}\right)$$

and

$$A_{k+p} = \sigma_1 \cdot \left(\sigma_2 + \sum_{l=p}^{k+p-1} \frac{l!(k-l)!}{(l-m+1)!(k-l-m+1)!} A_l\right) \quad (k \in \mathbb{N}). \tag{7}$$

Then

$$A_{k+p-1} = A_p \prod_{j=p}^{k+p-2} \sigma_3 \sigma_4 \quad (k+p-1 \in \mathbb{N} \setminus \{p\}). \tag{8}$$

where

$$\sigma_1 = \frac{2(k+p-m+1)!\{(p+m-1)(1-\beta(p+m))-\gamma_p\}}{(k+p)!(k+2p+\Upsilon)},$$
$$\sigma_2 = \frac{(p+m-2)!(k+2p+m-1)!}{(p-1)!(k+2p)!},$$
$$\sigma_3 = \frac{(k+p-m+1)(k+2p-1+\Upsilon)}{(k+p)(k+2p+\Upsilon)}$$

and

$$\sigma_4 = \frac{\frac{(p+m-2)!(k+2p+m-1)!}{(p-1)!(k+2p)!} + \sum_{l=p}^{k+p-1} \frac{l!(k-l)!}{(l-m+1)!(k-l-m+1)!} A_l}{\frac{(p+m-2)!(k+2p+m-2)!}{(p-1)!(k+2p-1)!} + \sum_{l=p}^{k+p-2} \frac{l!(k-l)!}{(l-m+1)!(k-l-m+1)!} A_l},$$

with

$$\Upsilon = \beta\{(k+p-m)(k+p-m-1)-(p+m-1)(p+m)\}.$$

Proof. From (7) a simple calculation yields

$$\frac{A_{k+p}}{A_{k+p-1}} = \sigma_3 \sigma_4. \tag{9}$$

Thus, for $k \ge 2p$, we deduce form (9) that

$$\begin{aligned} A_{k+p-1} &= \frac{A_{k+p-1}}{A_{k+p-2}} \cdots \frac{A_{p+2}}{A_{p+1}} \frac{A_{p+1}}{A_p} A_p \\ &= A_p \prod_{j=p}^{k+p-2} \sigma_3 \sigma_4. \end{aligned}$$

This completes the proof.

Next two Lemmas can be derived by working in similar way as in [42], here we omit the proof.

Lemma 3. Let

$$p+m-1+\beta\lambda\left(\lambda - \frac{p}{2} + 1\right) - \lambda + \beta\left(\frac{p}{2} - (p+m)(p+m-1)\right) > 0. \tag{10}$$

Suppose also that $f(z) \in \Sigma(p)$ is given by (1). If

$$\sum_{k=p}^{\infty}\frac{k!}{(k-m+1)!}\left(\left(k-m+1-\beta(k-m+1)(k-m)+\gamma_p\right)\right)|a_k| \\ \leq \frac{(p+m-2)!}{(p-1)!}\left(p+m-1-\beta(p+m)(p+m-1)-\gamma_p\right) \tag{11}$$

where (and throughout this paper unless otherwise mentioned) the parameter γ_p is constrained as follows:

$$\gamma_p = \lambda - \beta\lambda\left(\lambda - \frac{p}{2}+1\right) - \frac{1}{2}p\beta \tag{12}$$

then $f(z)\in \mathbf{H}_p^m(\beta,\lambda)$.

Lemma 4. Let $f(z)\in\Sigma(p)$, also suppose that γ_p is defined by (12) and the condition (10) holds true. Then $f(z)\in \mathbf{H}_p^{m+}(\beta,\lambda)$ if and only if

$$\sum_{k=p}^{\infty}\frac{k!}{(k-m+1)!}\left(\left(k-m+1-\beta(k-m+1)(k-m)+\gamma_p\right)\right)|a_k| \\ \leq \frac{(p+m-2)!}{(p-1)!}\left(p+m-1-\beta(p+m)(p+m-1)-\gamma_p\right). \tag{13}$$

3. Main Results

We begin with proving the following coefficients estimates for the functions belonging to the class $\mathbf{H}_p^m(\beta,\lambda)$.

Theorem 1. Let γ_p be defined by (12), if $f\in \mathbf{H}_p^m(\beta,\lambda)$ with $0<\beta<\frac{2}{5}$, $0\leq\lambda\leq p$. Then

$$|a_p|\leq A_p \quad \text{and} \quad |a_{k+p-1}|\leq A_{k+p-1}.$$

Proof. Suppose that

$$q^{(m-1)}(z) = -\frac{zf^{(m)}(z)}{f^{(m-1)}(z)} - \beta\frac{z^2 f^{(m+1)}(z)}{f^{(m-1)}(z)} + \beta\lambda\left(\lambda-\frac{p}{2}+1\right)+\frac{1}{2}p\beta-\lambda>0. \tag{14}$$

Then by definition of the function class $\mathbf{H}_p^m(\beta,\lambda)$, we know that $q^{(m-1)}(z)$ is analytic in E and

$$\Re\left(q^{(m-1)}(z)\right)>0 \qquad (z\in E)$$

with

$$q^{(m-1)}(0) = (p+m-1)\left(1-\beta(p+m)\right)-\gamma_p>0.$$

Thus by using (12) and (14), we have

$$q^{(m-1)}(z)f^{(m-1)}(z) = -zf^{(m)}(z) - \beta z^2 f^{(m+1)}(z) - \gamma_p f^{(m-1)}(z). \tag{15}$$

Noting that

$$h^{(m-1)}(z) = \frac{q^{(m-1)}(z)}{(p+m-1)\left(1-\beta(p+m)\right)-\gamma_p}\in P,$$

if we put

$$q^{(m-1)}(z) = c_0 + \sum_{k=p}^{\infty}\frac{k!}{(k-m+1)!}c_k z^{k-m+1},$$

where

$$c_0 = (p+m-1)\left(1-\beta(p+m)\right)-\gamma_p,$$

now, by Lemma 1, we know that

$$|c_k| \le 2\big((p+m-1)(1-\beta(p+m))-\gamma_p\big) \qquad (k \in \mathbb{N}),$$

it follows form (15) that

$$\begin{aligned}
&\left(c_0+\sum_{k=p}^{\infty}\frac{k!}{(k-m+1)!}c_k z^{k-m+1}\right)\cdot\left(\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}}+\sum_{k=p}^{\infty}\frac{k!}{(k-m+1)!}a_k z^{k-m+1}\right)\\
&=-\left(\frac{(-1)^{m}(p+m-1)!}{(p-1)!z^{p+m-1}}+\sum_{k=p}^{\infty}\frac{k!}{(k-m)!}a_k z^{k-m+1}\right)-\beta\left(\frac{(-1)^{m+1}(p+m)!}{(p-1)!z^{p+m-1}}+\sum_{k=p}^{\infty}\frac{k!}{(k-m-1)!}a_k z^{k-m+1}\right)\\
&-\gamma_p\left(\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}}+\sum_{k=p}^{\infty}\frac{k!}{(k-m+1)!}a_k z^{k-m+1}\right),
\end{aligned} \tag{16}$$

by the virtue of (16), we get

$$\begin{aligned}
&c_0\frac{p!}{(p-m+1)!}a_p+\frac{(-1)^{m-1}(p+m-2)!(2p+m-1)!}{(p-1)!(2p)!}c_{2p+m-1}\\
&=-\frac{p!}{(p-m)!}a_p-\beta\frac{p!}{(p-m-1)!}-\gamma_p\frac{p!}{(p-m+1)!}a_p
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&c_0\frac{(k+p)!}{(k+p-m+1)!}a_{k+p}+\frac{(-1)^{m-1}(p+m-2)!(k+2p+m-1)!}{(p-1)!(k+2p)!}c_{k+2p+m-1}\\
&+\sum_{l=p}^{k+p-1}\frac{l!(k-l)!}{(l-m+1)!(k-l-m+1)!}a_l c_{k-l}\\
&=-\frac{(k+p)!}{(k+p-m)!}a_{k+p}-\beta\frac{(k+p)!}{(k+p-m-1)!}a_{k+p}-\gamma_p\frac{p!}{(k+p-m+1)!}a_{k+p.}
\end{aligned} \tag{18}$$

From (17) we obtain

$$|a_p| \le A_p. \tag{19}$$

Moreover, from (18), we conclude that

$$|a_{k+p}| \le \sigma_1\cdot\left(\sigma_2+\sum_{l=p}^{k+p-1}\frac{l!(k-l)}{(l-m+1)!(k-l-m+1)!}|a_l|\right) \qquad (k+p \in \mathbb{N}). \tag{20}$$

Next, we define the sequence $\{A_k\}_{k=p}^{\infty}$ as follows:

$$A_p=\frac{(p+m-2)!(p-m+1)!(2p+m-1)!}{(2p)!p!(p-1)!}\cdot\left(\frac{(p+m-1)(1-\beta(p+m))-\gamma_p}{(1+\beta(1-2m))p}\right)$$

and

$$A_{k+p}=\sigma_1\cdot\left(\sigma_2+\sum_{l=p}^{k+p-1}\frac{l!(k-l)!}{(l-m+1)!(k-l-m+1)!}A_l\right) \qquad (k+p \in \mathbb{N}). \tag{21}$$

In order to prove that

$$|a_{k+p-1}| \le A_{k+p-1} \qquad (k+p-1 \in \mathbb{N}).$$

We make use of the principle of mathematical induction. By noting that

$$|a_p| \le A_p.$$

Let us consider

$$|a_l| \le A_l \qquad (l=p,p+1,\ldots,k+p-1;\ k \in \mathbb{N}).$$

Combining (20) and (21), we get

$$\begin{aligned}\left|a_{k+p}\right| &\le \sigma_1 \cdot \left(\sigma_2 + \sum_{l=p}^{k+p-1} \frac{l!(k-l)!}{(l-m+1)!(k-l-m+1)!}\left|a_l\right|\right) \\ &\le \sigma_1 \cdot \left(\sigma_2 + \sum_{l=p}^{k+p-1} \frac{l!(k-l)!}{(l-m+1)!(k-l-m+1)!} A_l\right) \\ &= A_{k+p} \qquad\qquad (k+p \in \mathbb{N}).\end{aligned}$$

Hence by the principle of mathematical induction, we have

$$\left|a_{k+p-1}\right| \le A_{k+p-1} \qquad (k+p-1 \in \mathbb{N}) \tag{22}$$

as desired.

By virtue of Lemma 2 and (21), we know that (8) holds true. Combining (22) and (8) we readily get coefficient estimates asserted by Theorem 1.

Remark 1. In its special case when $m=1$ and $p=1$, Theorem 1 would yield the known result due to Wang *et al.* [39].

Following the earlier work based upon the familiar concept of neighborhood of analytic functions by Goodman [43] and Ruscheweyh [44] and recently by Liu and Srivastava [12-14]. Suppose that γ_p is given by (12) and the condition (10) of Lemma 3 holds true, we here introduce the δ-neighborhood of function $f \in \Sigma(p)$ of the form (1) by means of the following definition

$$N_\delta\left(f^{(m-1)}\right) := \left\{\begin{array}{c} g \in \Sigma(p) \ : \ g^{(m-1)}(z) = \frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}} \\ +\sum_{k=p}^{\infty} \frac{k!}{(k-m+1)!} b_k z^{k-m+1} \quad \text{and} \\ \sum_{k=p}^{\infty} \frac{k!}{(k-m+1)!}\left(\frac{k-(m-1)+\beta(k-m+1)(k-m)+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}\right) \le \delta \quad (\delta \ge 0) \end{array}\right\}. \tag{23}$$

Making use of this definition, we prove the following result.

Theorem 2. Let the condition (10) holds true. If $f \in \Sigma(p)$ satisfies the condition

$$\frac{f^{(m-1)}(z) * \varepsilon z^{-(p+m-1)}}{1+\varepsilon} \in H_p^m(\beta,\lambda) \qquad (\varepsilon \in \mathbb{C}; |\varepsilon| < \delta; \delta > 0). \tag{24}$$

Then

$$N_\delta\left(f^{(m-1)}\right) \subset \mathbf{H}_p^m(\beta,\lambda). \tag{25}$$

Proof. By noting that (5) can be written as

$$\left|\frac{\frac{zf^{(m)}(z)}{f^{m-1}(z)} + \beta\frac{z^2 f^{(m+1)}(z)}{f^{(m-1)}(z)} + 1}{\frac{zf^{(m)}(z)}{f^{m-1}(z)} + \beta\frac{z^2 f^{(m+1)}(z)}{f^{(m-1)}(z)} + 2\gamma_p - 1}\right| < 1. \tag{26}$$

It is easily seen from (26) that a function $g(z) \in \mathbf{H}_p^m(\beta,\lambda)$ if and only if, for a complex number σ with $|\sigma| = 1$, we have

$$\frac{zg^{(m)}(z) + \beta z^2 g^{(m+1)}(z) + g^{(m-1)}(z)}{zg^{(m)}(z) + \beta z^2 g^{(m+1)}(z) + \left(2\gamma_p - 1\right)g^{(m-1)}(z)} \ne \sigma \quad (z \in E),$$

which is equivalent to

$$\frac{\left(g^{(m-1)} * h^{(m-1)}\right)(z)}{z^{-(p+m-1)}} \ne 0 \qquad (z \in E), \tag{27}$$

where

$$\begin{aligned} h^{(m-1)}(z) &= \frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}} + \sum_{k=p}^{\infty} \frac{k!}{(k-m+1)!} c_k z^{k-m+1} \\ c_k &:= \frac{\xi_1 - \xi_2\sigma}{\xi_3 + \xi_4\sigma} \end{aligned} \tag{28}$$

with

$$\xi_1 = k-(m-1)+\beta(k-m+1)(k-m)+1,$$
$$\xi_2 = k-(m-1)+\beta(k-m+1)(k-m)+(2\gamma_p-1),$$
$$\xi_3 = 1-(p+m-1)+\beta(p+m-1)(p+m)$$

and

$$\xi_4 = (p+m-1)-\beta(p+m-1)(p+m)-(2\gamma_p-1).$$

It follows from (28) that

$$|c_k| = \left|\frac{\xi_1-\xi_2\sigma}{\xi_3+\xi_4\sigma}\right| \le \frac{\xi_1+\xi_2|\sigma|}{-\xi_3+\xi_4|\sigma|} \le \frac{k-(m-1)+\beta(k-m+1)(k-m)+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p} \qquad (|\sigma|=1).$$

If $f\in\Sigma(p)$ given by (1) satisfies the inclusion property (24), we deduce from (27) that

$$\frac{\left(f^{(m-1)}*h^{(m-1)}\right)(z)}{z^{-(p+m-1)}} \ne -\varepsilon \qquad (|\varepsilon|<\delta;\delta>0),$$

or equivalently,

$$\frac{\left(f^{(m-1)}*h^{(m-1)}\right)(z)}{z^{-(p+m-1)}} \ge \delta \qquad (z\in E). \tag{29}$$

We now suppose that $\varphi\in N_\delta\left(f^{(m-1)}\right)$, with

$$\varphi^{(m-1)}(z) = \frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}} + \sum_{k=p}^{\infty}\frac{k!}{(k-m+1)!}d_k z^{k-m+1}.$$

It follows form (23) that

$$\begin{aligned}\left|\frac{\left(\left(\varphi^{(m-1)}-f^{(m-1)}\right)*h^{(m-1)}\right)(z)}{z^{-(p+m-1)}}\right| &= \left|\sum_{k=p}^{\infty}\frac{k!}{(k-m+1)!}(d_k-a_k)c_k z^{k+p}\right| \\ &\le |z^{p+m-1}|\sum_{k=p}^{\infty}\frac{k-(m-1)+\beta(k-m+1)(k-m)+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}|d_k-a_k| < \delta.\end{aligned} \tag{30}$$

Combining (29) and (30), we can easily find that

$$\begin{aligned}\left|\frac{\left(\varphi^{(m-1)}*h^{(m-1)}\right)(z)}{z^{-(p+m-1)}}\right| &= \left|\frac{\left(\left[f^{(m-1)}+\left(\varphi^{(m-1)}-f^{(m-1)}\right)\right]*h^{(m-1)}\right)(z)}{z^{-(p+m-1)}}\right| \\ &\ge \left|\frac{\left(f^{(m-1)}*h^{(m-1)}\right)(z)}{z^{-(p+m-1)}}\right| - \left|\frac{\left(\left(\varphi^{(m-1)}-f^{(m-1)}\right)*h^{(m-1)}\right)(z)}{z^{-(p+m-1)}}\right| > 0,\end{aligned}$$

which implies that

$$\frac{\left(\varphi^{(m-1)}*h^{(m-1)}\right)(z)}{z^{-(p+m-1)}} \ne 0 \qquad (z\in E).$$

Therefore, we have

$$\varphi\in \mathbb{H}_p^m(\beta,\lambda).$$

This complete the required proof.

Remark 2. If we set $m=1$ and $p=1$, in Theorem 2, we are led to the result similar to that given by Wang *et al.* [39].

Next, we derive the partial sum of the class $\mathbb{H}_p^m(\beta,\lambda)$. For some interesting investigations involving the partial sums in analytic function theory, see [5, 13-15].

Theorem 3. Let $f\in\Sigma(p)$ be given by (1) and define the partial sums $f_{n+p-1}^{(m-1)}(z)$ of $f^{(m-1)}(z)$ by

$$f_{n+p-1}^{(m-1)}(z)=\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}}+\sum_{k=p}^{n+p-1}\mathbf{M}a_k z^{k-m+1}\quad (n\in \mathrm{N}). \tag{31}$$

If

$$\sum_{k=p}^{\infty}\mathbf{M}\left(\frac{k-(m-1)+\beta(k-m+1)(k-m)+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}\right)|a_k|\le\frac{(p+m-2)!}{(p-1)!}, \tag{32}$$

where

$$\mathbf{M}=\frac{k!}{(k-m+1)!}$$

and γ_p is given by (12), and the condition (10) holds true. Then

$$f^{(m-1)}(z)\in \mathbf{H}_p^m(\beta,\lambda);$$

$$\Re\left(\frac{f^{(m-1)}(z)}{f_{n+p-1}^{(m-1)}(z)}\right)\ge\frac{n-2(m-1)+\mathbf{L}+\beta(p+m-1)(p+m)+2\gamma_p}{n-(m-1)+p+\mathbf{L}+\gamma_p} \tag{33}$$

and

$$\Re\left(\frac{f_{n+p-1}^{(m-1)}(z)}{f^{(m-1)}(z)}\right)\ge\frac{n-(m-1)+\mathbf{L}+p+\gamma_p}{n+\mathbf{L}+2p-\beta(p+m-1)(p+m)}\qquad (n\in \mathrm{N};z\in E); \tag{34}$$

where

$$\mathbf{L}=\beta(n+p-m)(n+p-m+1).$$

The bounds in (33) and (34) are sharp.

Proof. Suppose that

$$f_p^{(m-1)}(z)=\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}}.$$

We note that

$$\frac{f_p^{(m-1)}(z)+\varepsilon\left(\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}}\right)}{1+\varepsilon}=\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}}\in \mathbf{H}_p^m(\beta,\lambda).$$

From (32), we find that

$$\sum_{k=p}^{\infty}\mathbf{M}\left(\frac{k-(m-1)+\beta(k-m+1)(k-m)+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}\right)|a_k-0|\le\frac{(p+m-2)!}{(p-1)!},$$

which implies that

$$f^{(m-1)}(z)\in \mathrm{N}_p\left(\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}}\right).$$

By virtue of Theorem 2, we deduce that

$$f^{(m-1)}(z)\in \mathrm{N}_p\left(\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!z^{p+m-1}}\right)\subset \mathbf{H}_p^m(\beta,\lambda).$$

Next, it is easy to verify that

$$\frac{n-(m-1)+p+\mathbf{L}+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}>\frac{n-(m-1)+\beta(n-m+1)(n-m)+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}$$
$$>\frac{(p+m-2)!}{(p-1)!}\qquad (n\in \mathrm{N}).$$

Therefore, we have

$$\begin{aligned}&\sum_{k=p}^{n+p-1}\mathbf{M}|a_k|+\frac{n-(m-1)+p+\mathbf{L}+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}\sum_{k=n+p}^{\infty}\mathbf{M}|a_k|\\&\le\sum_{k=p}^{\infty}\mathbf{M}\left(\frac{k-(m-1)+\beta(k-m+1)(k-m)+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}\right)|a_k|\le\frac{(p+m-2)!}{(p-1)!}.\end{aligned}\tag{35}$$

If we set

$$\begin{aligned}h_p^{(m-1)}(z)&=\frac{n-(m-1)+p+\mathbf{L}+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}\\&\quad\cdot\left(\frac{f^{(m-1)}(z)}{f_{n+p-1}^{(m-1)}(z)}-\frac{n-2(m-1)+\mathbf{L}+\beta(p+m-1)(p+m)+2\gamma_p}{n-(m-1)+p+\mathbf{L}+\gamma_p}\right)\\&=1+\frac{\frac{n-(m-1)+p+\mathbf{M}+\gamma_m}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_m}\sum_{k=n+p}^{\infty}\mathbf{M}a_kz^{k+p}}{\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!}+\sum_{k=p}^{n+p-1}\mathbf{M}a_kz^{k+p}}.\end{aligned}\tag{36}$$

It follows from (35) and (36) that

$$\begin{aligned}&\left|\frac{h_p^{(m-1)}(z)-1}{h_p^{(m-1)}(z)+1}\right|\le\frac{\mathbf{N}\sum_{k=n+p}^{\infty}\mathbf{M}|a_k|}{2\left(\frac{(p+m-2)!}{(p-1)!}\right)-2\sum_{k=p}^{n+p-1}\mathbf{M}|a_k|-\mathbf{N}\sum_{k=n+p}^{\infty}\mathbf{M}|a_k|}\\&\le\frac{(p+m-2)!}{(p-1)!}\qquad(z\in E),\end{aligned}\tag{37}$$

where

$$\mathbf{N}=\frac{n-(m-1)+p+\mathbf{L}+\gamma_p}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}.$$

The last inequality in (37) now shows that

$$\Re\left(h_p^{(m-1)}(z)\right)\ge 0\qquad(z\in E).\tag{38}$$

Furthermore, by taking

$$f^{(m-1)}(z)=\frac{1}{z^{p+m-1}}-\frac{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}{n+p-(m-1)+\mathbf{L}+\gamma_p}z^{n+p-m+1}\tag{39}$$

then

$$\begin{aligned}\frac{f^{(m-1)}(z)}{f_{n+p-1}^{(m-1)}(z)}&=1-\frac{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}{n-(m-1)+p+\mathbf{L}+\gamma_p}z^{n+2p}\\&\to\frac{n-2(m-1)+\mathbf{L}+\beta(p+m-1)(p+m)+2\gamma_p}{n+p-(m-1)+\mathbf{L}+\gamma_p}\qquad(z\to 1^-),\end{aligned}$$

which implies that the bound in (33) is the best possible for each $n\in\mathbb{N}$.
Similarly, we suppose that

$$\begin{aligned}w_p^{(m-1)}(z)&=\frac{n+\mathbf{L}+2p-\beta(p+m-1)(p+m)}{(p+m-1)-\beta(p-m+1)(p+m)-\gamma_p}\\&\quad\cdot\left(\frac{f_{n+p-1}^{(m-1)}(z)}{f^{(m-1)}(z)}-\frac{n-(m-1)+\mathbf{L}+p+\gamma_p}{n+\mathbf{L}+2p-\beta(p+m-1)(p+m)}\right)\\&=1-\frac{\mathbf{J}\sum_{k=n+p}^{\infty}\mathbf{M}a_kz^{k+p}}{\frac{(-1)^{m-1}(p+m-2)!}{(p-1)!}+\sum_{k=p}^{\infty}\mathbf{M}a_kz^{k+p}},\end{aligned}\tag{40}$$

where

$$\mathbf{J}=\frac{n+\mathbf{L}+2p-\beta(p+m-1)(p+m)}{(p+m-1)-\beta(p-m+1)(p+m)-\gamma_p}.\tag{41}$$

In the view of (35) and (40) we conclude that

$$\left|\frac{w_p^{(m-1)}(z)-1}{w_p^{(m-1)}(z)+1}\right| \le \frac{\mathbf{J}\sum_{k=n+p}^{\infty}\frac{k!}{(k-m+1)!}|a_k|}{2\left(\frac{(p+m-2)!}{(p-1)!}\right)-2\sum_{k=p}^{n+p-1}\mathbf{M}|a_k|-\varpi\sum_{k=n+p}^{\infty}\mathbf{M}|a_k|} \le \frac{(p+m-2)!}{(p-1)!} \qquad (z\in E), \tag{42}$$

where

$$\varpi = \frac{n-2(m-1)+\mathbf{L}+\beta(p+m-1)(p+m)}{(p+m-1)-\beta(p+m-1)(p+m)-\gamma_p}.$$

The inequality in (42) now implies that

$$\Re\left(w_p^{(m-1)}(z)\right) \ge 0 \qquad (z\in E). \tag{43}$$

Combining (40) and (43), we readily get the assertion (34). The bound in (34) is sharp with the extremal function given in (39).

In what follows, we turn to quotients involving derivatives. The proof of Theorem 4 below is similar to that of Theorem 3, we here choose to omit the analogous details.

Theorem 4. Let $f\in\Sigma(p)$ be given by (1) and define the partial sums $f_{n+p-1}^{(m-1)}(z)$ of $f^{(m-1)}(z)$ by (31). If the conditions (10) and (32) hold, where γ_p given by (12) then

$$\Re\left(\frac{f^{(m)}(z)}{f_{n+p-1}^{(m)}(z)}\right) \ge \frac{(n+p)\left(2-m-p+\beta(p+m-1)(p+m)+\gamma_p\right)+\mathbf{L}-m+1+\gamma_p}{n+p-(m-1)+\mathbf{L}+\gamma_p} \tag{44}$$

and

$$\Re\left(\frac{f_{n+p-1}^{(m)}(z)}{f^{(m)}(z)}\right) \ge \frac{n+p-(m-1)+\mathbf{L}+\gamma_p}{(n+p)\left(p+m-\beta(p+m-1)(p+m)-\gamma_p\right)+\mathbf{L}-m+1+\gamma_p}. \tag{45}$$

The bounds in (44) and (45) are sharp with the extremal function given by (39).

Remark 3. If we put $m=1$ and $p=1$, in Theorem 3 and Theorem 4 above, we obtain the results which are precisely the same to those proved earlier by Wang *et al.* [41].

Theorem 5. Let

$$\beta_1 \ge \beta_2 \ge 1 \quad \text{and} \quad 1-\frac{p}{2} \le \lambda_1 \le \lambda_2 < p.$$

Then

$$\mathbf{H}_p^m(\beta_1,\lambda_1) \subset \mathbf{H}_p^m(\beta_2,\lambda_2). \tag{46}$$

Proof. Suppose that $f\in\mathbf{H}_p^m(\beta_1,\lambda_1)$. Then it follows from the definition that

$$\Re\left(\mathbf{F}(f,m,\beta)\right) < \lambda_1\left[\beta_1\left(\lambda_1-\frac{p}{2}+1\right)-1\right]+\frac{1}{2}p\beta_1 \qquad (z\in E). \tag{47}$$

Since $\beta_1\ge\beta_2\ge 1$ and $1-\frac{p}{2}\le\lambda_1\le\lambda_2<p$, we observe that

$$\lambda_1\left[\beta_1\left(\lambda_1-\frac{p}{2}+1\right)-1\right]+\frac{1}{2}p\beta_1 < \lambda_2\left[\beta_1\left(\lambda_2-\frac{p}{2}-1\right)+1\right]+\frac{1}{2}p\beta_1. \tag{48}$$

It follows from (47) and (48) that

$$\Re\left(\mathbf{F}(f,m,\beta)\right) < \lambda_2\left[\beta_1\left(\lambda_2-\frac{p}{2}+1\right)-1\right]+\frac{1}{2}p\beta_1 \qquad (z\in E), \tag{49}$$

which shows that $f(z)\in\mathbf{H}_p^m(\beta_1,\lambda_2)$, subsequently, we see that $f(z)\in\mathbf{MS}_p^*(\lambda_2)$ that is

$$\Re\left\{\frac{zf^{(m)}(z)}{f^{(m-1)}(z)}\right\} < -\lambda_2. \tag{50}$$

Now by setting

$$\mu = \frac{\beta_2}{\beta_1},$$

so that

$$0 < \mu \leq 1.$$

It is easy to verify from (49) and (50) that

$$\begin{aligned}
&\Re\left(\mathbf{F}(f,m,\beta) - \lambda_2\left[\beta_2\left(\lambda_2 - \frac{p}{2} + 1\right) - 1\right] - \frac{1}{2}p\beta_2\right) \\
&= \mu\Re\left(\mathbf{F}(f,m,\beta) - \lambda_2\left[\beta_1\left(\lambda_2 - \frac{p}{2} + 1\right) - 1\right] - \frac{1}{2}p\beta_1\right) + (1-\mu)\Re\left(\frac{zf^{(m)}(z)}{f^{(m-1)}(z)} + \lambda_2\right) \\
&< 0 \qquad\qquad (z \in E),
\end{aligned}$$

that is

$$f(z) \in \mathbf{H}_p^m(\beta_2, \lambda_2).$$

This evidently completes the proof.

From the Theorem 5 and the definition of the function class $\mathbf{H}_p^{m+}(\beta,\lambda)$ we easily get the following inclusion relationship.

Corollary 1. Let

$$\beta_1 \geq \beta_2 \geq 1 \quad and\ 1 - \frac{p}{2} \leq \lambda_1 \leq \lambda_2 < p.$$

then

$$\mathbf{H}_p^{m+}(\beta_1, \lambda_1) \subset \mathbf{H}_p^{m+}(\beta_2, \lambda_2) \subset \mathbf{MS}_p^*(\lambda_2).$$

By use of Lemma 4 on Theorem 5, we obtain the following result.

Corollary 2. Let $f \in \mathbf{H}_p^{m+}(\beta,\lambda)$. Suppose also that γ_p be defined by (12) and condition (10) holds true. Then

$$|a_k| \leq \frac{(k-m+1)!(p+m-2)!\left(p+m-1-\beta(p+m)(p+m-1)-\gamma_p\right)}{k!(p-1)!\left(k-m+1-\beta(k-m+1)(k-m)+\gamma_p\right)}.$$

Each of these inequalities is sharp, with the extremal function given by

$$f_{k+p-1}^{(m-1)}(z) = (-1)^{m-1}\frac{(p+m-2)!}{(p-1)!z^{p+m-1}} - \frac{(k-m+1)!\left(p+m-1-\beta(p+m)(p+m-1)-\gamma_p\right)}{k!\left(k-m+1-\beta(k-m+1)(k-m)+\gamma_p\right)}z^{k-m+1}.$$

4. Conclusion

Here, in our present investigation, we have successfully applied certain higher-order derivatives to the subclass of meromorphically regular p-valent functions and defined a new subclass of meromorphic starlike functions in punctured open unit disk E^*. Many properties and characteristic of this newly defined functions class such as sufficient conditions, coefficient bounds, neighborhoods, partial sums and inclusion relationships have studied. We also highlighted a numbers of known consequences, which are already present in literature.

Here, in this last section on conclusion, we reiterate the fact that the results for the meromorphic starlike functions which we have considered in this article, can easily be translated into the corresponding results for the quantum calculus by applying some obvious parametric and argument variations. Indeed, as observed earlier by Srivastava *et al.* [45-48].

Author Contributions

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