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Research Paper

Application of Complex Functional of Quality in Optimal Control of Spacecraft Motion

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Abstract. The problem of optimal control of the reorientation of a spacecraft as a solid body from an arbitrary initial position into a prescribed final angular position is considered and solved. The case is studied in detail when the minimized index combines, in a given proportion, the integral of modulus of angular momentum and duration of maneuver. It is proved that the accepted optimality criterion guarantees the motion of a spacecraft with modulus of angular momentum not exceeding the required value. Formalized equations and expressions for the synthesis of the optimal rotation program are obtained using quaternion models. It is shown that the optimal solution corresponds to the strategy "acceleration - rotation with constant modulus of angular momentum-braking", the angular momentum and the controling moment are perpendicular during optimal rotation between acceleration and braking. On the basis of necessary optimality conditions, the main properties, laws, and key characteristics (parameters, constants, integrals of motion) of the optimal solution of the control problem, including the turn time and the maximum angular momentum for the optimal motion, are determined. An estimation of the influence of the bounded controling moment on the character of the optimal motion and on the indicators of quality is made. The construction of an optimal control program of rotation is based on the quaternion variables and Pontryagin's maximum principle. The value of maximal angular momentum magnitude is calculated by condition of transversality. The designed method is universal and invariant relative to the moments of inertia. For dynamically symmetric spacecraft, a complete solution of the reorientation problem in closed form is presented. An example and results of mathematical modeling of the motion of a spacecraft under optimal control are presented, demonstrating the practical feasibility of the method for controlling spacecraft's spatial orientation.

Keywords: Spacecraft attitude, Quaternion, Optimal control, Criterion of quality, Maximum principle, Transfersality conditions.

1. Introduction

The optimal control problem of transferring the spacecraft into the required angular position from arbitrary initial attitude was solved. Spacecraft motion around the center of mass is given by quaternion of attitude [1]. Designing the optimal rotation program is based on quaternion models, Pontryagin's maximum principle, and universal variables [2]. Now, spacecrafts are used in many areas of scientific occupations and industry. In particular, astrophysical researches and other scientific discoveries would be impossible without spacecrafts [3-5]. For example, in April 2018, NASA launched the Transiting Exoplanet Survey Satellite (TESS), a space telescope that helps to study the exoplanets, or NASA's James Webb Space Telescope measuring atmospheric properties and compositions of small planets, and also Atmospheric Remote-sensing Infrared Exoplanet Large-survey (ARIEL) space telescope of European Space Agency [3]. Success of mission and duration of performance in a working point of orbit (orbital position) are determined by successful control of motion, by an efficiency of attitude control (an improved system of spacecraft attitude is especially important for the spacecrafts with instruments and devices for astronomy measurements and for satellites of Earth supervision).

The problems of the controlled motion of a spacecraft around center of mass has been studied in numerous papers [1-2, 6-26]; a controlling an angular motion of a spacecraft in most detailed form is presented only for the flat rotations around one of principal central axes of inertia [6] and spatial rotations of a spherically symmetric body [1]. The diverse methods are used for constructing control program of spacecraft orientation (in particular, the concept of inverse problem of dynamics [7-8] or the algorithm of fuzzy logic [9]). The finding the optimal solution of spacecraft's motion control is known also [10-21]. Time-optimal maneuvers are more popular [11-18]. Some solutions are obtained for axisymmetric spacecraft [15, 16, 19] (some authors did replacement of variables and reduced an initial control problem to reorientation problem for spheric-symmetrical body [19]). Terminal control for orbital orientation of a spacecraft are known also [22]. For special case of formulation of reorientation problem, dynamical control problem was solved for a spacecraft with arbitrary moments of inertia, and analytical solution together with constructive scheme of solving the boundary-value problem of the maximum principle was presented [20]. Attitude control of the spacecrafts with inertial actuators has specific features [23-25], and the patented method is known [26].

Below we investigate the problem of optimal control of a spacecraft (as solid body) in course of spatial turn from an arbitrary initial attitude into a given final angular position. Difference of the presented research work is use of new criterion of quality

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which limits magnitude of spacecraft's angular momentum during slew maneuver by known value. The kinematic problem of a turn was solved completely. Also, we considered optimal turn when initial and final angular velocities are zero and the controlling torque is restricted. We estimated how the duration of gaining and suppression of angular momentum influences general costs and turn's time. Issues of cost-efficiency for spacecraft motion control are still relevant and topical today, so the turn problem solved in the paper is practically important.

2. Equations of Angular Motion and Formulation of Control Problem

Angular motion of a spacecraft as a rigid body is described by dynamic equations [6]:

$$J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 = M_1 , \quad J_2\dot{\omega}_2 + (J_1 - J_3)\omega_1\omega_3 = M_2 , J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2 = M_3 \tag{1}$$

where J_i are the central principal moments of inertia of a spacecraft, M_i are projections of torque **M** onto the principal central axes of the spacecraft's inertia ellipsoid, ω_i are projections of the spacecraft's absolute angular velocity vector **ω** onto the axes of the body basis **E** formed by the principal central axes of spacecraft's inertia ellipsoid ($i = \overline{1,3}$). Angular position of a spacecraft is described by the known equation [1]

$$2\dot{\Lambda} = \Lambda \circ \boldsymbol{\omega} \tag{2}$$

where **ω** is the vector of absolute angular velocity of a spacecraft; Λ is the quaternion reflected spacecraft attitude relative to inertial basis I (we assume $\|\Lambda(0)\| = 1$). The quaternion Λ, specifying the current orientation of a spacecraft, is assumed to be normalized [1] for convenience. Equation (2) has the boundary conditions $\Lambda(0)=\Lambda_{in}$ and $\Lambda(T)=\Lambda_f$. The quaternions Λ_{in} and Λ_f that specify directions of spacecraft axes at initial and final moments of time have any arbitrary a priori given values satisfying the condition $\|\Lambda_{in}\| = \|\Lambda_f\| = 1$. Spacecraft motion is considered optimum if the value

$$G = T + k\int_0^T (J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2)dt \tag{3}$$

is minimal, where k=constant > 0 is constant positive (non-zero) coefficient; T is the time of termination of the turn. Optimal control of a spatial turn consists in transfer of a spacecraft from position Λ_{in} into position Λ_f according to the equation (2) with the minimum value of the functional (3). The chosen criterion of optimality combines (in given proportion) the time and the integral of square of angular momentum modulus. The accepted criterion of optimality guarantees spacecraft's motion with angular momentum which does not exceed the required magnitude. A finding an economical control are topical now.

3. Solving the Problem of the Optimal Control

When determining the optimal law of motion with respect to criterion (3), it is assumed that angular velocity **ω**(*t*) is a piecewise continuous function of time. The accepted functional (3) does not contain (in an explicit form) components M_i of the moment of forces. Therefore the projections of angular velocity ω_i ($i = \overline{1,3}$) are considered as the controlling variables (controls). For solving the formulated problem, we use Pontryagin's maximum principle [28] and the universal variables [2] (because the optimized functional does not include positional coordinates). For the formulated above optimization problem, the Hamiltonian is

$$H = r_1\omega_1 + r_1\omega_1 + r_1\omega_1 - k(J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2) - 1$$

where r_i are the universal variables (as the components of vector **r**) which satisfy the equations [2]

$$\dot{r}_1 = \omega_3 r_2 - \omega_2 r_3 , \quad \dot{r}_2 = \omega_1 r_3 - \omega_3 r_1 , \quad \dot{r}_3 = \omega_2 r_1 - \omega_1 r_2 , \tag{4}$$

The Hamiltonian H is written, ignoring the constraint $\|\Lambda\| = 1$ since $\|\Lambda(t)\| \equiv 1$ under any **ω**(*t*) for equations (2) (of course, $\|\Lambda_{in}\| = \|\Lambda_f\| = 1$). The optimal function **r**(*t*) is computed by the attitude quaternion Λ(*t*) using the following formulas [1, 2]:

$$\mathbf{r} = \tilde{\Lambda} \circ \mathbf{c}_E \circ \Lambda , \text{ where } \quad \mathbf{c}_E = \text{const} = \Lambda_{in} \circ \mathbf{r}(0) \circ \tilde{\Lambda}_{in}$$

For the vector **r** of universal variables $|\mathbf{r}| = \text{const} \neq 0$. The function H is maximal if the relations

$$\omega_i = r_i/(2kJ_i^2) \tag{5}$$

are satisfied (note that k≠0 in the index (3)). As is known, the functions r_i and ω_i should satisfy the conditions of transversality which are $\mathbf{r}(0)\neq 0$, $\mathbf{r}(T)\neq 0$ (since left and right endpoints of the trajectory Λ(*t*) are fixed) and H=0 because the maneuver end time T is not fixed and the Hamiltonian H is independent of time in explicit form [29]. After substitution equations (5) in expression for H and the requirement H=0, we have the equation

$$(r_1^2 / J_1^2 + r_2^2 / J_2^2 + r_3^2 / J_3^2) / 4k - 1 = 0$$

through which we obtain the following key properties of the controlled motion:

$$r_1^2 / J_1^2 + r_2^2 / J_2^2 + r_3^2 / J_3^2 = \text{const} = 4k , \quad J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2 = \text{const} \quad , \quad J_1^4\omega_1^2 + J_2^4\omega_2^2 + J_3^4\omega_3^2 = \text{const} \tag{6}$$

Last property follows directly from the demands (5) (they formalize condition of maximum for the Hamiltonian H). The condition of transversality H=0 is valid at each instant of time [29].

The problem of optimal control is reduced to finding the solution to the system of differential equations (2), (4) under the condition that the control $\boldsymbol{\omega}$ is chosen basing on the condition (5) with the simultaneous satisfaction of the condition of transversality H=0 and the boundary conditions $\Lambda(0)=\Lambda_{in}$, $\Lambda(T)=\Lambda_f$ (the conditions of transversality $\mathbf{r}(0)\neq 0$ and $\mathbf{r}(T)\neq 0$ are satisfied automatically, as it follows from first equality (6) written for optimal motion). The system of differential equations (4) for the variables r_i, together with the requirement of maximizing the Hamiltonian H and the condition H=0, provides the necessary optimality conditions.

We remind that the coefficient $k\neq 0$. If we take the ort $\mathbf{p}=\mathbf{r}/|\mathbf{r}|$ then

$$r_0 = 2\sqrt{k} / C \, ; |\mathbf{L}| = \sqrt{1/k}$$

where $r_0=|\mathbf{r}|$; $C=\sqrt{p_{10}^2/J_1^2+p_{20}^2/J_2^2+p_{30}^2/J_3^2}$; p_{i0} are the components of the vector $\mathbf{p}_0=\mathbf{p}(0)$; $\mathbf{L}$ is angular momentum. For the components p_i of the vector $\mathbf{p}$, we have the following equations [2]

$$\dot{p}_1=\omega_3 p_2-\omega_2 p_3\,,\quad \dot{p}_2=\omega_1 p_3-\omega_3 p_1\,,\quad \dot{p}_3=\omega_2 p_1-\omega_1 p_2 \tag{7}$$

and $p_1^2/J_1^2+p_2^2/J_2^2+p_3^2/J_3^2=\text{const}$, since $|\mathbf{r}|$= const. [2].

The equations for optimal angular velocities ω_i can be formalized in following form:

$$\dot{\omega}_1=\omega_2\omega_3(J_2^2-J_3^2)/J_1^2\,,\quad \dot{\omega}_2=\omega_1\omega_3(J_3^2-J_1^2)/J_2^2\,,\quad \dot{\omega}_3=\omega_1\omega_2(J_1^2-J_2^2)/J_3^2 \tag{8}$$

because the desired solution $\boldsymbol{\omega}(t)$ satisfies the conditions (4), (5) during optimal turn.

The boundary-value problem of the maximum principle is to determine such value of the vector $\mathbf{p}_0$ at which the solution $\Lambda(t)$ of the motion equation (2) and differential equations (4) (with the simultaneous satisfying the equalities (5) at each instant of time) satisfies the maneuver's conditions $\Lambda(0)=\Lambda_{in}$ and $\Lambda(T)=\Lambda_f$ (the quantity r_0 is calculated unambiguously by the vector $\mathbf{p}_0$ and the coefficient k). Optimal vector $\mathbf{p}_0$ is determined only by the values Λ_{in}, Λ_f and J_1, J_2, J_3.

If there were no restrictions on the moment of the forces, then optimal rotation of a spacecraft (in the sense of the criterion (3)) satisfy to the equations (7) and

$$\omega_i = b\, p_i / J_i^2\,, \tag{9}$$

b>0 is scalar value (from (7) we show that the vector $\mathbf{p}$ is immovable relative to inertial basis I); $|\mathbf{p}|$ = 1. Angular momentum $\mathbf{L}$ and the value b are connected by the formula

$$\mathbf{L}^2 = b^2(\,p_1^2/J_1^2+p_2^2/J_2^2+p_3^2/J_3^2\,)$$

In case of unlimited moments M_i at entire interval of motion $0<t<T$ spacecraft rotates with $|\mathbf{L}|$=const.; optimal turn of a spacecraft is carried out with constant modulus of angular momentum L_m. The optimal vectors $\boldsymbol{\omega}$ and $\mathbf{p}$ are connected by the relation

$$\omega_i = \frac{L_m p_i}{J_i^2\sqrt{p_1^2/J_1^2+p_2^2/J_2^2+p_3^2/J_3^2}} \qquad (i=\overline{1,3}), \tag{10}$$

at satisfying the conditions $\Lambda(0)=\Lambda_{in}$, $\Lambda(T)=\Lambda_f$ for solution $\Lambda(t)$ to equation (2), where L_m>0 is the magnitude (modulus) of angular momentum with which the turn of a spacecraft is made.

The optimal angular velocity $\boldsymbol{\omega}$ is related to the orientation quaternion Λ by the equality

$$\boldsymbol{\omega} = J^{-2} L_m\, \tilde{\Lambda} \circ \mathbf{c}_P \circ \Lambda / C$$

where $\mathbf{c}_P$=const. = $\Lambda_{in} \circ \mathbf{p}_0 \circ \tilde{\Lambda}_{in}$; J = diag (J_1, J_2, J_3) is the inertia tensor of a spacecraft.

Optimal control of a spatial turn consists of a short-term imparting the initial conditions of motion (the calculated angular velocity) to the spacecraft at the beginning of a turn, maintenance of spacecraft rotation with demanded (programmed) angular velocity $\boldsymbol{\omega}(t)$ at which the modulus of angular momentum of a spacecraft has constant value $|\mathbf{L}|$=const., and a short-term suppressing of available angular velocity to zero at the moment of time $t=T$, when $\Lambda(t)=\Lambda_f$ (at achievement by spacecraft of final position Λ_f). Key problem is the finding the law of variation of the vector $\mathbf{p}(t)$ and that at result of solving the system of the equations (2), (7), (10) with initial condition $\Lambda(0)=\Lambda_{in}$, the boundary condition $\Lambda(T)=\Lambda_f$ was satisfied at right endpoint; the value L_m is known $L_m=\sqrt{1/k}$ (the determination of the vector $\mathbf{p}(0)$ is a separate and rather complicated problem).

For reorientation maneuver, very important characteristic is integral

$$S = \int_0^T |\mathbf{L}(t)|\, dt \tag{11}$$

The value of characteristic S is determined only by the rotation conditions Λ_{in}, Λ_f and the spacecraft's principal central moments of inertia J_1, J_2, J_3. If time of reaching the calculated angular velocity which is equal

$$\omega_{i\ nom} = \frac{p_{i0}}{J_i^2 C\sqrt{k}}$$

and duration of suppressing the angular velocity to zero are infinitesimal, then duration of reorientation is $T = S\sqrt{k}$ because modulus of angular momentum between acceleration and braking is $|\mathbf{L}| = \sqrt{1/k}$, where the integral (11) is calculated as

$$S = t_{pr}\sqrt{J_1^2\omega_{1\,nom}^2 + J_2^2\omega_{2\,nom}^2 + J_3^2\omega_{3\,nom}^2}$$

where t_{pr} is the predicted time of achieving the condition $\Lambda(t_{pr})=\Lambda_f$ during rotation from the position $\Lambda(0)=\Lambda_{in}$ with initial angular velocity $\boldsymbol{\omega}(0)=\boldsymbol{\omega}_{nom}\neq 0$ (according to the equations (2), (8) in which $\boldsymbol{\omega}$ satisfies (10)). The value S and the vector $\mathbf{p}_0$, which satisfy optimal motion, are calculated together.

To be minimum index (3), spacecraft motion must satisfy the necessary conditions of optimality. Hence, for kinematically optimal rotation $|\mathbf{L}|$=const (see the dependences (5) together with H=0). Accordingly, for kinematically optimal turn

$$G = T + |\mathbf{L}|^2 T = S\,(1/|\mathbf{L}| + k|\mathbf{L}|) \quad \text{or} \quad G = 2S\sqrt{k}$$

(because $|\mathbf{L}| = \sqrt{1/k}$; $T = S\sqrt{k}$); G is minimal if $k|\mathbf{L}| = 1/|\mathbf{L}|$. Thus, the value G is minimal when the integral S is minimal, but S is minimum for motions (7), (9). Hence, the found control (7), (10) is indeed optimal with respect to criterion (3).

The system of equations (2), (7), and (10) which determines the optimal rotation has an analytical solution in elementary functions only for dynamically symmetric and dynamically spherical bodies. For a spherically symmetric spacecraft (when $J_1=J_2=J_3$), the solution $\mathbf{p}(t)$, $\boldsymbol{\omega}(t)$ have elementary form: $\mathbf{p}(t)$=const. and $\boldsymbol{\omega}(t)$=const., or in detail

$$p_i = \nu_i/\sqrt{\nu_1+\nu_2+\nu_3} \quad , \quad \text{and} \quad \omega_i = \frac{2\nu_i \arccos\,\nu_0}{T\sqrt{\nu_1+\nu_2+\nu_3}}$$

where $\nu_0, \nu_1, \nu_2, \nu_3$ are components of the reorientation quaternion $\Lambda_t = \tilde{\Lambda}_{in} \circ \Lambda_f$ [1]. The characteristic (11) is equal to $S = 2J_1 \arccos\nu_0$.

For a dynamically symmetric spacecraft (when, for example, $J_2=J_3$), the optimal control problem can be solved completely. Optimal solution $\boldsymbol{\omega}(t)$ can be written as follows [27]:

$$\omega_1 = \dot{\alpha} + \dot{\beta}\cos\vartheta\,, \quad \omega_2 = \dot{\beta}\sin\vartheta\,\sin(\dot{\alpha}\,t + \sigma)\,, \quad \omega_3 = \dot{\beta}\sin\vartheta\,\cos(\dot{\alpha}\,t + \sigma)$$

where σ = arctg (p_{20}/p_{30}); ϑ is the angle between the spacecraft's longitudinal axis and the vector $\mathbf{p}$ $(0 \le \vartheta \le \pi)$; $\dot{\alpha}$ is the angular velocity of its own rotation (around the longitudinal axis); and $\dot{\beta}$ is the angular velocity of the precession (around the vector $\mathbf{p}$). The characteristic (11) is equal

$$S = \sqrt{J_1^2(\alpha+\beta\cos\vartheta)^2 + J_2^2\beta^2\sin^2\vartheta}$$

where α and β are the angles of turn around the longitudinal axis and the vector $\mathbf{p}$, respectively (note $p_{10}=\cos\vartheta$). The optimal values of parameters $\mathbf{p}_0$, α, β, and ϑ are determined by the boundary angular positions Λ_{in} and Λ_f through the system of equations [27]

$$\cos\frac{\beta}{2}\cos\frac{\alpha}{2} - p_{10}\sin\frac{\beta}{2}\sin\frac{\alpha}{2} = \nu_0\,, \qquad \cos\frac{\beta}{2}\sin\frac{\alpha}{2} + p_{10}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} = \nu_1\,,$$

$$p_{20}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} + p_{30}\sin\frac{\beta}{2}\sin\frac{\alpha}{2} = \nu_2\,, \quad -p_{20}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} + p_{30}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} = \nu_3$$

concurrently with the condition $J_1^2(\alpha+\beta\cos\vartheta)^2 + J_2^2\beta^2\sin^2\vartheta \to \min$ $(|\alpha| \le \pi\,,\ 0 \le \beta \le \pi)$. The optimal values of the vector $\mathbf{p}_0$ and the angles α, β, and ϑ, satisfying the given boundary values Λ_{in} and Λ_f, can be determined using the known device [30].

4. Construction of a Typical Program of an Optimal Rotation

The problems in which the boundary values $\boldsymbol{\omega}(0)=\boldsymbol{\omega}(T)=0$ (such conditions of a spacecraft turn are most typical) are of practical importance. Of course, at times t=0 and t=T the angular velocity for a nominal program of spacecraft rotation, determined by the equations (10), are not equal to zero. Consequently, transition segments are unavoidable: acceleration of rotation as a transition from the state of rest (when $\boldsymbol{\omega}=0$) to the mode of rotation with an angular momentum of maximum value L_m, and braking, i.e., reduction of the spacecraft's angular momentum down to zero. Between acceleration of rotation and braking, the equations (7) and (10) are satisfied.

If the conditions of turn Λ_{in} , Λ_f , and the time T are such that times of acceleration and braking are very small (in comparison with the total time of turn T) and we may to neglect them, then one can consider as impulsive processes both imparting necessary angular momentum L_m to the spacecraft and reducing available angular momentum down to zero, and almost during all turn (between acceleration and braking) $|\mathbf{L}(t)|$=const=L_m with satisfaction of the equations (7), (10).

If the controlling moment $\mathbf{M}$ is limited, then a boost of spacecraft angular momentum to the required level $|\mathbf{L}|=L_m$ at the beginning of a turn and damping of available angular momentum to zero at the end of reorientation maneuver occupy some finite (distinct from zero) time. In general case, conditions of turn Λ_{in} and Λ_f may be such that one cannot neglect the transition segments (acceleration and braking). Quite often the vector $\mathbf{M}$ obey the condition [1, 26]

$$M_1^2 + M_2^2 + M_3^2 \le m_0^2 \qquad (12)$$

Punctual consecutive implementation of procedure of the maximum principle for dynamical problem of the optimal slew maneuver (when $\omega(0)=\omega(T)=0$ and the control torque $\mathbf{M}$ is limited) shows that maximal modulus of angular momentum is no more $\sqrt{1/k}$ for any instant of time $t \in [0, T]$ (independently of duration of acceleration and braking). I.e. always, during optimal rotation from the position Λ_{in} into the position Λ_f (in the sense of minimum (3)), the modulus $|\mathbf{L}(t)|$ have restriction by known upper level determined by the coefficient k of the minimized functional.

The laws of the fastest imparting and reduction of the angular velocity under the constraint (12) are known [26]. At the segment of acceleration, optimal control has the following form [26]

$$\mathbf{M} = m_0 J\omega / |J\omega|$$

If differentiate by time last equation, taking into account the equations (1), then we will obtain the following equations

$$\dot{M}_1 = \omega_3 M_2 - \omega_2 M_3\,, \quad \dot{M}_2 = \omega_1 M_3 - \omega_3 M_1\,, \quad \dot{M}_3 = \omega_2 M_1 - \omega_1 M_2$$

which shows that $\mathbf{M}$ is constant vector relative to the inertial basis I, and $|\mathbf{M}|=\text{const}=m_0$. At optimal motion, angular momentum of a spacecraft does not change the direction in inertial coordinate system. The magnitude of angular momentum varies according to the law $|\mathbf{L}|=m_0 t$.

At the segment of braking, optimal control is

$$\mathbf{M} = -m_0 J\omega / |J\omega|$$

(the controlling moment $\mathbf{M}$ makes with the angular momentum an angle of 180 degree) [26]. Angular momentum varies according to the law $|\mathbf{L}| = L_{oc} - m_0 (t - t_{br})$, where $L_{oc}=|J\omega(t_{br})|$; t_{br} is time of beginning of damping. For both acceleration and braking, optimal control (as fast response) is control under which the controlling moment is parallel to an angular momentum of at any moment of time.

The optimal motion of a spacecraft consists of segments on which the control moment maximum in magnitude acts (segments of acceleration and braking), and of a segment of rotation with constant (in modulus) angular momentum, equal to the designed value L_m. On the segment of maximal control moment, the angular momentum vector $\mathbf{L}$ has a permanent direction in the inertial space, but it is variable in magnitude (increase up to preset value on the acceleration segment, and decrease to zero on the braking segment), while moment $\mathbf{M}$ is immovable with respect to the reference basis I (the vectors $\mathbf{M}$ and $\mathbf{L}$ are parallel). During the spacecraft rotation with maximum angular momentum the parameters of motion are determined by equations (7), (10). In this case, the angular momentum vector $\mathbf{L}$ has a constant magnitude L_m, but its direction varies from a position preset at spacecraft acceleration to a position required during spacecraft braking. Since initial and final angular velocities are equal zero and the magnitude of control moment is constant $|\mathbf{M}|=\text{const}=m_0$, duration of stages of acceleration and braking will be identical. The optimal solution $\omega(t)$ during segment of nominal motion (between acceleration and braking) possesses the properties (6), the vectors $\mathbf{M}$ and $\mathbf{L}$ are orthogonal, the modulus of angular momentum is maximum and constant $|\mathbf{L}|=\text{const}=L_m$.

For rotations (7), (9), the integral of the modulus of the spacecraft's angular momentum S does not depend from time of turn T [21]. If the durations of the transition periods Δt_{ac} and Δt_{br} (acceleration and braking) are small and $\Delta t_{ac}+\Delta t_{br}<<T$, then the integral of the modulus of the angular momentum during the rotation time T barely changes and remains close to S, and the change of the modulus of the angular momentum during the acceleration and braking can be considered linear. Then we have the equality $(T-(\Delta t_{ac}+\Delta t_{br})/2)L_m=S$, where L_m is the modulus of the angular momentum at the phase of the nominal rotation (when $|\mathbf{L}|=\text{const}$); Δt_{ac} and Δt_{br} are the durations of the acceleration and extinction of the angular momentum. We have $\Delta t_{ac}+\Delta t_{rr}\geq 2L_m/m_0$, since $\tau=L_m/m_0$ is the minimal possible acceleration (braking) time with the restriction $|\mathbf{M}|\leq m_0$. Hence $L_m\geq S/(T-\tau)$ and $T\approx S/L_m+L_m/m_0$. We calculate value (3), taking into account the periods of acceleration and braking

$$G \approx S / L_m + L_m / m_0 + L_m k(S - L_m^2 / 3m_0) \tag{13}$$

(since the times of the acceleration Δt_{ac} and braking Δt_{br} are equal).

The optimal case will be the L_m for which the value G is minimal. Of course, we have $T\geq\Delta t_{ac}+\Delta t_{br}$ for any rotation, and hence $2T-\Delta t_{ac}-\Delta t_{br}=2S/L_m\geq\Delta t_{ac}+\Delta t_{br}>2L_m/m_0$. Therefore, L_m must satisfy the condition $L_m<\sqrt{m_0 S}$. From the necessary condition of the extremum, $dG/dL_m=0$, we obtain the equation with one unknown L_m:

$$L_m^4 k / m_0 - L_m^2(1 / m_0 + kS) + S = 0$$

whose solution is $L_m=\sqrt{1/k}$ (we ignore the solution $L_m=\sqrt{m_0 S}$, since it corresponds to the local maximum of the function (13) and to the situation when $T\leq\Delta t_{ac}+\Delta t_{br}$). We look for the minimum of function (13) (as well as (3)) in the interval $0<L_m<\sqrt{m_0 S}$. Hence $L_{opt}=\sqrt{1/k}$ is the optimal value of the parameter L_m of the optimal rotation program if $\Delta t_{ac}\neq 0$ and $\Delta t_{br}\neq 0$ (if, of course, $\sqrt{1/k}/m_0 << T_{opt}= S\sqrt{k}$ or $m_0 kS>>1$); T_{opt} is duration of ideal turn when $m_0\to\infty$ and $\Delta t_{ac}+\Delta t_{br}\to 0$. The value G is

$$G \approx 2S\sqrt{k} + 2\sqrt{1/k}/3m_0$$

If we assume $k = 1/L_{ad}^2$, then optimization of motion program in accordance with criterion (3) give satisfaction of the inequality $|\mathbf{L}(t)|\leq L_{ad}$ for any instant of time, where L_{ad} is maximal-admissible magnitude of spacecraft's angular momentum. Thus, in our optimization problem, we found the main properties, laws and key characteristics (parameters, constants, integrals of motion) of optimal solution of control problem using the conditions of transversality as very important and unique mathematical instrument. We have demonstrated that chosen criterion of optimality guarantees spacecraft rotation with angular momentum modulus not exceeding the required value.

Thus, the optimal control problem for spatial reorientation of a spacecraft from an arbitrary angular position into a given angular position is considered and solved definitively. An analytic solution of the proposed problem is presented, formal equations and computational expressions for constructing the optimal reorientation program were obtained. To solve the formulated problem, the maximum principle is applied basing on universal variables [2], and use of quaternions significantly simplifies computational procedures and reduces the computational costs of control algorithm, which makes it suitable for onboard realization. The reorientation problem in kinematic statement has been solved completely, in closed form. The main characteristic properties of optimal motion and the type of trajectory, which is optimal with respect to the chosen criterion, were determined. It was demonstrated that ideal optimal solution is control when angular momentum and controlling moment are perpendicular, and the gaining of the calculated value of angular momentum and suppressing the angular momentum are made instantaneously (i.e. impulsively). If the controlling torque is limited, acceleration and braking are made maximally rapidly, as far as the actuators of the spacecraft allow it, and analytical formulas were written for duration of acceleration and braking, and turn's time is found also. Direction of spacecraft's angular momentum is constant in the inertial coordinate system within segments of acceleration and braking. The universality of the designed control method is proved by the following factors: it does not depend from actuators type; mass and size of a spacecraft; configuration and distribution of spacecraft's masses; altitude of working orbit (and from others, for example, from periodicity of reorientation, angle of a turn).

5. Example of Numerical Solving the Control Problem and Results of Mathematical Modeling

Let us provide a numerical solution of the spacecraft's optimal control problem with respect to a programmed rotation. We consider maneuver from the initial attitude Λ_{in}, when the body axes coincide with the axes of the supporting basis **I**, into the given final position Λ_f with the elements

$$\lambda_0 = 0.258819,\ \lambda_1 = 0.723196,\ \lambda_2 = 0.4,\ \lambda_3 = 0.5$$

Let us assume that the spacecraft's principal central inertia moments have the values:

$$J_1 = 3140 \text{ kg m}^2,\ J_2 = 12000 \text{ kg m}^2, \text{ and } J_3 = 12760 \text{ kg m}^2.$$

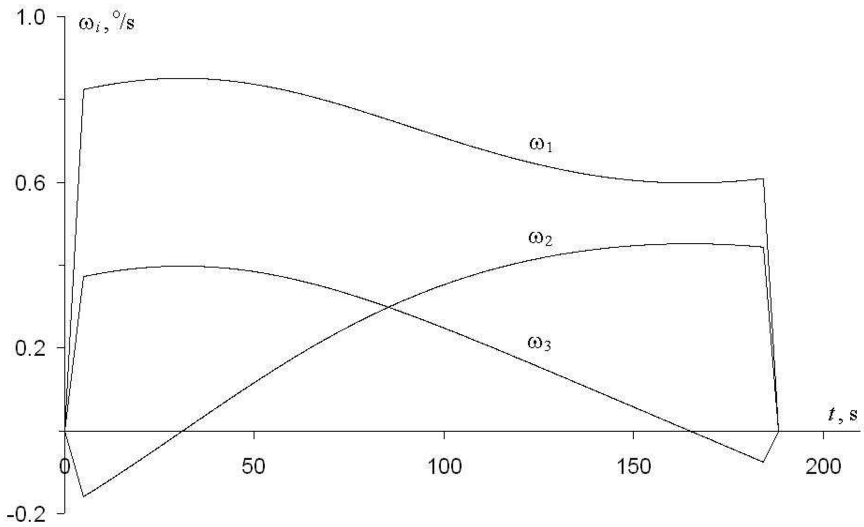

Fig. 1. Optimal variation of the programmed angular velocities

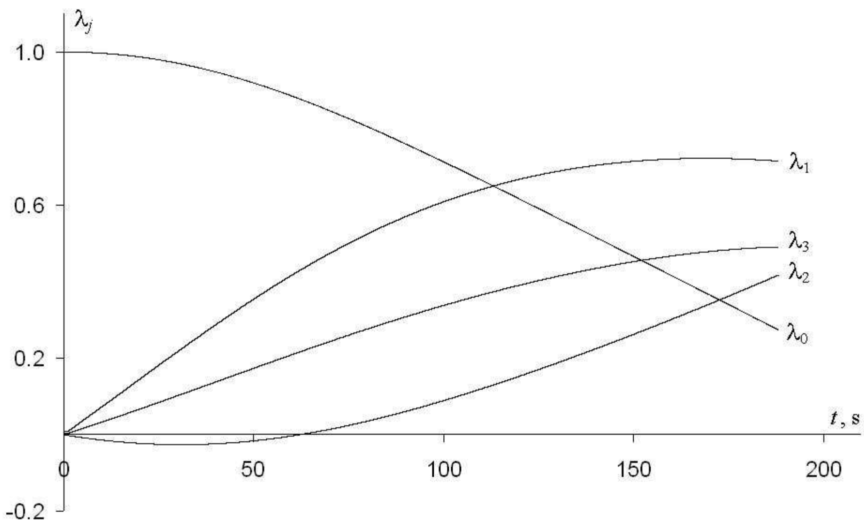

Fig. 2. Variation of the parameters of attitude during optimal maneuver

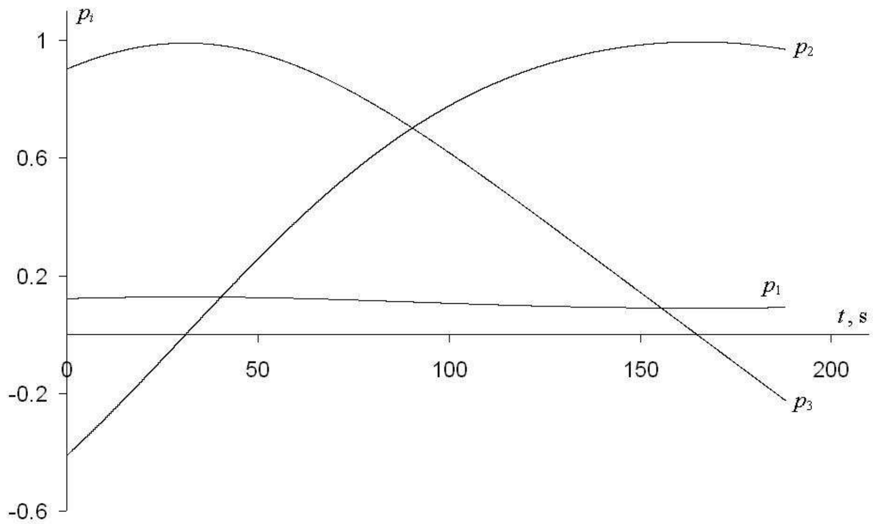

Fig. 3. The components of unit vector **p** as time functions

The coefficient in functional (3) is k=10^{-4} s^2/(kg^2 m^4). As a result of solving the kinematic reorientation problem on the transition from the position $\Lambda(0)=\Lambda_{in}$ into the position $\Lambda(T)=\Lambda_f$ (the optimal rotation problem in the impulse setting), we obtained the value of the vector $\mathbf{p}_0$={0.123191; −0.397949; 0.909099}. The maximal magnitude of the angular momentum (the programmed level) is L_m=100 N m s. The results of the mathematical modeling of the rotation process within the optimal control are shown in Figures 1–3 (for m_0=20 N m). The turn's duration was T=188 s. In Figure 1, we present the graphs of the changing angular velocities $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ in time. In Figure 2, we present the graphs of the changing components of the quaternion $\Lambda(t)$, determining the spacecraft's current attitude during rotation: $\lambda_0(t)$, $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$. Figure 3 shows the dynamics of the changing components $p_1(t)$, $p_2(t)$, and $p_3(t)$ of the vector **p**.

If durations of acceleration and braking are much smaller the duration of turn T, then the torque **M** is directed strictly against angular momentum **L** at spacecraft braking, and the instant when braking begins can be predicted with high accuracy. Duration of rotation damping is $\tau=|\mathbf{L}|/m_0$. The moment of the beginning of braking segment is determined by the condition [25]:

$$4\arcsin\frac{K\sqrt{q_2^2+q_3^2}}{\sqrt{(J_2\omega_2)^2+(J_3\omega_3)^2}}=\frac{K^2\sqrt{\omega_2^2+\omega_3^2}}{m_0\sqrt{(J_2\omega_2)^2+(J_3\omega_3)^2}}$$

where q_j are the components of quaternion of mismatch $\tilde{\Lambda}(t)\circ\Lambda_f$ $(j=0,1,2,3)$; $K=|J\boldsymbol{\omega}|$ is the magnitude of spacecraft's angular momentum. At braking segment, cancellation of angular momentum is carried out according to linear law: $|\mathbf{L}(t)|=L_m-m_0(t-t_{br})$, where t_{br} is the instant of the beginning of braking. Determination of the time instant t_{br} according to actual (the measured values) kinematic parameters of motion (angular mismatch and angular velocity) improves the accuracy of bringing the spacecraft into the required state $\Lambda=\Lambda_f$, $\boldsymbol{\omega}$=0.

The obtained solution of the formulated problem of optimal reorientation can be useful for control of spacecraft attitude by inertial actuators (gyrodynes) since this solution guarantees spacecraft's rotation with angular momentum which does not exceed the required modulus.

6. Conclusion

In this research, new control method of spacecraft attitude was obtained; the used criterion of quality is new and has special form what is principal difference from the known works. The designed method of spacecraft's motion control was described in detail. It was demonstrated that ideal optimal solution is control when angular momentum and controlling moment are perpendicular (during kinematic-optimal turn). Importance and significance of the executed investigations consist in the fact that the chosen criterion of quality bounds magnitude of angular momentum and minimizes reorientation time under this condition. Presence of ready formulas, for synthesis of optimal motion program during a slew maneuver, does the carried out research as practically significant and suitable for direct use in practice of spaceflights. The solved problem is very topical since the designed control algorithm of reorientation maneuver guarantees a motion with modulus of angular momentum not exceeding the required value which is determined by coefficients of the minimized functional. Therefore, time of stopping of rotation is not more a priori known value (because time of braking is proportional to modulus of angular momentum for restriction (12)). A procedure for implementing the control mode is described. Expressions for computing the temporal characteristics of the reorientation maneuver and the condition for finding the deceleration start moment based on factual kinematic motion parameters with use of terminal control principles are presented, it leads to high orientation precision. Estimations of the relative growth in the functional of quality due to the limited controlling moment (with respect to ideal rotation when torque is unbounded) were done. Example and results of mathematical modeling for spacecraft motion under optimal control are given. The obtained results demonstrate that the designed control method of spacecraft's three-dimensional reorientation is feasible in practice. Notice, the recent solutions [16-18] are not applicable for the general case of three-dimensional turn of arbitrary spacecraft; the work [22] describes synthesis of terminal reorientation control only for the spacecraft which moves along a circular orbit. But the method designed in present article is universal control, it does not depend on a ratio (proportion) of moments of inertia or final position of a spacecraft. Importance of the proposed mode of reorientation consists not only in aspect of angular momentum costs but in security sense because rotation with modulus of angular momentum not exceeding the given value allows us to stop rotation of a spacecraft within known duration (it is very topical in different critical situations).

Conflict of Interest

The author declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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