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Research Paper

On the Dynamics of the Logistic Delay Differential Equation with Two Different Delays

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Abstract. Here, we study the logistic delay differential equation with two different delays. First of all, we disscuse the local stability and Hopf bifurcation conditons. The method of steps is used to get a discretized analogue of the original system. Local stability and bifurcation analysis of the discretized system is investigated. Finally, we carry out some numerical simulations such as bifurcation diagram, Lyapunov exponent and phase portraits to verify the theoretical results and to illustrate complex dynamics of the considered system.

Keywords: Logistic equation; Time delay; Local stability; Bifurcation; Chaos.

1. Introduction

Delay differential equations (DDs) are applicable in science and engineering. Great attention is paid to qualitative analysis of solutions of ordinary and DDs [1-3]. Models with only one delay are frequently used when the other delays are insignificant to dynamical behaviors, but this is not the situation in many cases [4]. Furthermore, there are systems those are not stable with single delay, however, the system becomes stable when a second delay added to the system [5]. Therefore, models with multiple delays are of great interest. DDs with multiple delays are defined by the equation [6]:

$$\frac{dx}{dt} = f\left(t, x(t), x(t-\tau_i)\right), \qquad t \in [0, T],$$

where $i = 0,1,2,\dots,n$ are nonnegative integers and τ_i are nonnegative real constants.

In the last years, DDs with two delays have received alot of attention from specialists because of their great importance in physics and biology, and their rich dynamics[7-8]. This sort of equations arises in a large number of fields [9-16].

For the logistic equation there is an instantaneous dependence on changes in population size. However, there are cases in which the logistic model must contain time delays to represent processes like gestation and maturation times that are not instantaneous [17].

There are important results introduced in [18] concerning with the logistic delay differential equation

$$\frac{du}{dt} = -c\tau x(t) + \rho\tau x(t-1)\left(1 - x(t-1)\right),$$

where c, ρ and $\tau > 0$.

In this work, we generalize the results introduced in [18] for the logistic DDs with two different delays

$$\frac{dx}{dt} = -x(t) + \rho x(t-1)\left(1 - x(t-2)\right), \qquad t \in [0, T], \tag{1.1}$$

with $x(t) = x_0,\ t \le 0$.

Moreover, we illustrate more dynamics concerning with (1.1). Engineering principles are applied to ecological systems by many "ecological engineers" [19]. Population dynamics palys an important role in ecological engineering [20], the population represents

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humans, biological lifeforms in ecological systems, chemical compounds, farm lands [21]. The delay equation (1.1) is very important in population dynamics[17-18] that is the reason why ecological engineers pay attention to it.

2. Main Results

2.1. Solution of (1.1)

Theorem 1 Problem (1.1) has a unique solution $x \in C[0,T]$, $0 \le x(t) \le 1$.

Proof. Define the operator $F: C[0,T] \to C[0,T]$ by

$$Fx(t) = x_0 e^{-t} + \rho \int_0^t e^{s-t} x(s-1)(1 - x(s-2))ds.$$

Let x, y be two solutions, then

$$|Fx - Fy| \le \rho \int_1^t (e^{s-t}) \big| x(s-1)(1-x(s-2)) - y(s-1)(1-y(s-2)) \big| ds$$

$$\le \rho \int_0^t e^{s-t} |x(s-1) - y(s-1)| + e^{s-t} |x(s-2) - y(s-2)| ds$$

$$\le \rho [\max_{[1,T]} |x(s-1) - y(s-1)| \int_1^t e^{s-t} ds + \max_{[2,T]} |x(s-2) - y(s-2)| \int_2^t e^{s-t} ds].$$

Hence,

$$\| Fx - Fy \|_{[0,T]} \le \rho \| x - y \|_{[0,T]} \left(2 - e^{-(t-1)} - e^{-(t-2)}\right)$$

$$\le 2\rho \| x - y \|_{[0,T]}.$$

If $\rho < 1/2$, then F is contraction and the solution of (1.1) exists uniquely.

2.2. Stability and bifurcation

There are two fixed points of (1.1) given by the solution of $-x_{fix} + \rho x_{fix}(1 - x_{fix}) = 0$.

These are $(x_1)_{fix} = 0$ and $(x_2)_{fix} = 1 - 1/\rho$.

Stability conditions of $(x_1)_{fix} = 0$ can be obtained by analysing the eigenvalues of the linearized system [22].

The linearized equation at the neighborhood of $(x_1)_{fix} = 0$ is

$$\frac{dx}{dt} = -x(t) + \rho x(t-1). \tag{2.1}$$

Assume that $x(t) = e^{\lambda t}$, the characteristic equation reads

$$\lambda + 1 - \rho e^{-\lambda} = 0. \tag{2.2}$$

Lemma 1 [23] All roots of $\lambda + a + be^{-\lambda} = 0$ have negative real parts if and only if $a > -1$, $a + b > 0$, $b < \sqrt{a^2 + \xi^2}$, where c, b are real and $\xi = -a \ \tan\xi$, $0 < \xi < \pi$ if $a \ne 0$ and $\xi = \pi/2$ if $a = 0$.

Applay Lemma 1 to equation (2.2) with $a = 1$, $b = -\rho$, we can get the next theorem.

Theorem 2 The fixed point $(x_1)_{fix} = 0$ of (1.1) is unstable if $\rho < r_0$ or $\rho > 1$, where $r_0 = -\sqrt{1 + (\xi)^2}$, $\xi = -tan(\xi)$, $0 < \xi < \pi$, and is stable if $r_0 < \rho < 1$.

Theorem 3 When ρ passes through $\rho = r_0 = -\sqrt{1 + \xi^2}$, $\xi = -tan(\xi)$, $0 < \xi < \pi$, (1.1) admits a Hopf bifurcation from the fixed point $x_{fix} = 0$ to a periodic orbit.

Proof. Assume that $\lambda = i\omega_0$, $\omega_0 \in R^+$ is a pure imaginary solution of (2.2) for $\rho = \rho_*$. Then $i\omega_0 + 1 - \rho_* e^{-i\omega_0} = 0$ and $1 - \rho_* cos(\omega_0) = 0$, $\omega_0 - \rho_* sin(\omega_0) = 0$, $1 = \rho_* cos(\omega_0)$, $\omega_0 = \rho_* sin(\omega_0)$, and $\omega_0^2 + 1 = \rho_*^2 [cos(\omega_0)^2 + sin(\omega_0)^2] = \rho_*^2$, where $\rho_* = \pm\sqrt{1 + \omega_0^2}$, and $\omega_0 = -tan(\omega_0)$.

By Theorem 4, $\rho_* = -\sqrt{1 + \omega_0^2}$ is the critical value of ρ, where ω_0 is the root of $\omega_0 = -tan(\omega_0)$, $0 < \omega_0 < \pi$.

Now, we are left with the condition $d(Re(\lambda)) / d\rho |_{\rho = \rho_*} \ne 0$.

Let $\lambda = k(\rho) + i\omega(\rho)$, then from (2.2), we obtain $k + i\omega + 1 - \rho e^{-k - i\omega} = 0$. Hence,

$$k+1-\rho e^{-k}\cos\omega=0, \tag{2.3}$$

$$\omega+\rho e^{-k}\sin\omega=0. \tag{2.4}$$

Differentiate (2.3) and (2.4) with respect to ρ, we obtain

$$\frac{dk}{d\rho}-e^{-k}\cos(\omega)+\rho e^{-k}\cos(\omega)\frac{dk}{d\rho}+\rho e^{-k}\sin(\omega)\frac{d\omega}{d\rho}=0,$$

$$\frac{d\omega}{d\rho}+e^{-k}\sin(\omega)+\rho e^{-k}\cos(\omega)\frac{d\omega}{d\rho}-\rho e^{-k}\sin(\omega)\frac{dk}{d\rho}=0.$$

Solving for $dk/d\rho$, we obtain

$$\frac{d(Re(\lambda))}{d\rho}|_{\rho=\rho_*}=\frac{d(Re(\lambda))}{d\rho}|_{k=0,\omega=\omega_0,\rho=\rho_*}$$

$$=\frac{\cos(\omega_0)+\rho_*}{(1+\rho_*\cos(\omega_0))^2+(\rho_*\sin(\omega_0))^2}$$

$$=\frac{\rho_*\cos(\omega_0)+\rho_*^2}{\rho_*[(1+\rho_*\cos(\omega_0))^2+(\rho_*\sin(\omega_0))^2]}$$

$$=\frac{1+\rho_*^2}{\rho_*[(1+\rho_*\cos(\omega_0))^2+(\rho_*\sin(\omega_0))^2]}\neq 0.$$

(2) The linearized equation at the neighborhood of $(x_2)_{fix}=1-1/\rho$ is

$$\frac{dy}{dt}=-y(t)+y(t-1)-(\rho-1)y(t-2), \tag{2.5}$$

where $y(t)=y(t)-(1-1/\rho)$.
The characteristic equation is

$$\lambda+1-e^{-\lambda}+(\rho-1)e^{-2\lambda}=0. \tag{2.6}$$

Theorem 4 When ρ passes through $\rho=\rho_*=1-\sqrt{(\cos(\omega_0)-1)^2+(\omega_0+\sin(\omega_0))^2}$, $\omega_0=(\cos(\omega_0)-1)tan(2\omega_0)-\sin(\omega_0)$, then (1.1) admits a Hopf bifurcation from the fixed point $x_{fix}=1-1/\rho$ to a periodic orbit.

Proof. Assume that $\lambda=i\omega_0$, $\omega_0\in R^+$ is a pure imaginary solution of (2.6) for $\rho=\rho_*$. Then, we obtain

$$i\omega_0+1-e^{-i\omega_0}+(\rho_*-1)e^{-i2\omega_0}=0.$$

Hence, it follows that

$$1-\cos(\omega_0)+(\rho_*-1)\cos(2\omega_0)=0,$$

$$\omega_0+\sin(\omega_0)-(\rho_*-1)\sin(2\omega_0)=0,$$

$$(\rho_*-1)=\rho=\rho_*=(\cos(\omega_0)-1)^2+(\omega_0+\sin(\omega_0))^2,$$

$$\rho_*=1\pm\sqrt{(\cos(\omega_0)-1)^2+(\omega_0+\sin(\omega_0))^2},$$

$$\frac{\omega_0+\sin(\omega_0)}{\cos(\omega_0)-1}=\frac{\sin(2\omega_0)}{\cos(2\omega_0)}, \quad \omega_0=(\cos(\omega_0)-1)tan(2\omega_0)-\sin(\omega_0)$$

Now, we are left with the condition $d(Re(\lambda))/d\rho|_{\rho}\neq 0$.
Let $\lambda=k(\rho)+i\omega(\rho)$, using (2.6) we get

$$k+i\omega+1-e^{-k-i\omega}+(\rho-1)e^{-2(k-i\omega)}=0,$$

then,

$$k+1-e^{-k}\cos\omega+(\rho-1)e^{-2k}\cos(2\omega)=0, \tag{2.7}$$

$$\omega+e^{-k}\sin\omega-(\rho-1)e^{-2k}\sin 2\omega=0. \tag{2.8}$$

Differentiate (2.7) and (2.8) with respect to ρ, we obtain

$$\frac{dk}{d\rho}+e^{-k}\cos(\omega)\frac{dk}{d\rho}+e^{-k}\sin(\omega)\frac{d\omega}{d\rho}-2(\rho-1)e^{-2k}\cos(2\omega)\frac{dk}{d\rho}+e^{-2k}\cos(2\omega)-2(\rho-1)e^{-2k}\sin(2\omega)\frac{d\omega}{d\rho}=0,$$

$$\frac{d\omega}{d\rho}-e^{-k}\sin(\omega)\frac{dk}{d\rho}+e^{-k}\cos(\omega)\frac{d\omega}{d\rho}-e^{-2k}\sin(2\omega)+2(\rho-1)e^{-2k}\sin(2\omega)\frac{dk}{d\rho}-2(\rho-1)e^{-2k}\cos(2\omega)\frac{d\omega}{d\rho}=0.$$

Solving for $dk/d\rho$, we obtain

$$\frac{d(\mathrm{Re}(\lambda))}{d\rho}\Big|_{\rho=\rho_*}=\frac{dk}{d\rho}\Big|_{k=0,\omega=\omega_0,\rho=\rho_*}$$

$$=\frac{2(\rho_*-1)-\sin(2\omega_0)\sin(\omega_0)-\cos(2\omega_0)-\cos(2\omega_0)\cos(\omega_0)}{[1+\cos(\omega_0)-2(\rho_*-1)\cos(2\omega_0)]^2+[\sin(\omega_0)-2(\rho_*-1)\sin(2\omega_0)]^2]}$$

$$=\frac{2(\rho_*-1)-\cos(\omega_0)-\cos(2\omega_0)}{[1+\cos(\omega_0)-2(\rho_*-1)\cos(2\omega_0)]^2+[\sin(\omega_0)-2(\rho_*-1)\sin(2\omega_0)]^2]}.$$

Using (2.7) at $k=0,\ \rho=\rho_*,\ \omega=\omega_0$, we can get

$$\frac{d(\mathrm{Re}(\lambda))}{d\rho}\Big|_{\rho=\rho_*}=\frac{\rho_*[2-\cos(2\omega_0)]-3}{[1+\cos(\omega_0)-2(\rho_*-1)\cos(2\omega_0)]^2+[\sin(\omega_0)-2(\rho_*-1)\sin(2\omega_0)]^2]}.$$

It is clear that for $0<\rho_*<1$ and $\rho_*[2-\cos(2\omega_0)]<3$, $d(\mathrm{Re}(\lambda))/d\rho\,|_{\rho=\rho_*}\neq 0$.

Then at $\rho_*=1-\sqrt{(\cos(\omega_0)-1)^2+(\omega_0+\sin(\omega_0))^2}$, the condition $d(\mathrm{Re}(\lambda))/d\rho\,|_{\rho}\neq 0$ is satisfied.

2.3. The discretized system

The delay equation (1.1) can be writen as

$$\frac{dx}{dt}=-x(t)+\rho x(t-1)(1-y(t-1)), \tag{2.9}$$

$$y(t)=x(t-1), \tag{2.10}$$

with $x(t)=x_0,\ t\le 0$.

Step method is used to get a discretized analogue of the system (2.9)- (2.10) as follows[24]:

1-Let $t\in(0,1]$, then

$$y_1(t)=x_0,$$

$$x_1(t)=x_0e^{-t}+\rho\int_0^t e^{s-t}x(s-1)(1-(y(s-1)))ds,$$

$$=x_0e^{-t}+\rho x_0(1-y_1)(1-e^{-t}).$$

Let $t=1$, then

$$y_1(1)=x_0,$$

$$x_1(1)=x_0e^{-1}+\rho x_0(1-y_1)(1-e^{-1}).$$

2-For $t\in(1,2]$, when $t\le 1$, take $x(t)=x_1=x_1(1)$, $y_1(t)=y_1(1)=y_1$.
Then,

$$y_2(t)=x_1(t),$$

$$x_2(t)=x_1e^{-(t-1)}+\rho\int_1^t(e^{s-t})x_1(1-(y_1))ds,$$

$$=x_1e^{-(t-1)}+\rho x_1(1-y_1)(1-e^{-(t-1)}).$$

Let $t=2$, then

$$y_2(1)=x_1,$$

$$x_2(2)=x_1(1)e^{-1}+\rho x_1(1)(1-y_1(1))(1-e^{-1}).$$

2-For $t\in(2,3]$, when $t\le 2$, take $x(t)=x_2=x_2(2)$, $y_2(t)=y_2(2)=y_2$.
Then,

$$y_3(t)=x_2(t),$$

$$x_3(t)=x_1e^{-(t-1)}+\rho\int_2^t(e^{s-t})x_2(1-(y_2))ds,$$

$$=x_2e^{-(t-2)}+\rho x_1(1-y_2)(1-e^{-(t-2)}).$$

Let $t=3$, then

$$y_3(3)=x_2,$$

$$x_3(3)=x_2e^{-1}+\rho x_2(1-y_2(1))(1-e^{-1}).$$

Repeating this process, we get

$$y_{n+1}(t)=x_n(t),$$

$$x_{n+1}(t)=x_ne^{-(t-n)}+\rho x_n(1-y_n)(1-e^{-(t-n)}).$$

Let $t=n+1$, then

$$y_n=x_n,$$
$$x_{n+1}=x_ne^{-1}+\rho x_n(1-y_n)(1-e^{-1}). \tag{2.11}$$

2.4. Local stability and bifurcation analysis of the discretized system

The system (2.11) has two fixed points $(x_1^*,y_1^*)=(0,0)$, $(x_2^*,y_2^*)=(1-1/\rho,1-1/\rho)$.

(1) At $(x_1^*,y_1^*)=(0,0)$:

The Jacobian matrix at $(x_1^*,y_1^*)=(0,0)$ reads

$$J(0,0)=\begin{pmatrix}e^{-1}+\rho(1-e^{-1}) & 0\\ 1 & 0\end{pmatrix}.$$

The characteristic equation $\lambda^2-\lambda(e^{-1}+\rho(1-e^{-1}))=0$ has two roots $\lambda_1=0$ and $\lambda_2=e^{-1}+\rho(1-e^{-1})$.
We can see that $\lambda_2=1$ at $\rho=1$, $\lambda_2>1$ for $\rho>1$ and $\lambda_2<1$ when $\rho<1$, then we have
Proposition 1 $(x_1^*,y_1^*)=(0,0)$ is a saddle if $\rho>1$, a sink if $\rho<1$, a nonhyperbolic if $\rho=1$.

The bifurcation due to the existence of an eigenvalue $\lambda=1$ is called a fold bifurcation and its condition implies that $det\left(J\left(0,0,\rho^*\right)-I_2\right)=0$, where I_2 is the unit 2x2 matrix [25].

Lemma 2 If $\rho=1$, then system (2.11) admits a fold bifurcation at $\left(x_1^*,y_1^*\right)=(0,0)$.

Proof. The condition

$$det\left(J\left(0,0,\rho^*\right)-I_2\right)=0$$

gives

$$det\begin{pmatrix} e^{-1}+\rho^*\left(1-e^{-1}\right)-1 & 0 \\ 1 & -1 \end{pmatrix}=0,$$

$$1-e^{-1}-\rho^*\left(1-e^{-1}\right)=0, \quad \rho^*\left(1-e^{-1}\right)=1-e^{-1},$$

then $\rho^*=1$.

(2) At $\left(x_2^*,y_2^*\right)=(1-1/\rho,1-1/\rho)$:

The Jacobian matrix at $\left(x_2^*,y_2^*\right)=(1-1/\rho,1-1/\rho)$ reads $J(1-1/\rho,1-1/\rho)=\begin{pmatrix} 1 & -(\rho-1)\left(1-e^{-1}\right) \\ 1 & 0 \end{pmatrix}$.

The characteristic equation reads

$$\lambda^2-\lambda+(\rho-1)\left(1-e^{-1}\right)=0.$$

Lemma 3 [26] Let $f(\lambda)=\lambda^2+p\lambda+q$. If $F(1)>0$, and $f(\lambda)=0$ has two roots λ_1, λ_2, then

1. $|\lambda_1|<1$ and $|\lambda_2|<1$ if and only if $f(-1)>0$ and $q<1$;
2. $|\lambda_1|<1$ and $|\lambda_2|>1$ (or $|\lambda_1|>1$ and $|\lambda_2|<1$) if and only if $f(-1)<0$;
3. $|\lambda_1|>1$ and $|\lambda_2|>1$ if and only if $f(-1)>0$ and $q>1$;
4. $\lambda_1=-1$ and $|\lambda_2|\neq 1$ if and only if $f(-1)=0$ and $p\neq 0,2$;
5. λ_1 and λ_2 are complex and $|\lambda_{1,2}|=1$ if and only if $p^2-4q<0$ and $q=1$.

Proposition 2 $\left(x_2^*,y_2^*\right)=(1-1/\rho,1-1/\rho)$ is sink if $1<\rho<\dfrac{2}{1-e^{-1}}$ and is source if $\rho>\dfrac{2}{1-e^{-1}}$.

The bifurcation due to the existence of $\lambda_{1,2}=e^{\pm i\theta_0}$, $0<\theta_0<\pi$ is called a Neimark-Sacker bifurcation [25].

Lemma 4 If $\rho=1+\dfrac{1}{1-e^{-1}}$, then system (2.11) admits a Neimark-Sacker bifurcation at $\left(x_1^*,y_1^*\right)=(0,0)$.

Proof. The characteristic equation

$$\lambda^2-\lambda+(\rho-1)\left(1-e^{-1}\right)=0$$

has two roots

$$\lambda_{1,2}=\frac{1\pm\sqrt{1-4(\rho-1)\left(1-e^{-1}\right)}}{2}. \tag{2.12}$$

We can see that, when $1-4(\rho-1)\left(1-e^{-1}\right)<0$, the two roots are complex.

Then for $\rho>1+\dfrac{1}{4\left(1-e^{-1}\right)}$, we can write $\lambda_{1,2}=\dfrac{1\pm i\sqrt{4(\rho-1)\left(1-e^{-1}\right)-1}}{2}$.

Suppose that $\lambda_{1,2}=e^{\pm i\theta_0}$, $0<\theta_0<\pi$, for some parameter value $\rho=\rho^*>1+\dfrac{1}{4\left(1-e^{-1}\right)}$, then

$$\lambda_1\lambda_2=\frac{1-1+4\left(\rho^*-1\right)\left(1-e^{-1}\right)}{4}=1, \quad \left(\rho^*-1\right)\left(1-e^{-1}\right)=1, \quad \rho^*=1+\frac{1}{1-e^{-1}}. \tag{2.13}$$

Thus at $\rho=\rho^*=1+\dfrac{1}{1-e^{-1}}$, we have $\lambda_{1,2}=e^{\pm\frac{i\pi}{3}}$ and the system admits a Neimark-Sacker bifurcation.

2.5. Numerical simulations

In order to verify the theoretical results that we obtained, we carry out numerical simulations.
Figure 1 verifies the analysis obtained in Section 2.4 by the bifurcation and Lyapunov exponent diagrams. Figure 2 illustrates the complex dynamics of (2.11) by giving phase portraits at values of ρ at which the map is chaotic.

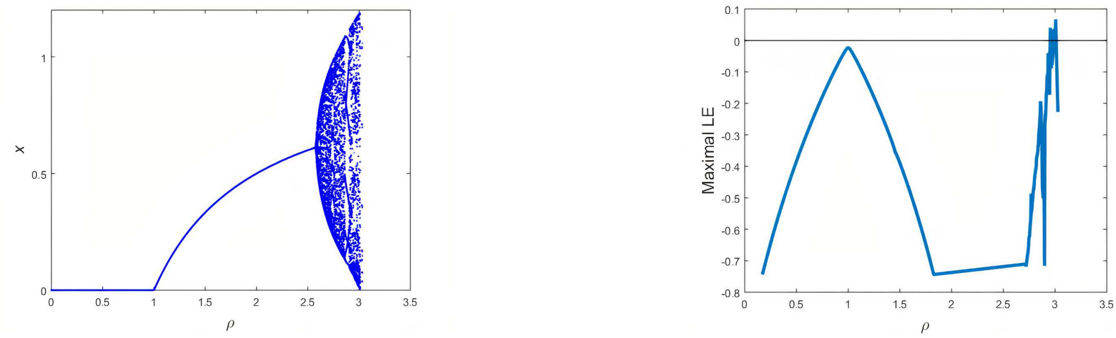

Fig. 1. Bifurcation and Lyapunov exponent diagrams

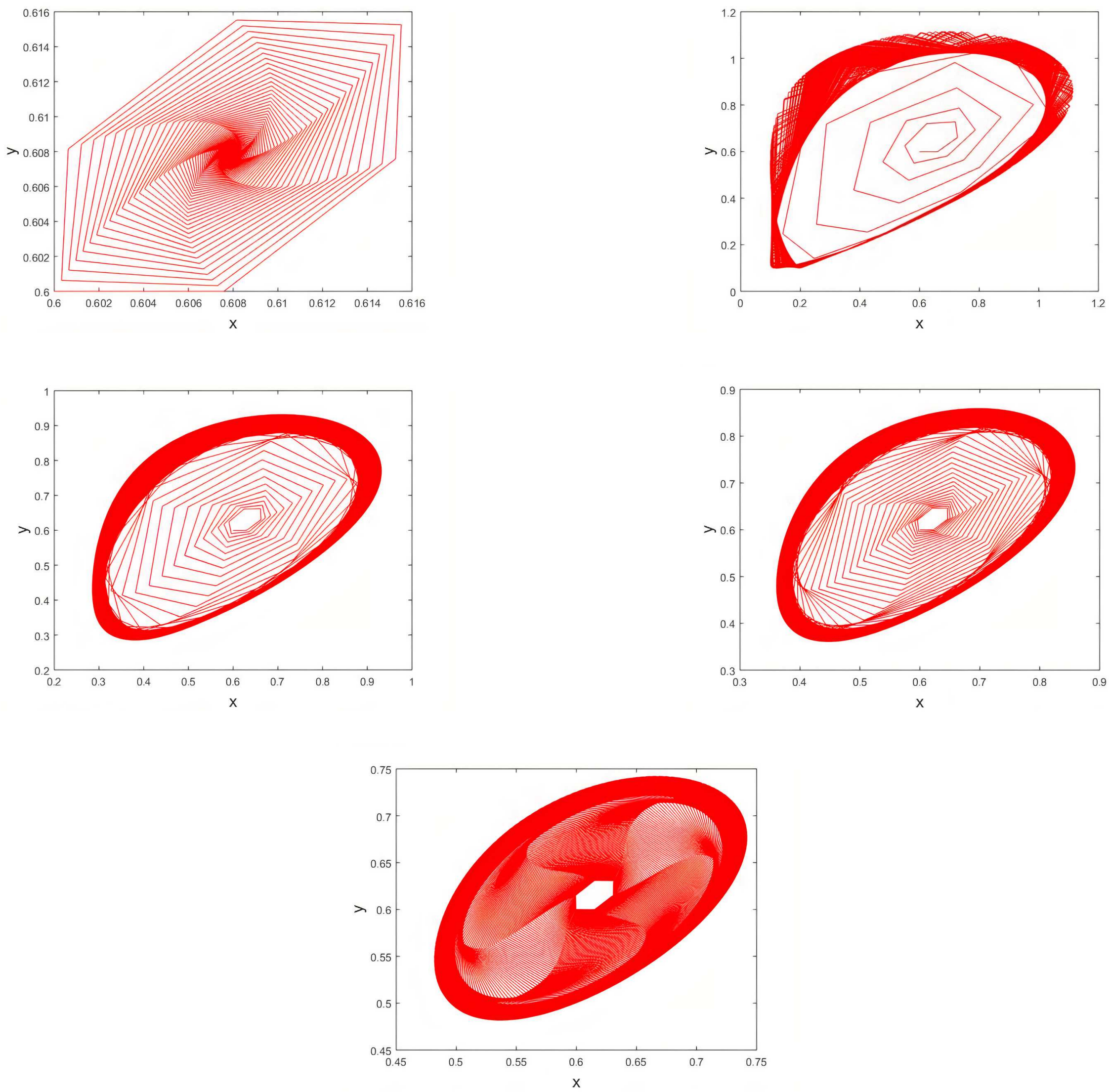

Fig. 2. The phase portraits at values of ρ

3. Conclusion

In this paper, the dynamics of the logistic DDs with two delays are studied. After obtaining the fixed points, the local stability is investigated. Secondly, we show that the considered system admits the Hopf bifurcation. We obtained a discrete analogue of the considered system by applying the method of steps. After obtaining the fixed points of the discretized system, the local stability is investigated. The discretized system admits fold and Neimark-Sacker bifurcations. Finally, numerical simulations is performed to verify the theoretical analysis obtained and to illustrate complex dynamics of the system.

Author Contributions

All authors are equally contributed to the paper.

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