J. Appl. Comput. Mech., 7(2) (2021) 559-577
DOI: 10.22055/JACM.2020.35224.2601

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

A Robust Three-Level Time-Split MacCormack Scheme for Solving Two-Dimensional Unsteady Convection-Diffusion Equation

Eric Ngondiep[1,2]

[1] Department of Mathematics and Statistics, College of Science, Al-Imam, Muhammad Ibn Saud, Islamic University (IMSIU), 90950 Riyadh, Saudi Arabia

[2] Hydrological Research Centre, Institute for Geological and Mining Research, 4110 Yaounde-Cameroon

Received September 29 2020; Revised November 03 2020; Accepted for publication November 03 2020.
Corresponding author: E. Ngondiep (engondiep@imamu.edu.sa; ericngondiep@gmail.com)

Abstract. A three-level time-split MacCormack method has been developed for solving the two-dimensional time-dependent convection-diffusion equation. The differential operator "splits" the two-dimensional problem into two pieces so that each subproblem is easy to solve using the original MacCormack procedure. The obtained scheme is temporal second-order convergent and spatial fourth-order accurate. This considerably reduces the computational cost of the algorithm. Under a suitable time-step restriction, both stability and convergence of the proposed technique are analyzed in the $L^{\infty}(0, T; L^2)$-norm. A large set of numerical examples which provide the convergence rate of the new algorithm are presented. Overall, the considered approach is found to lie in the class of robust numerical schemes for solving general systems of partial differential equations.

Keywords: 2D unsteady convection-diffusion equation, One-dimensional operators (splitting), Explicit MacCormack scheme, Explicit time-split method, Stability and convergence rate.

1. Introduction and motivation

Bolus dispersion and time-dependence can be more easily implemented using the Eulerian approach. The general concept of an Eulerian model is to solve a convection-diffusion equation for the aerosol in an idealized version of the lung geometry, using ideas first developed for modeling gas transport in the lung [1,2]. The aerosol not only is convected through the lung by the air motion but also diffuses relative to the air due to Brownian motion [3, 4]. As with the Lagrangian models, purely Eulerian deposition models that simultaneously treat the entire respiratory tract are restricted to one or two spatial dimension due to computational cost, with depth into the lung being the spatial dimension that is used. The one-dimensional models can be derived by considering the equation for mass conservation of the aerosol particles. To illustrate this situation, we introduce a coordinate system (x, y, z) with x representing depth into the lung and y, z, denoting the cross section of the air-filled portions of all parts of the lung at a depth x. Furthermore, considering a short section of lung from depth x to depth $x + dx$, integrating the first term on the left-hand side over the two spatial dimensions y and z and breaking up the other two terms into integrations over two types of surfaces: airway surfaces S_{yz} and airway lumen S_x, we obtain the following figure.

In this work, a numerical model for computation of convection-diffusion transport of nonconservative substances in two-dimensional environment is developed. This equation describes the convection and diffusion of various physical quantities, for example, heat, mass and energy [5-8]. The applications of transport equation have been given in [9-11]. Specifically, the two-dimensional transport equation has been used to describe many other physical situations such as heat transfer, dispersion of pollutants in rivers and streams, flows in porous media, spreads of contaminants in coastal seas, thermal pollution in river systems and estuaries [12-16]. The method is based on a three-level time-split explicit MacCormack approach, in which the convection and diffusion terms are calculated separately in one step. Special attention is paid to the convection operator, which introduces some difficulties in many numerical schemes, which in many cases, has proven to cause instability models. Assuming that the initial and boundary conditions u_0 and φ, respectively, are regular enough and satisfy the condition $\varphi(x,y,0) = u_0(x,y)$, for any $(x,y) \in \partial\Omega$, we consider the following two-dimensional convection-diffusion equation,

$$\frac{\partial u}{\partial t} + \mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) - a\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \quad (x,y) \in \Omega, \ t \in (0,T]; \tag{1}$$

subjects to the initial condition

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega; \tag{2}$$

and the boundary condition

$$u(x,y,t) = \varphi(x,y,t); \quad (x,y) \in \partial\Omega, \ t \in (0,T]; \tag{3}$$

Published online: November 23 2020

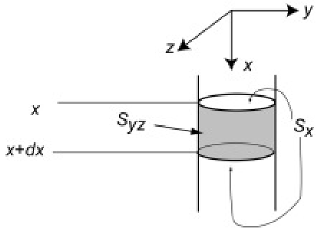

Fig. 1. A portion of a lung airway at depth x in the lung is shown. S_x is the airway lumen surface at depth x, and S_{yz} is the airway surface itself.

where μ and a are two physical parameters, Ω =(0,1) $\times$ (0,1), $\partial\Omega$ is the boundary of Ω.

A major class of transport problems do not have analytical solution due to the complexities involved in their formulation (like nonlinearity). Numerical simulations to these equations have attracted a lot of attention and which have resulted in various numerical techniques including finite difference, finite element, boundary element and time-splitting methods. Since very less works have been carried out to obtain analytical solutions of two-dimensional transport equations therefore a lot of works are needed to develop efficient numerical schemes for solving the model equation (1) – (3). In literature, several finite difference schemes which include predictor-corrector, time-splitting and collocation radial basis functions have been used for solving Poisson equation, time-dependent convection-diffusion and advection-diffusion equation are developed and deeply analyzed (for example, see [17-28]) (for example, see [17-28]).

In some recent works [29-31], the authors proposed a three-level time-split MacCormack method for solving two-dimensional reaction-diffusion and heat conduction equations. The method applies to time-dependent equations of the form: $u_t = A_1(u) + A_2(u)$ where u_t denotes $\partial u/\partial t$ and A_j $(j=1,2)$ are differential operators, so that the subproblems $u_t = A_1(u)$ and $u_t = A_2(u)$ are each solved independently using the explicit MacCormack method. Furthermore, the solution of the original problem is advanced by alternating between (approximately) solving each of the two subproblems. The great advantage of this approach is the decomposition of the errors in the resulting numerical approximation into errors obtained by splitting the exact solution operator and the other one due to numerically solving the subproblems.

The two-step explicit MacCormack scheme has been widely used in numerical solutions of a large class of problems in computational fluid dynamics [32-33]. In this technique, the first order time derivative is approximated at the predictor and corrector phases using forward difference representation, with alternate one-side differencing for first-order space derivatives. This is more convenient in the search of numerical solutions of systems of equations with nonlinear convective jacobian matrices using second order one-side explicit schemes, such as Lax-Wendroff approach [34-36]. One difficulty observed from this method is that it is not appropriate for problems with high Reynold numbers, where the viscous regions become very thin. In [37], MacCormack developed a hybrid version of his scheme, so called the MacCormack rapid solver method for overcoming this drawback. Most recently, the authors applied the rapid solver method and a two-level explicit approach for solving a certain class of ordinary/partial differential equations [38-41]. Their results have shown that the hybrid version of MacCormack has a good stability condition and it is too faster than a wide class of numerical schemes for solving steady and unsteady flows at high to low Reynolds numbers [37].

With the information gleaned from the two-step MacCormack and MacCormack rapid solver methods, we are ready to analyze a three-level time-split MacCormack approach applied to the initial-boundary value problem (1)-(3). In [43-46] explicit, implicit and explicit-implicit methods have been analyzed. In some cases, the approach used the explicit scheme and the implicit difference scheme for the solution of the convection and diffusion processes, respectively. In the other cases, the time derivative was approximated using an explicit scheme while the implicit method has been used to approximate the spatial derivatives. The authors [47-49] applied B-splines techniques and time-split methods in conjunction with trigonometric B-splines approach, MacCormack, Crank-Nicolson and Runge-Kutta schemes to obtain solutions with good convergence rate (at least second-order). Unfortunately, almost all the results provided in the above works were obtained by a large set of numerical experiments. The theoretical analysis has not been considered. The following paragraph provides a detail description of our contribution.

A three-level time-split MacCormack approach we study for the two-dimensional convection-diffusion equation (1) – (3) is new, a predictor-corrector method, second order accurate in time, fourth order convergent in space and it is motivated by: (a) the form of the time step restriction $\max\{2ak/h^2, |\mu|k/h\} \leq 1$. In fact, the considered problem (1) – (3) can model both linear hyperbolic and parabolic equations. When a = 0 and $\mu \neq 0$, we obtain a $2D$ wave equation. A wide set of explicit numerical schemes for solving the corresponding equation are stable under the time step limitation $|\mu| k/h \leq 1\$$. For $a \neq 0$ and $\mu = 0$, the obtained equation is a two-dimensional heat conduction equation. It is well known that several explicit schemes for solving this equation is stable under the time step requirement $4ak/h^2 \leq 1$. All these requirements are known in literature as Courant-Friedrich-Lewy condition. (b) a time-split MacCormack approach is more practical (indeed: second order accurate in time and fourth order convergent in space. Furthermore, the explicit MacCormack algorithm is a widely used scheme for solving problem in computational fluid dynamics). (c) Both theoretical and numerical results are deeply analyzed. We recall that an explicit time-split MacCormack algorithm "splits" the original MacCormack scheme into a sequence of one-dimensional operators, thereby achieving a good stability condition. In other words, the splitting makes it possible to advance the solution in each direction with the maximum allowable time step (for instance, see [50-52]). For an overview of this technique, we refer the readers to [25, 52] and references therein. Setting $\Delta t_x = \Delta t$ and $\Delta t_y = \Delta t/2p$ where p is any positive integer, a high order convergent scheme for solving the initial-boundary value problem (1) – (3) can be constructed in the following manner

$$u_{ij}^{n+1} = \left[L_y\left(\frac{\Delta t}{2p}\right)\right]^p L_x(\Delta t)\left[L_y\left(\frac{\Delta t}{2p}\right)\right]^p u_{ij}^n, \tag{4}$$

where $L_x(\Delta t_x)$ and $L_y(\Delta t_y)$ are one-dimensional difference operators whose explicit expressions will be defined in the following. We remind that our study consists to compute approximate solutions of problem (1)-(3) using equation (4) (with $p=1$). For the convenient of writing the notations $u)_{ij}^n = u_{ij}^n$ and $[u+v]_{ij}^n = u_{ij}^n + v_{ij}^n$ are used in this paper.

It is important to mentioning that although the proposed model has the same order of accuracy (stable, temporal second-order and spatial fourth-order convergent) when computing an approximate solution of two-dimensional unsteady convection-diffusion equation (1) – (3) and the problem studied in [25], the considered technique is at least twice more faster when applied to the initial-boundary value problem (1) – (3) than to two-dimensional linear time-dependent convection-diffusion-reaction equation deeply analyzed in [25]. Indeed, the source/sink terms (so called zero-order reaction rate coefficient and first-order reaction rate, respectively) slow the convergence speed of the algorithm. Furthermore, the numerical experiments presented in this work are performed with different values of convective and diffusive coefficients. Also in that case, we observe satisfactory results regarding the accuracy of the proposed procedure. Finally, compared to the numerical evidences presented in [25], some tables indicating the decay of energy and physical tests are considered where the computed solution, the analytical ones and the errors can be visualized by iso-values. This information represents the new contribution of this paper with respect to the analysis developed in [25]. Furthermore, it is worth noticing to recall that the proposed method is fast and requires least computational costs than a broad range of numerical schemes [10, 11, 17, 19, 21, 22, 26, 47].

The aim of this paper is to analyze the following four items:

1. Detailed description of a three-level time-split MacCormack scheme for solving the initial-boundary value problem (1) – (3),
2. Stability analysis of the numerical scheme,
3. Error estimates of the method,
4. Numerical examples which consider the convergence rate, confirms the theoretical results and shows the efficiency and effectiveness of the method.

Items 1-4, represent our original contributions since as far as we know, there is no available work in literature which solves the initial-boundary value problem (1) – (3), using a time-split MacCormack method.

The rest of the paper is organized as follows: Section 2 considers a full description of a three-level time-split MacCormack method for solving the parabolic equations (1) – (3). In section 3, we study the stability of the scheme under the time-step restriction given above, while section 4 presents a deep analysis of the error estimates and the convergence of the method. A wide set of numerical experiments which provide the convergence rate of the new algorithm and confirms the theoretical result (on the stability) are presented in section 5. We draw the conclusion and present our future investigations in section 6.

2. Notation and full description of a three-level time-split MacCormack method

In this section we set notation and provide a complete description of a three-level time-split MacCormack scheme for solving the two-dimensional initial-boundary value problem (1) – (3). It is worth mentioning that a time-split MacCormack "splits" the original MacCormack scheme into a sequence of one-dimensional difference operations, thereby achieving a less restrictive stability condition. In other words, the splitting makes it possible to advance the solution in each direction with the maximum allowable time step ([25, 29, 52, 31] and [54] page 231).

Suppose N and M be two positive integers. Let $k:=\Delta t=T/N$, $h:=\Delta x=\Delta y=1/M$, be the time step and grid spacing, respectively. Setting $t^n=kn$, t^{*}: time used at the beginning of the next step in a time-split MacCormack scheme and t^{**}: time used at the last step of a time-split MacCormack approach, then $t^{*}\in(t^n,t^{n+1})$, $t^{**}\in(t^{*},t^{n+1})$, $n=0,1,2,\dots,N-1$, $x_i=ih$, $y_j=jh$; $0\le i,j\le M$ Furthermore, let $\Omega_k=\{t^n, 0\le n\le N\}$, $\bar{\Omega}_h=\{(x_i,y_j),\ 0\le i,j\le M\}$, $\Omega_h=\bar{\Omega}_h\cap\Omega$ and $\partial\Omega_h=\bar{\Omega}_h\cap\partial\Omega$.

Let $\mathcal{U}_h=\{u_{ij}^n;\ n=0,1,\dots,N;\ \text{i,j}=0,1,2,\dots,M\}$, be the space of grid functions defined on $\Omega_h\times\Omega_k$. Now, following the MacCormack approach, we can set

$$\delta_x u^n_{i+\frac{1}{2},j}=\frac{u^n_{i+1,j}-u^n_{ij}}{h};\quad \delta^x u^n_{ij}=\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2h};\quad \delta_y u^n_{i,j+\frac{1}{2}}=\frac{u^n_{i,j+1}-u^n_{ij}}{h};\quad \delta^y u^n_{ij}=\frac{u^n_{i,j+1}-u^n_{i,j-1}}{2h};$$

$$\delta_x^2 u^n_{ij}=\frac{\delta_x u^n_{i+\frac{1}{2},j}-\delta_x u^n_{i-\frac{1}{2},j}}{h}\quad\text{and}\quad \delta_y^2 u^n_{ij}=\frac{\delta_y u^n_{i,j+\frac{1}{2}}-\delta_y u^n_{i,j-\frac{1}{2}}}{h}. \tag{5}$$

Defining the following norms and scalar products

$$\|u^n\|_{L^2}=h\left(\sum_{i,j=1}^{M-1}|u^n_{ij}|^2\right)^{\frac{1}{2}};\ \|\delta_x u^n\|_{L^2}=h\left(\sum_{j=1}^{M-1}\sum_{i=0}^{M-1}|\delta_x u^n_{i+\frac{1}{2},j}|^2\right)^{\frac{1}{2}},\ \|\delta_y u^n\|_{L^2}=h\left(\sum_{j=0}^{M-1}\sum_{i=1}^{M-1}|\delta_y u^n_{i,j+\frac{1}{2}}|^2\right)^{\frac{1}{2}},\quad \|\delta_\lambda^2 u^n\|_{L^2}=h\left(\sum_{i,j=1}^{M-1}|\delta_\lambda^2 u^n_{ij}|^2\right)^{\frac{1}{2}}, \tag{6}$$

where $\lambda=x$ or y. Furthermore, the scalar products are defined as

$$(u^n,v^n)=h^2\sum_{i,j=1}^{M-1}u^n_{ij}v^n_{ij},\ <\delta_x u^n,\delta_x v^n>_x=h^2\sum_{j=1}^{M-1}\sum_{i=0}^{M-1}\delta_x u^n_{i+\frac{1}{2},j}\delta_x v^n_{i+\frac{1}{2},j},$$

and

$$<\delta_y u^n,\delta_y v^n>_y=h^2\sum_{j=0}^{M-1}\sum_{i=1}^{M-1}\delta_y u^n_{i,j+\frac{1}{2}}\delta_y v^n_{i,j+\frac{1}{2}}. \tag{7}$$

Finally, we introduce the following discrete norms

$$|\|u\||_{L^\infty(0,T;L^2(\Omega))}=\max_{0\le n\le N}\|u^n\|_{L^2(\Omega)},\quad |\|u\||_{L^2(0,T;L^2(\Omega))}=\left(k\sum_{n=0}^{N}\|u^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}},$$

and

$$|\|u\||_{L^1(0,T;L^2(\Omega))}=k\sum_{n=0}^{N}\|u^n\|_{L^2(\Omega)};\ \text{for } u\in\mathcal{U}_h. \tag{8}$$

Now, expanding the Taylor series for the function u about (x_i,y_j,t^n) at the predictor and corrector steps with time step $k/2$ provides

$$u_{ij}^{\bar{*}}:=u(x_i,y_j,t^{\bar{*}})=u^n_{ij}+\frac{k}{2}u_t)^n_{ij}+\frac{k^2}{8}u_{2t})^n_{ij}+O(k^3);\ u_{ij}^{\bar{\bar{*}}}:=u(x_i,y_j,t^{\bar{\bar{*}}})=u^n_{ij}+\frac{k}{2}u_t)^{\bar{*}}_{ij}+\frac{k^2}{8}u_{2t})^{\bar{*}}_{ij}+O(k^3), \tag{9}$$

where $t^{\bar{*}}$ is the time used at the predictor step and $t^{\bar{\bar{*}}}$ is the one used at the corrector step which is also the final time provided at this stage. In fact, $t^{\bar{\bar{*}}}$ is just a notation, it can be considered as t^{*}. For the sake of readability and to avoid any confusion, we used $\overline{\overline{**}}$ to indicate the value provided at the corrector step since the approximate solution constructed at the first phase of a time-split MacCormack scheme should consider the index $*$. This must indicate the difference between the value provided at the corrector step and the approximate solution (which is usually called corrected value) obtained at the first stage of the algorithm. Using the definition of the operator $L_y(k/2)$, let consider the equation

$$u_t + \mu u_y - a u_{yy} = 0 , \tag{10a}$$

which is equivalent to

$$u_t = -\mu u_y + a u_{yy} , \tag{10b}$$

Since the right side of the convection-diffusion equation (1) equals zero, we assume in the rest of this paper that the function u is regular enough, has partial derivatives in space up to order fourth. Using equation (10); it is not difficult to see that

$$u_{2t} = \left(a u_{yy} - \mu u_y\right)_t = a^2 u_{4y} - 2a\mu u_{3y} + \mu^2 u_{2y} .$$

This fact together with equation (9) provide

$$u_{ij}^{\bar{*}} = u_{ij}^{n} + \frac{k}{2}[a u_{yy} - \mu u_y]_{ij}^{n} + \frac{k^2}{8}\left[a^2 u_{4y} - 2a\mu u_{3y} + \mu^2 u_{yy}\right]_{ij}^{n} + O(k^3) , \tag{11}$$

$$u_{ij}^{\bar{\bar{*}}} = u_{ij}^{n} + \frac{k}{2}[a u_{yy} - \mu u_y]_{ij}^{\bar{*}} + \frac{k^2}{8}\left[a^2 u_{4y} - 2a\mu u_{3y} + \mu^2 u_{yy}\right]_{ij}^{\bar{*}} + O(k^3) . \tag{12}$$

Applying the Taylor series for the function u about (x_i, y_j, t^n) with mesh size h using forward and backward difference representations results in

$$u_{y,ij}^{n} = \delta_y u_{i,j+\frac{1}{2}}^{n} + O(h), u_{yy,ij}^{n} = \delta_y^2 u_{ij}^{n} + O(h^2), u_{3y,ij}^{n} = \delta_y^2(\delta_y u_{i,j+\frac{1}{2}}^{n}) + O(h), u_{4y,ij}^{n} = \delta_y^4 u_{ij}^{n} + O(h^2) , \tag{13a}$$

$$u_{y,ij}^{\bar{*}} = \delta_y u_{i,j-\frac{1}{2}}^{\bar{*}} + O(h), u_{yy,ij}^{\bar{*}} = \delta_y^2 u_{ij}^{\bar{*}} + O(h^2), u_{3y,ij}^{*} = \delta_y^2(\delta_y u_{i,j-\frac{1}{2}}^{*}) + O(h), u_{4y,ij}^{\bar{*}} = \delta_y^4 u_{ij}^{\bar{*}} + O(h^2) , \tag{13b}$$

where $\delta_y w_{ij}^{l}$ and $\delta_y^2 w_{ij}^{l}$ are given by relation (5). Substituting the first four equations of relation (13) into equation (11) and the last four equations of relation (13) into equation (12) to get

$$u_{ij}^{\bar{*}} = u_{ij}^{n} + \frac{k}{2}[a\delta_y^2 u_{ij}^{n} - \mu \delta_y u_{i,j+\frac{1}{2}}^{n}] + k^2 \rho_{ij}^{n} + O(k^3 + kh) , \tag{14}$$

and

$$u_{ij}^{\bar{\bar{*}}} = u_{ij}^{n} + \frac{k}{2}[a\delta_y^2 u_{ij}^{\bar{*}} - \mu \delta_y u_{i,j-\frac{1}{2}}^{\bar{*}}] + k^2 \rho_{ij}^{\bar{*}} + O(k^3 + kh) , \tag{15}$$

where

$$\rho_{ij}^{n} = \frac{1}{8}\left[a^2 \delta_y^2(\delta_y^2 u_{ij}^{n}) - 2a\mu \delta_y^2(\delta_y u_{i,j+\frac{1}{2}}^{n}) + \mu^2 \delta_y^2 u_{ij}^{n}\right] ,$$

and

$$\rho_{ij}^{\bar{*}} = \frac{1}{8}\left[a^2 \delta_y^2(\delta_y^2 u_{ij}^{\bar{*}}) - 2a\mu \delta_y^2(\delta_y u_{i,j-\frac{1}{2}}^{\bar{*}}) + \mu^2 \delta_y^2 u_{ij}^{\bar{*}}\right] . \tag{16}$$

The term $\delta_y^2 u_{ij}^{\bar{*}}$ and $\delta_y u_{i,j-\frac{1}{2}}^{\bar{*}}$ should be expressed as a function of $\delta_y^2 u_{ij}^{n}$ and $\delta_y u_{i,j-\frac{1}{2}}^{n}$. It comes from equation (14) that

$$\delta_y^2 u_{ij}^{\bar{*}} = \delta_y^2 u_{ij}^{n} + \frac{k}{2}[a\delta_y^2(\delta_y^2 u_{ij}^{n}) - \mu \delta_y^2(\delta_y u_{i,j+\frac{1}{2}}^{n})] + O(k^2 + kh) , \tag{17}$$

and

$$\delta_y u_{i,j-\frac{1}{2}}^{\bar{*}} = \delta_y u_{i,j-\frac{1}{2}}^{n} + \frac{k}{2}[a\delta_y^2(\delta_y u_{i,j-\frac{1}{2}}^{n}) - \mu \delta_y(\delta_y u_{ij}^{n})] + O(k^2 + kh) . \tag{18}$$

Combining equations (14)-(18), direct computations give

$$u_{ij}^{\bar{\bar{*}}} = u_{ij}^{n} + \frac{k}{2}\left(a\delta_y^2 u_{ij}^{n} - \mu \delta_y u_{i,j-\frac{1}{2}}^{n}\right) + k^2\left(\hat{\rho}_{ij}^{n} + \rho_{ij}^{\bar{*}}\right) + O(k^3 + kh) , \tag{19}$$

where

$$\hat{\rho}_{ij}^{n}=\frac{1}{4}\left[a^{2}\delta_{y}^{2}(\delta_{y}^{2}u_{ij}^{n})-2a\mu\delta_{y}^{2}(\delta^{y}u_{ij}^{n})+\mu^{2}\delta_{y}^{2}u_{ij}^{n}\right]. \tag{20}$$

Taking the average of $u_{ij}^{\bar{*}}$ and $u_{ij}^{\bar{\bar{*}}}$ yields

$$\frac{u_{ij}^{\bar{*}}+u_{ij}^{\bar{\bar{*}}}}{2}=u_{ij}^{n}+\frac{k}{2}\left(a\delta_{y}^{2}u_{ij}^{n}-\mu\delta^{y}u_{ij}^{n}\right)+\frac{k^{2}}{2}\left(\rho_{ij}^{n}+\hat{\rho}_{ij}^{n}+\rho_{ij}^{\bar{*}}\right)+O(k^{3}+kh^{2}), \tag{21}$$

where $\delta^{y}u_{ij}^{n}=(\delta_{y}u^{n}_{i,j+\frac{1}{2}}+\delta_{y}u^{n}_{i,j-\frac{1}{2}})/2$, ρ_{ij}^{α} and $\hat{\rho}_{ij}^{n}$ are given by relations (16) and (20), respectively. The second order error term $O(h^{2})$ in equation (21) comes from the approximation $u_{y})_{ij}^{n}=\delta^{y}u_{ij}^{n}+O(h^{2})$.
On the other hand, to define the operator $L_{x}(k)$, we should consider the equation $u_{t}+\mu u_{x}-au_{xx}=0$, which is equivalent to

$$u_{t}=-\mu u_{x}+au_{xx}, \tag{22}$$

It comes from equation (22), that

$$u_{2t}=(au_{xx}-\mu u_{x})_{t}=a^{2}u_{4x}-2a\mu u_{3x}+\mu^{2}u_{2x}. \tag{23}$$

The application of the Taylor series expansion for the function u about (x_{i},y_{i},t^{*}) (where $t^{*}\in(t^{n},t^{n+1})$, is the time used at the beginning of the next step in a time-split MacCormack scheme) with mesh size h using both forward and backward difference representations gives

$$u_{x,ij}^{*}=\delta_{x}u^{*}_{i+\frac{1}{2},j}+O(h),u_{xx,ij}^{*}=\delta_{x}^{2}u_{ij}^{*}+O(h^{2}),u_{3x,ij}^{*}=\delta_{x}^{2}(\delta_{x}u^{*}_{i+\frac{1}{2},j})+O(h),u_{4x,ij}^{*}=\delta_{x}^{4}u_{ij}^{*}+O(h^{2}), \tag{24a}$$

$$u_{x,ij}^{\overline{**}}=\delta_{x}u^{\overline{**}}_{i-\frac{1}{2},j}+O(h),u_{xx,ij}^{\overline{**}}=\delta_{x}^{2}u_{ij}^{\overline{**}}+O(h^{2}),u_{3x,ij}^{\overline{**}}=\delta_{x}^{2}(\delta_{x}u^{\overline{**}}_{i-\frac{1}{2},j})+O(h),u_{4x,ij}^{\overline{**}}=\delta_{x}^{4}u_{ij}^{\overline{**}}+O(h^{2}), \tag{24b}$$

where $\delta_{x}u^{l}_{i\pm\frac{1}{2},j}$ and $\delta_{x}^{2}u_{ij}^{l}$ are defined by equation (5). Also, expanding the Taylor series for the function u at the predictor and corrector steps about (x_{i},y_{j},t^{*}) with time step k using forward difference, it is not difficult to observe that

$$u_{ij}^{\overline{**}}=u_{ij}^{*}+ku_{t})_{ij}^{*}+\frac{k^{2}}{2}u_{2t})_{ij}^{*}+O(k^{3}),\ u_{ij}^{\overline{\overline{**}}}=u_{ij}^{*}+ku_{t})_{ij}^{\overline{**}}+\frac{k^{2}}{2}u_{2t})_{ij}^{\overline{**}}+O(k^{3}). \tag{25}$$

Plugging equations (22), (23), (24) and (25), simple calculations provide

$$u_{ij}^{\overline{**}}=u_{ij}^{*}+k[a\delta_{x}^{2}u_{ij}^{*}-\mu\delta_{x}u^{*}_{i+\frac{1}{2},j}]+k^{2}\gamma_{ij}^{*}+O(k^{3}+kh), \tag{26}$$

and

$$u_{ij}^{\overline{\overline{**}}}=u_{ij}^{*}+k[a\delta_{x}^{2}u_{ij}^{\overline{**}}-\mu\delta_{x}u^{\overline{**}}_{i-\frac{1}{2},j}]+k^{2}\gamma_{ij}^{\overline{**}}+O(k^{3}+kh), \tag{27}$$

where

$$\gamma_{ij}^{*}=\frac{1}{2}\left[a^{2}\delta_{x}^{2}(\delta_{x}^{2}u_{ij}^{*})-2a\mu\delta_{x}^{2}(\delta_{x}u^{*}_{i+\frac{1}{2},j})+\mu^{2}\delta_{x}^{2}u_{ij}^{*}\right],\quad \gamma_{ij}^{\overline{**}}=\frac{1}{2}\left[a^{2}\delta_{x}^{2}(\delta_{x}^{2}u_{ij}^{\overline{**}})-2a\mu\delta_{x}^{2}(\delta_{x}u^{\overline{**}}_{i-\frac{1}{2},j})+\mu^{2}\delta_{x}^{2}u_{ij}^{\overline{**}}\right], \tag{28}$$

In order to obtain a simple expression of terms $\delta_{x}^{2}u_{ij}^{\overline{**}}$ and $\delta_{x}u^{\overline{**}}_{i,j-\frac{1}{2}}$, we should use equation (26). Performing direct computations, it is easy to see that

$$\delta_{x}^{2}u_{ij}^{\overline{**}}=\delta_{x}^{2}u_{ij}^{*}+k[a\delta_{x}^{2}(\delta_{x}^{2}u_{ij}^{*})-\mu\delta_{x}^{2}(\delta_{x}u^{*}_{i+\frac{1}{2},j})]+O(k^{2}+kh),$$

and

$$\delta_{x}u^{\overline{**}}_{i-\frac{1}{2},j}=\delta_{x}u^{*}_{i-\frac{1}{2},j}+k[a\delta_{x}^{2}(\delta_{x}u^{*}_{i-\frac{1}{2},j})-\mu\delta_{x}^{2}u_{ij}^{*}]+O(k^{2}+kh).$$

This fact, together with relation (27) yields

$$u_{ij}^{\overline{\overline{**}}}=u_{ij}^{*}+k\left(a\delta_{x}^{2}u_{ij}^{*}-\mu\delta_{x}u^{*}_{i-\frac{1}{2},j}\right)+k^{2}\left(\hat{\gamma}_{ij}^{*}+\gamma_{ij}^{\overline{**}}\right)+O(k^{3}+kh), \tag{29}$$

where

$$\hat{\gamma}_{ij}^{*}=a^{2}\delta_{x}^{2}(\delta_{x}^{2}u_{ij}^{*})-2a\mu\delta_{x}^{2}(\delta^{x}u_{ij}^{*})+\mu^{2}\delta_{x}^{2}u_{ij}^{*}. \tag{30}$$

A combination of equations (26) and (29) results in

$$\frac{u_{ij}^{\overline{**}} + u_{ij}^{\overline{\overline{**}}}}{2} = u_{ij}^{*} + k\left(a\delta_x^2 u_{ij}^{*} - \mu\delta^x u_{ij}^{*}\right) + \frac{k^2}{2}\left(\gamma_{ij}^{*} + \hat{\gamma}_{ij}^{*} + \gamma_{ij}^{\overline{**}}\right) + O(k^3 + kh^2). \tag{31}$$

In way similar, starting with the one-dimensional equation: $u_t + \mu u_y - a u_{yy} = 0$, applying the Taylor series for the function u about (x_i, y_j, t^{**}) (where t^{**} is the time used at the last step in a time-split MacCormack approach) at the predictor and corrector stages with time step $k/2$ and grid spacing h, using forward and backward difference representations to get

$$\frac{u_{ij}^{\overline{n+1}} + u_{ij}^{\overline{\overline{n+1}}}}{2} = u_{ij}^{**} + \frac{k}{2}\left(a\delta_y^2 u_{ij}^{**} - \mu\delta^y u_{ij}^{**}\right) + \frac{k^2}{2}\left(\theta_{ij}^{**} + \hat{\theta}_{ij}^{**} + \theta_{ij}^{\overline{n+1}}\right) + O(k^3 + kh^2), \tag{32}$$

where

$$\theta_{ij}^{\mu} = \frac{1}{8}\left[a^2\delta_y^2(\delta_y^2 u_{ij}^{\mu}) - 2a\mu\delta_y^2(\delta^y u^{\mu}_{i,j\pm\frac{1}{2}}) + \mu^2\delta_y^2 u_{ij}^{\mu}\right],\ \hat{\theta}_{ij}^{**} = \frac{1}{4}\left[a^2\delta_y^2(\delta_y^2 u_{ij}^{\mu}) - 2a\mu\delta_y^2(\delta^y u_{ij}^{\mu}) + \mu^2\delta_y^2 u_{ij}^{\mu}\right] \tag{33}$$

where we set $\mu = **, \overline{n+1}$ and $\delta^y u_{ij}^{\mu} = (\delta_y u^{\mu}_{i,j+\frac{1}{2}} + \delta_y u^{\mu}_{i,j-\frac{1}{2}})/2$. Furthermore, $\overline{n+1}$ means the dummy time index at the predictor step while $\overline{\overline{n+1}}$ indicates the one considered at the corrector step, $u^{\overline{n+1}}$ is the value computed at the predictor step and $u^{\overline{\overline{*}}}$ is the one provided at the corrector step.

To construct a three-level explicit time-split MacCormack method for solving the initial-boundary value problem (1)-(3), we must follow the ideas presented in [29-31, 51]. More specifically, we should neglect the terms of second order together with the infinitesimal term $O(k^3 + kh^2)$ in equations (21); (31) and (32). In addition, the terms u_{ij}^{*}, u_{ij}^{**} and u_{ij}^{n+1} are defined as the average of predicted and corrected values, that is,

$$u_{ij}^{*} = \frac{u_{ij}^{\overline{*}} + u_{ij}^{\overline{\overline{*}}}}{2},\ u_{ij}^{**} = \frac{u_{ij}^{\overline{**}} + u_{ij}^{\overline{\overline{**}}}}{2},\ u_{ij}^{n+1} = \frac{u_{ij}^{\overline{n+1}} + u_{ij}^{\overline{\overline{n+1}}}}{2}. \tag{34}$$

Thus, equations

$$u_{ij}^{*} = L_y(k/2)u_{ij}^{n},\ u_{ij}^{**} = L_x(k)u_{ij}^{*},\ u_{ij}^{n+1} = L_y(k/2)u_{ij}^{**}, \tag{35}$$

are by definition equivalent to

$$u_{ij}^{*} = u_{ij}^{n} + \frac{k}{2}\left(a\delta_y^2 u_{ij}^{n} - \mu\delta^y u_{ij}^{n})\right),\ u_{ij}^{**} = u_{ij}^{*} + k\left(a\delta_x^2 u_{ij}^{*} - \mu\delta^x u_{ij}^{*})\right),\ u_{ij}^{n+1} = u_{ij}^{**} + \frac{k}{2}\left(a\delta_y^2 u_{ij}^{**} - \mu\delta^y u_{ij}^{**})\right). \tag{36}$$

Since the operator $L_y(k/2)L_x(k)L_y(k/2)$ is symmetric, this fact together with relations (21), (31) and (32) show that the obtained method is a three-level approach, an explicit predictor-corrector scheme, second order accurate in time and fourth order convergent in space. This theoretical analysis is confirmed by a large set of numerical evidences (see section 5). From the definition of the linear operators δ^x, δ^y, δ_x^2 and δ_y^2 given by (5), equation (36) can be rewritten as follows. For $n = 0,1,\dots,N-1$;

$$u_{ij}^{*} = u_{ij}^{n} + \frac{k}{2}\left[\frac{a}{h^2}(u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}) - \frac{\mu}{2h}(u_{i,j+1}^{n} - u_{i,j-1}^{n})\right],\ i = 0,1,\dots,M;\ j = 1,2,\dots,M-1; \tag{37}$$

$$u_{ij}^{**} = u_{ij}^{*} + k\left[\frac{a}{h^2}(u_{i+1,j}^{*} - 2u_{ij}^{*} + u_{i-1,j}^{*}) - \frac{\mu}{2h}(u_{i+1,j}^{*} - u_{i-1,j}^{*})\right],\ i = 1,2,\dots,M-1;\ j = 0,1,\dots,M; \tag{38}$$

$$u_{ij}^{n+1} = u_{ij}^{**} + \frac{k}{2}\left[\frac{a}{h^2}(u_{i,j+1}^{**} - 2u_{ij}^{**} + u_{i,j-1}^{**}) - \frac{\mu}{2h}(u_{i,j+1}^{**} - u_{i,j-1}^{**})\right];\ i = 0,1,\dots,M;\ j = 1,2,\dots,M-1; \tag{38}$$

with the initial and boundary conditions, for $i, j = 0,1,\dots,M$,

$$u_{ij}^{0} = u_0(x_i, y_j);\ u_{i0}^{n} = \varphi_{i0}^{n};\ u_{iM}^{n} = \varphi_{iM}^{n};\ u_{0j}^{n} = \varphi_{0j}^{n};\ u_{Mj}^{n} = \varphi_{Mj}^{n};\ u_{0j}^{*} = \varphi_{0j}^{n+1};\ u_{Mj}^{*} = \varphi_{Mj}^{n+1};\ u_{j0}^{*} = \varphi_{j0}^{n+1}; \tag{39a}$$

$$u_{jM}^{*} = \varphi_{jM}^{n+1};\ u_{0j}^{**} = \varphi_{0j}^{n+1};\ u_{Mj}^{**} = \varphi_{Mj}^{n+1};\ u_{j0}^{**} = \varphi_{j0}^{n+1};\ u_{jM}^{**} = \varphi_{jM}^{n+1};\ u_{i0}^{N} = \varphi_{i0}^{N};\ u_{iM}^{N} = \varphi_{iM}^{N}; \tag{39b}$$

$$u_{0j}^{N} = \varphi_{0j}^{N}; \tag{40a}$$

and

$$u_{Mj}^{N} = \varphi_{Mj}^{N}. \tag{40b}$$

Equations (37)-(40) represent a full description of a three-level explicit time-split MacCormack method applied to convection-diffusion equation (1)-(3).

In the rest of this paper, we prove the stability, the error estimates and the convergence rate of a three-level time-split MacCormack approach under the time step restriction

$$\max\left\{\frac{2ak}{h^2},\frac{|\mu|k}{h}\right\}\leq 1\,, \tag{41}$$

where a and μ are physical parameters given in equation (1). We recall that k is the time step and h is the grid size. We assume that the exact solution $\bar{u}$ satisfies

$$|||\bar{u}|||_{L^\infty(0,T;L^2(\Omega))}+|||\bar{u}|||_{H^1(0,T;H^3(\Omega))}+|||\bar{u}|||_{H^2(0,T;H^1(\Omega))}+|||\bar{u}|||_{H^2(0,T;L^2(\Omega))}+|||\bar{u}|||_{L^2(0,T;H^4(\Omega))}\leq\varpi\,, \tag{42}$$

where ϖ is a positive constant independent of the time step k and space step h.

3. Stability analysis of a three-level time-split MacCormack scheme

In this section we analyze the stability of a three-level time-split MacCormack scheme (37)-(40) for solving the two-dimensional parabolic equation (1)-(3). The proof of the main result on the stability given by Theorem 1 requires an intermediate result (namely Lemma 1).

Theorem 1. (Stability analysis of the time-split MacCormack). Let u be the numerical solution provided by the scheme (37)-(40). Under the time step restriction (41), it holds

$$\max_{0\leq n\leq N}\|u^n\|_{L^2}\leq\varpi+(\varpi+\|u^1\|_{L^2})\exp\left(\frac{5\mu^2T}{2a}\sum_{l=0}^{2}\left(\frac{\mu^2}{a}k\right)^l\right),$$

where μ and a are two physical parameters and ϖ is given by relation (42). The following result (namely Lemma 1) is too important in the proof of Theorem 1.

Lemma 1. Setting $u_{ij}^n=u(x_i,y_j,t^n)$ be the numerical solution provided by the scheme (37)-(40), $\bar{u}_{ij}^n=\bar{u}(x_i,y_j,t^n)$ be the exact solution of the initial-boundary value problem (1)-(3) and let $e_{ij}^n=u_{ij}^n-\bar{u}_{ij}^n$ be the error. We recall that $\bar{u}_{ij}^{*}=(\bar{u}_{ij}^{\bar{*}}+\bar{u}_{ij}^{\bar{\bar{*}}})/2$ and $\bar{u}_{ij}^{**}=(\bar{u}_{ij}^{\bar{**}}+\bar{u}_{ij}^{\bar{\bar{**}}})/2$ satisfy relations (21) and (31), respectively. u_{ij}^{*} and u_{ij}^{**} are given by equations (37) and (38), respectively. The following equalities hold:

$$a<\delta_x^2e_{ij}^n,e_{ij}^n>_x=h^2\sum_{j,i=1}^{M-1}\frac{a}{h^2}\left(e_{i+1,j}^n-2e_{ij}^n+e_{i-1,j}^n\right)e_{ij}^n=-a\|\delta_xe^n\|_{L^2}^2\,, \tag{43}$$

$$a<\delta_y^2e_{ij}^n,e_{ij}^n>_y=h^2\sum_{j,i=1}^{M-1}\frac{a}{h^2}\left(e_{i,j+1}^n-2e_{ij}^n+e_{i,j-1}^n\right)e_{ij}^n=-a\|\delta_ye^n\|_{L^2}^2\,, \tag{44}$$

$$\sum_{j,i=1}^{M-1}\left(e_{i,j+1}^n-e_{i,j-1}^n\right)e_{ij}^n=0\,, \tag{45}$$

and

$$\|\delta_xe^n\|_{L^2},\ \|\delta_ye^n\|_{L^2}\leq 2h^{-1}\|e^n\|_{L^2}\,. \tag{46}$$

where the operators δ_x and δ_y are given by relation (5).

Proof (of Theorem 1). Combining equations (21), (34) and (37), direct computations provide

$$e_{ij}^{*}=e_{ij}^n+\frac{k}{2}\left(a\delta_y^2e_{ij}^n-\mu\delta^ye_{ij}^n\right)+\frac{k^2}{2}\left(\rho_{ij}^n+\hat{\rho}_{ij}^n+\rho_{ij}^{\bar{*}}\right)+O(k^3+kh^2)\,, \tag{47}$$

where ρ_{ij}^α ($\alpha=n,\bar{*}$) and $\hat{\rho}_{ij}^n$ are defined by (16) and (20), respectively. Utilizing the definition of the operator δ_y^2 equation (47) is equivalent to

$$e_{ij}^{*}=e_{ij}^n+\frac{k}{2}\left(\frac{a}{h^2}(e_{i,j+1}^n-2e_{ij}^n+e_{i,j-1}^n)-\frac{\mu}{2h}(e_{i,j+1}^n-e_{i,j-1}^n)\right)+\frac{k^2}{2}\left(\rho_{ij}^n+\hat{\rho}_{ij}^n+\rho_{ij}^{\bar{*}}\right)+O(k^3+kh^2)\,. \tag{48}$$

Since the formulas can become quite heavy, for the sake of readability, we should neglect the higher order terms in time step k and grid spacing h. However, the truncation of the infinitesimal terms does not compromise the result on the stability analysis. This fact together with equation (48) yields

$$e_{ij}^{*}=e_{ij}^n+\frac{k}{2}\left(\frac{a}{h^2}(e_{i,j+1}^n-2e_{ij}^n+e_{i,j-1}^n)-\frac{\mu}{2h}(e_{i,j+1}^n-e_{i,j-1}^n)\right). \tag{49}$$

Performing simple calculations, equation (49) gives

$$(e_{ij}^{*})^2=(e_{ij}^n)^2+k\left[\frac{a}{h^2}(e_{i,j+1}^n-2e_{ij}^n+e_{i,j-1}^n)-\frac{\mu}{2h}(e_{i,j+1}^n-e_{i,j-1}^n)\right]e_{ij}^n+\frac{k^2}{4}\left(\frac{a}{h^2}(e_{i,j+1}^n-2e_{ij}^n+e_{i,j-1}^n)-\frac{\mu}{2h}(e_{i,j+1}^n-e_{i,j-1}^n)\right)^2. \tag{50}$$

Now, using inequality $(a\pm b)^2\leq 2(a^2+b^2)$ together with equality $2(a-b)b=a^2-b^2-(a-b)^2$ for any $a,b\in\mathbb{R}$, straightforward computations result in

$$(e_{i,j+1}^n-2e_{ij}^n+e_{i,j-1}^n)^2\leq 2\left[(e_{i,j+1}^n-e_{ij}^n)^2+(e_{i,j-1}^n-e_{ij}^n)^2\right]. \tag{51}$$

From estimate (51), it holds

$$\left[\frac{a}{h^2}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})-\frac{\mu}{2h}(e^n_{i,j+1}-e^n_{i,j-1})\right]^2\le 2\left[\frac{2a^2}{h^4}\left[(e^n_{i,j+1}-e^n_{i,j})^2+(e^n_{i,j}-e^n_{i,j-1})^2\right]+\left(\frac{\mu}{2h}(e^n_{i,j+1}-e^n_{i,j-1})\right)^2\right]. \quad (52)$$

A combination of estimates (50) and (52) gives

$$(e^*_{ij})^2\le(e^n_{ij})^2+k\left\{\frac{a}{h^2}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})e^n_{ij}-\frac{\mu}{2h}(e^n_{i,j+1}-e^n_{i,j-1})e^n_{ij}\right\}+\frac{k^2}{2}\left\{\frac{2a^2}{h^2}\left[\left(\delta_y e^n_{i,j+\frac{1}{2}}\right)^2+\left(\delta_y e^n_{i,j-\frac{1}{2}}\right)^2\right]+\frac{\mu^2}{2}\left[\left(\delta_y e^n_{i,j+\frac{1}{2}}\right)^2+\left(\delta_y e^n_{i,j-\frac{1}{2}}\right)^2\right]\right\}.$$

Summing this up from $i,j=1,2,...,M-1$, and rearranging of terms, we obtain

$$\sum_{i,j=1}^{M-1}(e^*_{ij})^2\le\sum_{i,j=1}^{M-1}(e^n_{ij})^2+\frac{ak}{h^2}\sum_{i,j=1}^{M-1}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})e^n_{ij}-\frac{\mu}{2h}\sum_{i,j=1}^{M-1}(e^n_{i,j+1}-e^n_{i,j-1})e^n_{ij}+k^2\left(\frac{a^2}{h^2}+\frac{\mu^2}{4}\right)\sum_{i,j=1}^{M-1}\left[\left(\delta_y e^n_{i,j+\frac{1}{2}}\right)^2+\left(\delta_y e^n_{i,j-\frac{1}{2}}\right)^2\right],$$

which implies

$$\sum_{i,j=1}^{M-1}(e^*_{ij})^2\le\sum_{i,j=1}^{M-1}(e^n_{ij})^2+\frac{ak}{h^2}\sum_{i,j=1}^{M-1}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})e^n_{ij}-\frac{\mu}{2h}\sum_{i,j=1}^{M-1}(e^n_{i,j+1}-e^n_{i,j-1})e^n_{ij}+2k^2\left(\frac{a^2}{h^2}+\frac{\mu^2}{4}\right)\sum_{i,j=1}^{M-1}\left(\delta_y e^n_{i,j+\frac{1}{2}}\right)^2. \quad (53)$$

Multiplying both sides of estimate (53) by h^2 and using equations (44)-(46) of Lemma 1, it is easy to see that

$$\| e^*\|^2_{L^2}\le\| e^n\|^2_{L^2}-ak\|\delta_y e^n\|^2_{L^2}+2k^2\left(\frac{a^2}{h^2}+\frac{\mu^2}{4}\right)\|\delta_y e^n\|^2_{L^2}=\| e^n\|^2_{L^2}-ak\left(1-\frac{2ak}{h^2}\right)\|\delta_y e^n\|^2_{L^2}+\frac{\mu^2k^2}{2}\|\delta_y e^n\|^2_{L^2}.$$

From the time step restriction (41), i.e., $-1+2ak/h^2\le 0$, and estimate (46) of Lemma 1, it is not difficult to observe that

$$\| e^*\|^2_{L^2}\le\| e^n\|^2_{L^2}+\frac{\mu^2k^2}{2}\|\delta_y e^n\|^2_{L^2}\le\| e^n\|^2_{L^2}+\frac{2\mu^2k^2}{h^2}\| e^n\|^2_{L^2}\le\| e^n\|^2_{L^2}+\frac{\mu^2k}{a}\| e^n\|^2_{L^2}. \quad (54)$$

In way similar, using equations (31); (34) and (38), adapting the proof for $\| e^*\|^2_{L^2}$ and replacing k by $k/2$, it not hard to show that

$$\| e^{**}\|^2_{L^2}\le\| e^*\|^2_{L^2}+\frac{2\mu^2k}{a}\| e^*\|^2_{L^2}. \quad (55)$$

Finally, combining equations (32), (34) and (39), following the proof for $\| e^*\|^2_{L^2}$ but replacing n and $*$ by $**$ and $n+1$, respectively, it is not difficult to see that

$$\| e^{n+1}\|^2_{L^2}\le\| e^{**}\|^2_{L^2}+\frac{\mu^2k}{a}\| e^{**}\|^2_{L^2}. \quad (56)$$

We should find a relationship between $\| e^{n+1}\|^2_{L^2}$ and $\| e^n\|^2_{L^2}$, for any nonnegative integer $n\ge 1$. In fact, if $n\ge 0$, the initial condition $e^0_{ij}=0$, for $0\le i,j\le M$, can lead to a sequence of vanishing error $e^n=0$, which in practical application is not possible. Plugging estimates (54), (55) and (56), simple calculations yield

$$\| e^{n+1}\|^2_{L^2}\le\left(1+\frac{\mu^2k}{a}\right)^2\left(1+\frac{2\mu^2k}{a}\right)\| e^n\|^2_{L^2}=\| e^n\|^2_{L^2}+\frac{\mu^2k}{a}\left[4+\frac{5\mu^2k}{a}+\frac{2\mu^4k^2}{a^2}\right]\| e^n\|^2_{L^2}.$$

Summing this up from $n=1,2,..,p-1$, for any nonnegative integer p that satisfies $1\le p\le N$, it is easy to see that

$$\| e^p\|^2_{L^2}\le\| e^1\|^2_{L^2}+\frac{\mu^2k}{a}\left[4+\frac{5\mu^2k}{a}+\frac{2\mu^4k^2}{a^2}\right]\sum_{n=1}^{p-1}\| e^n\|^2_{L^2}. \quad (57)$$

Applying the Gronwall Lemma, estimate (57) gives

$$\| e^p\|^2_{L^2}\le\| e^1\|^2_{L^2}\exp\left\{\frac{\mu^2kp}{a}\left(4+\frac{5\mu^2k}{a}+\frac{2\mu^4k^2}{a^2}\right)\right\}. \quad (58)$$

Since $p\le N$ and $k=T/N$, so $\mu^2kp/a=\mu^2Tp/aN\le\mu^2T/a$. This fact, together with estimate (58) provide

$$\| e^p\|^2_{L^2}\le\| e^1\|^2_{L^2}\exp\left\{\frac{\mu^2T}{a}\left(4+\frac{5\mu^2k}{a}+\frac{2\mu^4k^2}{a^2}\right)\right\}\le\| e^1\|^2_{L^2}\exp\left\{\frac{5\mu^2T}{a}\sum_{l=0}^{2}\left(\frac{\mu^2k}{a}\right)^l\right\}.$$

Taking the square root, it is not hard to observe that

$$\| e^p\|_{L^2}\le\| e^1\|_{L^2}\exp\left\{\frac{5\mu^2T}{2a}\sum_{l=0}^{2}\left(\frac{\mu^2k}{a}\right)^l\right\}. \quad (59)$$

But, we have that $\| u^p\|_{L^2}-\|\bar{u}^p\|_{L^2}\le\| u^p-\bar{u}^p\|_{L^2}=\| e^p\|_{L^2}$ and $\| e^1\|_{L^2}\le\| u^1\|_{L^2}+\|\bar{u}^1\|_{L^2}$. This fact, together with estimate (59) yields

$$\| u^p\|_{L^2}\le\|\bar{u}^p\|_{L^2}+(\| u^1\|_{L^2}+\|\bar{u}^1\|_{L^2})\exp\left\{\frac{5\mu^2T}{2a}\sum_{l=0}^{2}\left(\frac{\mu^2k}{a}\right)^l\right\}.$$

The proof of Theorem 1 is completed thanks to estimate (42). Indeed, $\bar{u}$ is the exact solution of the initial-boundary value problem (1)-(3).

4. Convergence of the method

In this section we study the error estimates of a three-level time-split MacCormack method (37)-(40) for solving equations (1)-(3), under the time step restriction (41). Furthermore, we assume that the exact solution $\bar{u}$ satisfies relation (42).

Theorem 2 (Convergence of the time-split MacCormack). Consider u be the numerical solution provided by a three-level time-split MacCormack approach (37)-(40). Under the time step restriction (41), the error $e = u - \bar{u}$, satisfies

$$|||e|||_{L^\infty(0,T;L^2)} \leq \varpi + \| u^1 \|_{L^2})(1+k)^2\sqrt{CT\psi(k,h)}\exp\left\{\frac{CT}{2}(1+k)^5\right\}(k+h^4) ,$$

where ϖ is a positive constant independent of the time step k and grid spacing h, ϖ is given by relation (42) and

$$\psi(k,h) = 2 + k^2 + k^3 + h^2 + (1+k)(1+h^4+h^8) .$$

Some intermediate results (namely Lemmas 1, 2 and 3) play an important role in the proof of Theorem 2.

Lemma 2. [29-31]. Consider $v \in H^4(\Omega)$, be a function satisfying $v|_{[x_i,x_{i+1}]} \in \mathcal{C}^6[x_i,x_{i+1}]$, for $i = 0,1,2,...,M-1$. Then, it holds

$$\frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}) - v_{2x,i} = \frac{h^2}{12}v_{4x,i} - \frac{h^4}{720}\left[v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)})\right] , \text{ for } i = 1,2,...,M-1 ,$$

where $\theta_i^{(4)} \in (x_{i-1},x_i)$, $\theta_i^{(3)} \in (x_i,x_{i+1})$ and v_{mx} denotes the derivative of order m of v . Furthermore, for $i = 2,3,...,M-2$,

$$\frac{1}{h^4}(v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}) - v_{4x,i} = h^2\left\{\frac{1}{720}\left[v_{6x}(\theta_i^{(1)}) + v_{6x}(\theta_i^{(2)})\right] + \frac{241}{3220}\left[v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)})\right]\right\} ,$$

where $\theta_i^{(2)} \in (x_{i-2},x_{i-1})$, $\theta_i^{(4)} \in (x_{i-1},x_i)$, $\theta_i^{(3)} \in (x_i,x_{i+1})$ and $\theta_i^{(1)} \in (x_{i+1},x_{i+2})$.

Lemma 3. The terms ρ_{ij}^α and $\hat{\rho}_{ij}^n$ given by equations (16) and (20), respectively, can be bounded as

$$|\rho_{ij}^n|, |\rho_{ij}^{\bar{*}}|, |\hat{\rho}_{ij}^n| \leq \hat{C}_1\left[1+\hat{C}_2h^2+\hat{C}_3h^4\right], \tag{60}$$

where $\hat{C}_l$, $l = 1,2,3$, are positive constant independent of the time step k and the mesh size h .

Proof (of Lemma 3). For the proof of inequality $|\rho_{ij}^n| \leq \hat{C}_1\left[1+\hat{C}_2h^2+\hat{C}_3h^4\right]$, we refer the readers to [31, 29] for a similar proof. The proof of the other estimates are similar.

Using Lemmas 1, 2, 3, we are ready to prove Theorem 2.

Proof (of Theorem 2). Firstly, we recall that the error term provided by the numerical scheme (37)-(40) is given by $e_{ij}^n = u_{ij}^n - \bar{u}_{ij}^n$, where $\bar{u}$ represents the exact solution of equations (21), (31) and (32) and u satisfies relations (37)-(40).

It comes from equation (48) that

$$e_{ij}^* = e_{ij}^n + \frac{k}{2}\left(\frac{a}{h^2}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) - \frac{\mu}{2h}(e_{i,j+1}^n - e_{i,j-1}^n)\right) + \frac{k^2}{2}\left(\rho_{ij}^n + \hat{\rho}_{ij}^n + \rho_{ij}^{\bar{*}}\right) + O(k^3 + kh^2) ,$$

which can be rewritten as

$$e_{ij}^* = e_{ij}^n + \frac{k}{2}\left(\frac{a}{h^2}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) - \frac{\mu}{2h}(e_{i,j+1}^n - e_{i,j-1}^n)\right) + \frac{k^2}{2}\left(\rho_{ij}^n + \hat{\rho}_{ij}^n + \rho_{ij}^{\bar{*}}\right) + C_r(k^3 + kh^2) ,$$

where C_r is a parameter which does not depend on the time step k and the grid spacing h, and whose value can change from place to place, ρ_{ij}^α and $\hat{\rho}_{ij}^n$ are given by equations (16) and (20), respectively. Taking the square, simple calculations give

$$\begin{aligned}(e_{ij}^*)^2 = (e_{ij}^n)^2 + k\left\{\frac{a}{h^2}\left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n\right)e_{ij}^n - \frac{\mu}{2h}(e_{i,j+1}^n - e_{i,j-1}^n)e_{ij}^n\right\} + k^2\left(\rho_{ij}^n + \hat{\rho}_{ij}^n + \rho_{ij}^{\bar{*}}\right)e_{ij}^n \\ +2C_r(k^3 + kh^2)e_{ij}^n + \frac{k^3}{2}\left(\frac{a}{h^2}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) - \frac{\mu}{2h}(e_{i,j+1}^n - e_{i,j-1}^n)\right)\left(\rho_{ij}^n + \hat{\rho}_{ij}^n + \rho_{ij}^{\bar{*}}\right) + \\ C_r\left(\frac{a}{h^2}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) - \frac{\mu}{2h}(e_{i,j+1}^n - e_{i,j-1}^n)\right)(k^4 + k^2h^2) + \frac{k^4}{4}\left(\rho_{ij}^n + \hat{\rho}_{ij}^n + \rho_{ij}^{\bar{*}}\right)^2 + C_r^2(k^3 + kh^2)^2 \\ +C_r(k^5 + k^3h^2)\left(\rho_{ij}^n + \hat{\rho}_{ij}^n + \rho_{ij}^{\bar{*}}\right) + \frac{k^2}{4}\left(\frac{a}{h^2}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) - \frac{\mu}{2h}(e_{i,j+1}^n - e_{i,j-1}^n)\right)^2\end{aligned} \tag{61}$$

Applying the inequalities: $2ab \leq a^2 + b^2$, $(a \pm b)^2 \leq 2(a^2 + b^2)$ and $(a \pm b \pm c)^2 \leq 3(a^2 + b^2 + c^2)$, for every $a,b,c \in \mathbb{R}$, together with the time step restriction (41) (that is, $2ak \leq h^2$, equation (61) yields

$$(e_{ij}^*)^2 \leq (e_{ij}^n)^2 + k\left\{\frac{a}{h^2}\left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n\right)e_{ij}^n - \frac{\mu}{2h}(e_{i,j+1}^n - e_{i,j-1}^n)e_{ij}^n\right\} + k^2\left|\left(\rho_{ij}^n + \hat{\rho}_{ij}^n + \rho_{ij}^{\bar{*}}\right)e_{ij}^n\right| +$$

$$\frac{k^2}{4}\left\{\frac{a^2}{h^4}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})^2+\frac{\mu^2}{4h^2}\left(e^n_{i,j+1}-e^n_{i,j-1}\right)^2-\frac{a\mu}{h^3}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})\left(e^n_{i,j+1}-e^n_{i,j-1}\right)\right\}+$$
$$2C_r(k^3+kh^2)\,|\,e^n_{ij}\,|+\frac{k^3}{2}\left(\frac{a}{h^2}\,|\,e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}\,|+\frac{\mu}{2h}\,|\,e^n_{i,j+1}-e^n_{i,j-1}\,|\right)|\,\rho^n_{ij}+\hat{\rho}^n_{ij}+\rho^*_{ij}\,|+$$
$$C_r\left(\frac{a}{h^2}\,|\,e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}\,|+\frac{\mu}{2h}\,|\,e^n_{i,j+1}-e^n_{i,j-1}\,|\right)(k^4+k^2h^2)+\frac{3k^4}{4}\left(|\,\rho^n_{ij}\,|^2+|\,\hat{\rho}^n_{ij}\,|^2+|\,\rho^*_{ij}\,|^2\right)+C_r^2(k^3+kh^2)^2+$$
$$C_r(k^5+k^3h^2)\left(|\,\rho^n_{ij}\,|+|\,\hat{\rho}^n_{ij}\,|+|\,\rho^*_{ij}\,|\right),$$

which implies

$$(e^*_{ij})^2\le(e^n_{ij})^2+k\left\{\frac{a}{h^2}\left(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}\right)e^n_{ij}-\frac{\mu}{2h}(e^n_{i,j+1}-e^n_{i,j-1})e^n_{ij}\right\}+\frac{k}{4}(e^n_{ij})^2+k^3\left(\rho^n_{ij}+\hat{\rho}^n_{ij}+\rho^*_{ij}\right)^2$$
$$+\frac{k^2}{4}\left\{\frac{2a^2}{h^4}\left[(e^n_{i,j+1}-e^n_{ij})^2+(e^n_{ij}-e^n_{i,j-1})^2\right]+\frac{\mu^2}{4h^2}\left(e^n_{i,j+1}-e^n_{i,j-1}\right)^2-\frac{a\mu}{h^3}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})\right.$$
$$\left.(e^n_{i,j+1}-e^n_{i,j-1})\right\}+\frac{k^3}{16}\left(\frac{a}{h^2}\,|\,e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}\,|+\frac{\mu}{2h}\,|\,e^n_{i,j+1}-e^n_{i,j-1}\,|\right)^2+k^3(\rho^n_{ij}+\hat{\rho}^n_{ij}+\rho^*_{ij})^2+$$
$$\frac{k^3}{4}\left(\frac{a}{h^2}\,|\,e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}\,|+\frac{\mu}{2h}\,|\,e^n_{i,j+1}-e^n_{i,j-1}\,|\right)^2+C_r^2(k^{\frac{5}{2}}+k^{\frac{1}{2}}h^2)^2+\frac{3k^4}{4}\left(|\,\rho^n_{ij}\,|^2+|\,\hat{\rho}^n_{ij}\,|^2+|\,\rho^*_{ij}\,|^2\right)+$$
$$C_r^2(k^3+kh^2)^2+C_r^2(k^{\frac{5}{2}}+k^{\frac{1}{2}}h^2)^2+k(e^n_{ij})^2+\frac{1}{2}C_r^2(k^3+kh^2)^2+\frac{k^4}{2}\left(|\,\rho^n_{ij}\,|+|\,\hat{\rho}^n_{ij}\,|+|\,\rho^*_{ij}\,|\right)^2 \tag{62}$$
$$\le(e^n_{ij})^2+k\left\{\frac{a}{h^2}\left(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}\right)e^n_{ij}-\frac{\mu}{2h}(e^n_{i,j+1}-e^n_{i,j-1})e^n_{ij}\right\}+\frac{5k}{4}(e^n_{ij})^2+6k^3\left(|\,\rho^n_{ij}\,|^2+|\,\hat{\rho}^n_{ij}\,|^2+|\,\rho^*_{ij}\,|^2\right)$$
$$+\frac{a^2k^2}{2h^4}\left[(e^n_{i,j+1}-e^n_{ij})^2+(e^n_{ij}-e^n_{i,j-1})^2\right]+\frac{\mu^2k^2}{16h^2}\left(e^n_{i,j+1}-e^n_{i,j-1}\right)^2-\frac{a\mu k^2}{4h}(\delta_y e^n_{i,j+\frac{1}{2}}-\delta_y e^n_{i,j-\frac{1}{2}})$$
$$(\delta_y e^n_{i,j+\frac{1}{2}}+\delta_y e^n_{i,j-\frac{1}{2}})+\frac{5k^3}{8}\left(\frac{a^2}{h^4}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})^2+\frac{\mu^2}{4h^2}(e^n_{i,j+1}-e^n_{i,j-1})^2\right)+$$
$$4C_r^2(k^5+k^2h^4)+\frac{9k^4}{4}\left(|\,\rho^n_{ij}\,|^2+|\,\hat{\rho}^n_{ij}\,|^2+|\,\rho^*_{ij}\,|^2\right)+3C_r^2(k^6+k^2h^4)$$

It comes from the time step restriction (41), that is, $2ak/h^2\le 1$, together with estimate (62) that

$$(e^*_{ij})^2\le(e^n_{ij})^2+k\left\{\frac{a}{h^2}\left(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}\right)e^n_{ij}-\frac{\mu}{2h}(e^n_{i,j+1}-e^n_{i,j-1})e^n_{ij}\right\}+\frac{5k}{4}(e^n_{ij})^2$$
$$+\frac{ak}{4}\left[(\delta_y e^n_{i,j+\frac{1}{2}})^2+(\delta_y e^n_{i,j-\frac{1}{2}})^2\right]+\frac{\mu^2k}{16a}\left[(e^n_{i,j+1})^2+(e^n_{i,j-1})^2\right]-\frac{a\mu k^2}{4h}\left[(\delta_y e^n_{i,j+\frac{1}{2}})^2-(\delta_y e^n_{i,j-\frac{1}{2}})^2\right]$$
$$+\frac{5k}{8}\left(\frac{3}{4}[(e^n_{i,j+1})^2+4(e^n_{ij})^2+(e^n_{i,j-1})^2]+\frac{k\mu^2}{4a}[(e^n_{i,j+1})^2+(e^n_{i,j-1})^2]\right)+4C_r^2(k^5+kh^4)$$
$$+3C_r^2(k^6+k^2h^4)+3k^3\left(2+\frac{3k}{4}\right)\left(|\,\rho^n_{ij}\,|^2+|\,\hat{\rho}^n_{ij}\,|^2+|\,\rho^*_{ij}\,|^2\right)$$

Summing this up from $i,j=1,2,\ldots M-1$, to get

$$\sum_{i,j=1}^{M-1}(e^*_{ij})^2\le\sum_{i,j=1}^{M-1}(e^n_{ij})^2+\frac{ak}{h^2}\sum_{i,j=1}^{M-1}\left(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}\right)e^n_{ij}+\frac{ak}{4}\sum_{i,j=1}^{M-1}\left[(\delta_y e^n_{i,j+\frac{1}{2}})^2+(\delta_y e^n_{i,j-\frac{1}{2}})^2\right]+$$
$$\frac{5k}{4}\sum_{i,j=1}^{M-1}(e^n_{ij})^2-\frac{\mu k}{2h}\sum_{i,j=1}^{M-1}(e^n_{i,j+1}-e^n_{i,j-1})e^n_{ij}+\frac{15k}{32}\sum_{i,j=1}^{M-1}\left[(e^n_{i,j+1})^2+4(e^n_{ij})^2+(e^n_{i,j-1})^2\right]+$$
$$\frac{\mu^2k}{16a}\left(1+\frac{5k}{2}\right)\sum_{i,j=1}^{M-1}\left[(e^n_{i,j+1})^2+(e^n_{i,j-1})^2\right]-\frac{a\mu k^2}{4h}\sum_{i,j=1}^{M-1}\left[(\delta_y e^n_{i,j+\frac{1}{2}})^2-(\delta_y e^n_{i,j-\frac{1}{2}})^2\right] \tag{63}$$
$$+3k^3\left(2+\frac{3k}{4}\right)\sum_{i,j=1}^{M-1}\left(|\,\rho^n_{ij}\,|^2+|\,\hat{\rho}^n_{ij}\,|^2+|\,\rho^*_{ij}\,|^2\right)+C_r^2k\sum_{i,j=1}^{M-1}\left[3kh^4+4k^4+3k^5+4h^4\right]$$

Now, utilizing the boundary condition (40), $e^n_{iM}=e^n_{i0}=0$, for all $i=0,1,\ldots,M$, Lemmas 1 and 3, and multiplying both sides of inequality (63) by h^2, straightforward computations provide

$$h^2\sum_{i,j=1}^{M-1}(e^*_{ij})^2\le h^2\sum_{i,j=1}^{M-1}(e^n_{ij})^2-ak\,\|\,\delta_y e^n\,\|^2_{L^2(\Omega)}+\frac{ak}{2}\,\|\,\delta_y e^n\,\|^2_{L^2(\Omega)}+k\left[\frac{65}{16}+\frac{\mu^2}{8a}+\frac{5\mu^2k}{16a}\right]h^2\sum_{i,j=1}^{M-1}(e^n_{ij})^2$$
$$-\frac{a\mu k^2}{4h}h^2\sum_{i=1}^{M-1}\left[(\delta_y e^n_{i,M-\frac{1}{2}})^2-(\delta_y e^n_{i,\frac{1}{2}})^2\right]+C_r^2k(M-1)^2h^2\left[3kh^4+4k^4+3k^5+4h^4\right)]+$$

$$3k^3h^2(2+\frac{3k}{4})(M-1)^2\left[\hat{C}_1^2(1+\hat{C}_2h^2+\hat{C}_3h^4)^2+\hat{C}_1^2(1+\hat{C}_2h^2+\hat{C}_3h^4)^2+\hat{C}_1^2(1+\hat{C}_2h^2+\hat{C}_3h^4)^2\right]$$

$$\leq h^2\sum_{i,j=1}^{M-1}(e_{ij}^n)^2-ak\,\|\delta_y e^n\|_{L^2(\Omega)}^2+\frac{ak}{2}\|\delta_y e^n\|_{L^2(\Omega)}^2+k\left[\frac{65}{16}+\frac{\mu^2}{8a}+\frac{5\mu^2k}{16a}\right]h^2\sum_{i,j=1}^{M-1}(e_{ij}^n)^2$$

$$+\frac{a\mu k^2}{2h}\|\delta_y e^n\|_{L^2(\Omega)}^2+C_r^2k\left[3kh^4+4k^4+3k^5+4h^4\right]+9k^3(2+\frac{3k}{4})\hat{C}_1^2(1+\hat{C}_2h^2+\hat{C}_3h^4)^2 .$$

Since $h=1/M$, $kh^4\leq k^2+h^8$ and $(1+\hat{C}_2h^2+\hat{C}_3h^4)^2\leq 3(1+\hat{C}_2^2h^4+\hat{C}_3^2h^8)$, using the time step restriction (41), that is, $|\mu k|/h\leq 1$, this becomes

$$h^2\sum_{j,i=1}^{M-1}(e_{ij}^{*})^2\leq h^2\sum_{i,j=1}^{M-1}(e_{ij}^n)^2+k\left[\frac{65}{16}+\frac{\mu^2}{8a}+\frac{5\mu^2k}{16a}\right]h^2\sum_{i,j=1}^{M-1}(e_{ij}^n)^2+C_r^2k\left[3k^2+4k^4+3k^5+4h^4+3h^8\right]+\ 27k^3(2+\frac{3k}{4})\hat{C}_1^2(1+\hat{C}_2^2h^4+\hat{C}_3^2h^8) ,$$

which implies

$$\|e^{*}\|_{L^2}^2\leq\|e^n\|_{L^2}^2+\hat{C}_4k\left\{(1+k)\|e^n\|_{L^2}^2+k^2(1+k)(1+h^4+h^8)+k^2+k^4+k^5+h^4+h^8\right\} , \tag{64}$$

where we absorbed all the constants into a positive constant $\hat{C}_4$. In way similar, one shows that

$$\|e^{**}\|_{L^2}^2\leq\|e^{*}\|_{L^2}^2+\hat{C}_5k\left\{(1+k)\|e^{*}\|_{L^2}^2+k^2(1+k)(1+h^4+h^8)+k^2+k^4+k^5+h^4+h^8\right\} , \tag{65}$$

where all the constants have been absorbed into a constant $\hat{C}_5>0$, and

$$\|e^{n+1}\|_{L^2}^2\leq\|e^{**}\|_{L^2}^2+\hat{C}_6k\left\{(1+k)\|e^{**}\|_{L^2}^2+k^2(1+k)(1+h^4+h^8)+k^2+k^4+k^5+h^4+h^8\right\} , \tag{66}$$

where all the constants have been absorbed into a constant $\hat{C}_6>0$. Now, setting

$$\varphi_1(k,h)=1+k \text{ and } \varphi_2(k,h)=k^2(1+k)(1+h^4+h^8)+k^2+k^4+k^5+h^4+h^8 , \tag{67}$$

plugging estimates (64)-(67), direct calculations result in

$$\|e^{n+1}\|_{L^2}^2\leq\|e^n\|_{L^2}^2+k\left\{\hat{C}_4+\hat{C}_5+\hat{C}_6+k\left[\hat{C}_4\hat{C}_5+\hat{C}_6(\hat{C}_4+\hat{C}_5)+k\hat{C}_4\hat{C}_5\hat{C}_6\varphi_1(k,h)\right]\varphi_1(k,h)\right\}\times$$

$$\varphi_1(k,h)\|e^n\|_{L^2}^2+k\left\{\hat{C}_4+\hat{C}_5+\hat{C}_6+k\left[\hat{C}_4\hat{C}_5+\hat{C}_4\hat{C}_6+\hat{C}_5\hat{C}_6+k\hat{C}_4\hat{C}_5\hat{C}_6\varphi_1(k,h)\right]\varphi_1(k,h)\right\}\varphi_2(k,h) ,$$

where "$\times$" denotes the usual multiplication. Absorbing all the constants into a positive constant $\hat{C}_7$, this yields

$$\|e^{n+1}\|_{L^2}^2\leq\|e^n\|_{L^2(\Omega)}^2+\hat{C}_7k\left\{\left[1+k(1+k\varphi_1(k,h))\varphi_1(k,h)\right]\varphi_1(k,h)\|e^n\|_{L^2}^2+\left[1+k[1+k\varphi_1(k,h)]\varphi_1(k,h)\right]\varphi_2(k,h)\right\} .$$

Summing this up from $n=1,2,..,p-1$, for any nonnegative integer p such that, $1\leq p\leq N$, we obtain

$$\|e^p\|_{L^2}^2\leq\|e^1\|_{L^2}^2+\hat{C}_7k\left\{\left[1+k(1+k\varphi_1(k,h))\varphi_1(k,h)\right]\varphi_1(k,h)\sum_{n=1}^{p-1}\|e^n\|_{L^2}^2+p\left[1+k[1+k\varphi_1(k,h)]\varphi_1(k,h)\right]\varphi_2(k,h)\right\} . \tag{68}$$

Applying the Gronwall Lemma, estimate (68) provides

$$\|e^p\|_{L^2}^2\leq\hat{C}_7\|e^1\|_{L^2}^2\,pk\exp\left\{\hat{C}_7pk\left[1+k(1+k\varphi_1(k,h))\varphi_1(k,h)\right]\varphi_1(k,h)\right\}\left[1+k[1+k\varphi_1(k,h)]\varphi_1(k,h)\right]\times\varphi_2(k,h) . \tag{69}$$

Since $p\leq N$, $k=T/N$, so $\hat{C}_7kp=\hat{C}_7Tp/N\leq\hat{C}_7T$. Combining this and estimate (69) to get

$$\|e^p\|_{L^2}^2\leq\hat{C}_7T\|e^1\|_{L^2}^2\exp\left\{\hat{C}_7T\left[1+k(1+k\varphi_1(k,h))\varphi_1(k,h)\right]\varphi_1(k,h)\right\}\left[1+k[1+k\varphi_1(k,h)]\varphi_1(k,h)\right]\times\varphi_3(k,h)^2 , \tag{70}$$

where $\varphi_3(k,h)^2=\varphi_2(k,h)$, $\varphi_2(k,h)$ is given by equation (67). Taking the square root, it is easy to see that

$$\|e^p\|_{L^2}\leq\|e^1\|_{L^2}\sqrt{\hat{C}_7T\left[1+k[1+k\varphi_1(k,h)]\varphi_1(k,h)\right]}\exp\left\{\frac{\hat{C}_7T}{2}\left[1+k(1+k\varphi_1(k,h))\varphi_1(k,h)\right]\varphi_1(k,h)\right\}\times\varphi_3(k,h) . \tag{71}$$

Since $\varphi_3(k,h)^2=\varphi_2(k,h)$, it comes from equation (67) that

$$\varphi_3(k,h)^2=k^2(1+k)(1+h^4+h^8)+k^2+k^4+k^5+h^4+h^8\leq(k+h^4)^2\psi(k,h) ,$$

where

$$\psi(k,h)=2+k^2+k^3+h^2+(1+k)(1+h^4+h^8) .$$

Utilizing relation (67); simple computations gives

$$1+k\left[1+k\varphi_1(k,h)\right]\varphi_1(k,h)=1+k+2k^2+2k^3+k^4\le(1+k)^4,$$

and

$$\left[1+k\left(1+k\varphi_1(k,h)\right)\varphi_1(k,h)\right]\varphi_1(k,h)=1+2k+3k^2+4k^3+3k^4+k^5\le(1+k)^5.$$

Finally, taking the maximum over p of estimate (70), for $1\le p\le N$ (Indeed, $\|e^0\|_{L^2}=0$), the proof of Theorem 2 is completed thanks to inequality $\|e^1\|_{L^2}\le\|u^1\|_{L^2}+\|\bar{u}^1\|_{L^2}$ and equation (8).

5. Numerical experiments and convergence rate

In this section we construct an exact solution to the initial-boundary value problem (1)-(3). Furthermore, we present two other examples described in the literature [18, 20] to demonstrate the efficiency and utility of the proposed numerical scheme in bidimensional case. In each case we obtain satisfactory results, so our algorithm performances are not worse for multidimensional problems. The predicted convergence rate from the theoretical analysis is confirmed (see Section 2). This convergence rate is obtained by listing in Tables 1-9 the errors between the computed solution and the exact ones with different values of mesh size h and time step k, satisfying $k=h^2/2$. Finally, we look at the error estimates of our proposed method for the parameter $T=1$, $a=1$ and $\mu\in\{1,0.8,-1\}$.

Assuming that the exact solution to problem (1)-(3) is of the form $\bar{u}(x,y,t)=\left[1+\exp(ct+dx+by)\right]^{-n}$ where n is an integer. By simple calculations, it holds

$$\bar{u}_t(x,y,t)=-nc\exp(ct+dx+by)\left[1+\exp(ct+dx+by)\right]^{-n-1}, \tag{71}$$

$$\bar{u}_x(x,y,t)=-nd\exp(ct+dx+by)\left[1+\exp(ct+dx+by)\right]^{-n-1}, \tag{72}$$

$$\bar{u}_y(x,y,t)=-nb\exp(ct+dx+by)\left[1+\exp(ct+dx+by)\right]^{-n-1}, \tag{73}$$

and

$$\bar{u}_{xx}(x,y,t)=-nd^2\exp(ct+dx+by)[1-n\exp(ct+dx+by)]\left[1+\exp(ct+dx+by)\right]^{-n-2}. \tag{74}$$

In similar way

$$\bar{u}_{yy}(x,y,t)=-nb^2\exp(ct+dx+by)[1-n\exp(ct+dx+by)]\left[1+\exp(ct+dx+by)\right]^{-n-2}. \tag{75}$$

Combining equations (71)-(75), it is not hard to see that

$$\bar{u}_t+\bar{u}_x+\bar{u}_y-(\bar{u}_{xx}+\bar{u}_{yy})=-n\exp(ct+dx+by)\left(1+\exp(ct+dx+by)\right)^{-n-1}\Big\{c+b+d-(b^2+d^2)$$
$$\left[1-n\exp(ct+dx+by)\right]\left(1+\exp(ct+dx+by)\right)^{-1}\Big\}.$$

Setting $c+b+d=d^2+b^2$, this gives

$$\begin{aligned}\bar{u}_t+\bar{u}_x+\bar{u}_y-(\bar{u}_{xx}+\bar{u}_{yy})&=-n(d^2+b^2)\exp(ct+dx+by)\left(1+\exp(ct+dx+by)\right)^{-n-1}\\&\Big\{1-\left[1-n\exp(ct+dx+by)\right]\left[1+\exp(ct+dx+by)\right]^{-1}\Big\}.\end{aligned} \tag{76}$$

For $n=-1$, equation (76) becomes

$$\bar{u}_t+\bar{u}_x+\bar{u}_y-(\bar{u}_{xx}+\bar{u}_{yy})=0.$$

Setting $d^2+b^2=1$, this gives $b^2=1-d^2$. Since b^2 must be strictly greater than zero, this implies $d^2<1$. For $d=-\sqrt{2}/2$, we can take $b=\sqrt{2}/2$, and then $c=1$. So the exact solution is given by $\bar{u}(x,y,t)=1+\exp\left(t-\sqrt{2}x/2+\sqrt{2}y/2\right)$, for $t\in[0,1]$ and $(x,y)\in[0,1]^2$. The initial and boundary conditions are determined by this solution.

To analyze the convergence rate of our numerical scheme, we assume that the mesh size $h\in\left\{1/2^l,\ l=1,2,\dots,5\right\}$ and time step $k=2^{-l}$, $l=3,4,\dots,11$. Under the time step restriction (41), we set $k=h^2/2$ and we compute the error estimates: $\|\!|E(u)|\!\|_{L^2(0,T;L^2)}$, $\|\!|E(u)|\!\|_{L^\infty(0,T;L^2)}$ and $\|\!|E(u)|\!\|_{L^1(0,T;L^2)}$ related to the time-split method to see that the algorithm is stable, second order accurate in time and fourth order convergent in space. Furthermore, we plot the exact solution, the numerical solution and the errors versus n. It follows from this analysis that a three-level time-split MaCormack method is efficient and effective than a large class of numerical methods widely studied in literature [10, 11, 17, 19, 21, 22, 26, 47]. Finally, when h varies in the given range, we observe from Tables 1-9 that the infinitesimal terms (that is, the errors) $O(k^{\beta})+O(h^{\theta})$ are dominated by the h-terms $O(h^{\theta})$ (or k-terms $O(k^{\beta})$). So, the ratio r_u^m, where $m=1,2,\infty$ of the approximation errors on two adjacent mesh levels Ω_{2h} and Ω_h is approximately $(2h)^{\theta}/h^{\theta}=2^{\theta}$, where m refers to the $L^m(0,T;L^2(\Omega))$-error norm. Thus, we can simply use r_u^m to estimate the corresponding convergence rate with respect to h. Define the norms for the approximate solution u (indicated by "U" in the figures), the exact one $\bar{u}$ (represented by "u" in the figures) and the errors $E(u)$ (simply denoted by "E" in the figures) as follows

$$|||u|||_{L^2(0,T;L^2)}=\left[k\sum_{n=0}^{N}\|u^n\|_{L^2_f}^2\right]^{\frac{1}{2}},\ |||\bar{u}|||_{L^2(0,T;L^2)}=\left[k\sum_{n=0}^{N}\|\bar{u}^n\|_{L^2_f}^2\right]^{\frac{1}{2}},$$

$$|||E(u)|||_{L^2(0,T;L^2)}=\left[k\sum_{n=0}^{N}\|u^n-\bar{u}^n\|_{L^2_f}^2\right]^{\frac{1}{2}},\ |||E(u)|||_{L^1(0,T;L^2)}=k\sum_{n=0}^{N}\|u^n-\bar{u}^n\|_{L^2_f},$$

$$|||E(u)|||_{L^\infty(0,T;L^2)}=\max_{0\le n\le N}\|u^n-\bar{u}^n\|_{L^2_f}.$$

• **Test 1.** Let Ω be the unit square $(0,1)\times(0,1)$ and T be the final time, $T=1$. We assume that the parameters $a=\mu=1$, in such a way that the exact solution $\bar{u}$ is given by

$$\bar{u}(x,y,t)=1+\exp\left(t-\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y\right).$$

The initial and boundary conditions are determined by this solution. We take the mesh size and time step: $h\in\{2^{-l}, l=1,2,...,5\}$ and $k=2^{-l}, l=2,3,...,11$.

• **Test 2.** Now, let Ω be the unit square $(0,1)\times(0,1)$ and $T=1$. The diffusive term a is assumed equals 1 and the parameter $\mu=0.8$. The analytic solution $\bar{u}$ is defined in [18, 20] by

$$\bar{u}(x,y,t)=\frac{1}{1+4t}\exp\left(-a\frac{(x-\mu t-0.5)^2}{1+4t}-a\frac{(y-\mu t-0.5)^2}{1+4t}\right),$$

Tables 1-3. Analyzing of stability and convergence rate $O(h^{\theta}+\Delta t^{\beta})$ for time-split MacCormack by r_u^m, under the time step restriction (41), that is, $\max\{2kh^{-2},kh^{-1}\}\le 1$, with varying time step $k=\Delta t$ and mesh grid $h=\Delta x=\Delta y$.

Case: $k=\frac{1}{2}h^2$

h	$\|\|E(u)\|\|_{L^2}$	r_u^2	$\|\|E(u)\|\|_{L^\infty}$	r_u^∞	$\|\|E(u)\|\|_{L^1}$	r_u^1
2^{-1}	0.1170×10^{0}	----	0.1653×10^{0}	----	0.1133×10^{0}	----
2^{-2}	0.2870×10^{-1}	4.0767	0.4270×10^{-1}	3.8712	0.2750×10^{-1}	4.1200
2^{-3}	0.7700×10^{-2}	3.7273	0.1180×10^{-1}	3.6186	0.7400×10^{-2}	3.7162
2^{-4}	0.2000×10^{-2}	3.8500	0.3100×10^{-2}	3.8065	0.1900×10^{-2}	3.8947
2^{-5}	0.4862×10^{-3}	4.1135	0.7595×10^{-3}	4.0816	0.4632×10^{-3}	4.1019

Decay of energy

h	$\|\|u\|\|_{L^2}^2$	$\|\|\bar{u}\|\|_{L^2}^2$	$\|\|u\|\|_{L^\infty}^2$	$\|\|\bar{u}\|\|_{L^\infty}^2$	$\|\|u\|\|_{L^1}^2$	$\|\|\bar{u}\|\|_{L^1}^2$
2^{-1}	2.3995×10^{1}	2.3654×10^{1}	3.8764×10^{1}	3.8438×10^{1}	2.3995×10^{1}	2.3654×10^{1}
2^{-2}	1.42641×10^{1}	1.4150×10^{1}	2.5499×10^{1}	2.5330×10^{1}	1.4264×10^{1}	1.4150×10^{1}
2^{-3}	1.0938×10^{1}	1.0904×10^{1}	2.0042×10^{1}	1.9981×10^{1}	1.0938×10^{1}	1.0904×10^{1}
2^{-4}	9.5602	9.5509	1.7605×10^{1}	1.7587×10^{1}	9.5602	9.5509
2^{-5}	8.9336	8.9312	1.6463×10^{1}	1.6458×10^{1}	8.9336	8.9312

Case: $k=h$

h	$\|\|E(u)\|\|_{L^2}$	r_u^2	$\|\|E(u)\|\|_{L^\infty}$	r_u^∞	$\|\|E(u)\|\|_{L^1}$	r_u^1
2^{-1}	2.7878	----	3.6899	----	2.5392	----
2^{-2}	0.3174×10^{7}	0.8783×10^{-6}	0.6348×10^{7}	0.5813×10^{-6}	0.1591×10^{7}	0.1596×10^{-5}
2^{-3}	0.5816×10^{24}	0.5457×10^{-17}	1.6451×10^{24}	0.3859×10^{-17}	0.2057×10^{24}	0.7735×10^{-17}

Tables 4-6. Stability and convergence rate $O(h^{\theta}+\Delta t^{\beta})$ for time-split MacCormack by r_u^m, with varying spacing h and time step k, under the time step restriction (41), that is, $\max\{2kh^{-2},\mu kh^{-1}\}\leq 1$.

Case: $k=2^{-1}h^2$

h	$\|\|E(u)\|\|_{L^2}$	r_u^2	$\|\|E(u)\|\|_{L^\infty}$	r_u^∞	$\|\|E(u)\|\|_{L^1}$	r_u^1
2^{-1}	0.3050×10^{-1}	$---$	0.6500×10^{-1}	$---$	0.2410×10^{-1}	$---$
2^{-2}	0.9600×10^{-2}	3.1771	0.194×10^{-1}	3.3163	0.7800×10^{-2}	3.0897
2^{-3}	0.27×10^{-2}	3.5556	0.56×10^{-2}	3.4643	0.22×10^{-2}	3.5455
2^{-4}	0.7000×10^{-3}	3.8571	0.1500×10^{-2}	3.7333	0.6000×10^{-3}	3.6667
2^{-5}	0.2000×10^{-3}	3.5000	0.4000×10^{-3}	3.7500	0.1000×10^{-3}	6.0000

Decay of energy

h	$\|\|u\|\|_{L^2}^2$	$\|\|\bar{u}\|\|_{L^2}^2$	$\|\|u\|\|_{L^\infty}^2$	$\|\|\bar{u}\|\|_{L^\infty}^2$	$\|\|u\|\|_{L^1}^2$	$\|\|\bar{u}\|\|_{L^1}^2$
2^{-1}	3.6560×10^{-1}	3.7550×10^{-1}	1.2244×10^{0}	1.2244×10^{0}	3.6560×10^{-1}	3.7550×10^{-1}
2^{-2}	2.2890×10^{-1}	2.3400×10^{-1}	9.8910×10^{-1}	9.8910×10^{-1}	2.2890×10^{-1}	2.3400×10^{-1}
2^{-3}	1.8580×10^{-1}	1.8740×10^{-1}	8.6170×10^{-1}	8.6170×10^{-1}	1.8580×10^{-1}	1.8740×10^{-1}
2^{-4}	1.6800×10^{-1}	1.6840×10^{-1}	7.9700×10^{-1}	7.9700×10^{-1}	1.6800×10^{-1}	1.6840×10^{-1}
2^{-5}	1.5970×10^{-1}	1.5980×10^{-1}	7.6450×10^{-1}	7.6450×10^{-1}	1.5970×10^{-1}	1.5980×10^{-1}

Case: $k=h\leq\mu^{-1}h$

h	$\|\|E(u)\|\|_{L^2}$	r_u^2	$\|\|E(u)\|\|_{L^\infty}$	r_u^∞	$\|\|E(u)\|\|_{L^1}$	r_u^1
2^{-1}	0.1400	---	0.1828	---	0.1295	---
2^{-2}	0.1088×10^{7}	1.2868×10^{-17}	. 0.2176×10^{7} .	0.8393×10^{-7}	54.52×10^{7}	0.2375×10^{-7}
2^{-3}	0.3903×10^{24}	0.2788×10^{-17}	1.1039×10^{24}	0.1971×10^{-17}	0.1380×10	3.9507×10^{-17}

Tables 7-9. Stability analysis and convergence rate $O(h^{\theta}+\Delta t^{\beta})$ for time-split MacCormack by r_u^m with varying spacing h and time step Δt.

Case: $k=2^{-1}h^2$

h	$\|\|E(u)\|\|_{L^2}$	r_u^2	$\|\|E(u)\|\|_{L^\infty}$	r_u^∞	$\|\|E(u)\|\|_{L^1}$	r_u^1
2^{-1}	0.8640×10^{-1}	$---$	0.1238×10^{0}	$---$	0.831×10^{-1}	$---$
2^{-2}	0.2260×10^{-1}	3.8230	0.3380×10^{-1}	3.6627	0.2170×10^{-1}	3.8295
2^{-3}	0.6200×10^{-2}	3.6452	0.9500×10^{-2}	3.5579	0.6000×10^{-2}	3.6167
2^{-4}	0.16×10^{-2}	3.8750	0.2500×10^{-2}	3.800	0.1600×10^{-2}	3.7500
2^{-5}		4.0000	0.6000×10^{-3}	4.1667	0.4000×10^{-3}	4.0000

Decay of energy

h	$\|\|u\|\|_{L^2}^2$	$\|\|\bar{u}\|\|_{L^2}^2$	$\|\|u\|\|_{L^\infty}^2$	$\|\|\bar{u}\|\|_{L^\infty}^2$	$\|\|u\|\|_{L^1}^2$	$\|\|\bar{u}\|\|_{L^1}^2$
2^{-1}	1.2126×10^{1}	1.1980×10^{1}	2.3821×10^{1}	2.3697×10^{1}	1.2126×10^{1}	1.1980×10^{1}
2^{-2}	6.5672×10^{0}	6.5110×10^{1}	1.4545×10^{1}	1.4462×10^{1}	6.5672×10^{0}	6.5110×10^{0}
2^{-3}	4.7867×10^{0}	4.7691×10^{0}	1.0952×10^{1}	1.0919×10^{1}	4.7867×10^{0}	4.7691×10^{0}
2^{-4}	4.0754×10^{0}	4.0705×10^{0}	9.4014×10^{0}	9.3911×10^{0}	4.0754×10^{0}	4.0705×10^{0}
2^{-5}	3.7584×10^{0}	3.7571×10^{0}	8.6876×10^{0}	8.6847×10^{0}	3.7584×10^{0}	3.7571×10^{0}

Case: $k=h=\|\mu^{-1}\|h$

h	$\|\|E(u)\|\|_{L^2}$	r_u^2	$\|\|E(u)\|\|_{L^\infty}$	r_u^∞	$\|\|E(u)\|\|_{L^1}$	r_u^1
2^{-1}	3.4220	---	4.4885	---	3.1490	---

The initial and boundary conditions are given by the exact solution $\bar{u}$. Similar to Test 1, we take the mesh size and time step: $h = 2^{-l}, l = 1,2,\ldots,5$ and $k \in \{2^{-l}, l = 2,3,\ldots,11\}$.

- **Test 3.** Finally, let Ω be the unit square $(0,1)\times(0,1)$ be the unit square and $T = 1$. Assuming that the diffusive term $a = 1$ and $\mu = -1$, the exact solution $\bar{u}$ is given in [53] by

$$\bar{u}(x,y,t) = \exp(t)\left[\exp\left(\frac{\mu-\sqrt{5}}{2}x\right) + \exp\left(\frac{\mu-\sqrt{5}}{2}y\right)\right].$$

The initial and boundary conditions are given by the exact solution $\bar{u}$.
Similar to both **Tests 1, 2**, the time step and grid spacing are chosen such that: $k \in \{2^{-l}, l = 2,3,\ldots,11\}$ and $h = 2^{-l}, l = 1,2,\ldots,5$. We compute the error estimates: $E(u)$ related to a three-level time-split MacCormack approach to see that the method is second order convergent in time and fourth order accurate in space. Furthermore, we plot the errors together with the energies versus n to see the efficiency and effectiveness of the considered algorithm.

5.1. Stability and convergence of a three-level time-split MacCormack method: a = 1 and $\mu = 1$

The analysis on the convergence of the numerical scheme presented in Section 4 has suggested that our algorithm is first order convergent in time and fourth order accurate in space. If the result provided in Section 2, page 11 is to believe, this shows that the time-split MacCormack scheme is inconsistent. Surprisingly, it comes from **Tests 1-3**, more precisely **Figures 2-4** and **Tables 1-9**, that the three-level time-split MacCormack technique is stable, second order accurate in time and fourth order convergent in space under the time step restriction (41) , which confirms the theoretical result provided in Section 2, page 11. Thus, the considered method applied to initial-boundary value problem (1)-(3) is: stable, consistent, second order convergent in time and fourth order convergent in space. Furthermore, it is worth noticing to mention that the proposed method is fast and requires least computational costs than a broad range of numerical schemes [10,11,17,19,21,22,26,47] for solving equations (1)-(3). Finally, numerical examples suggest that the new algorithm can be a robust tools for the integration of general systems of hyperbolic/parabolic PDEs.

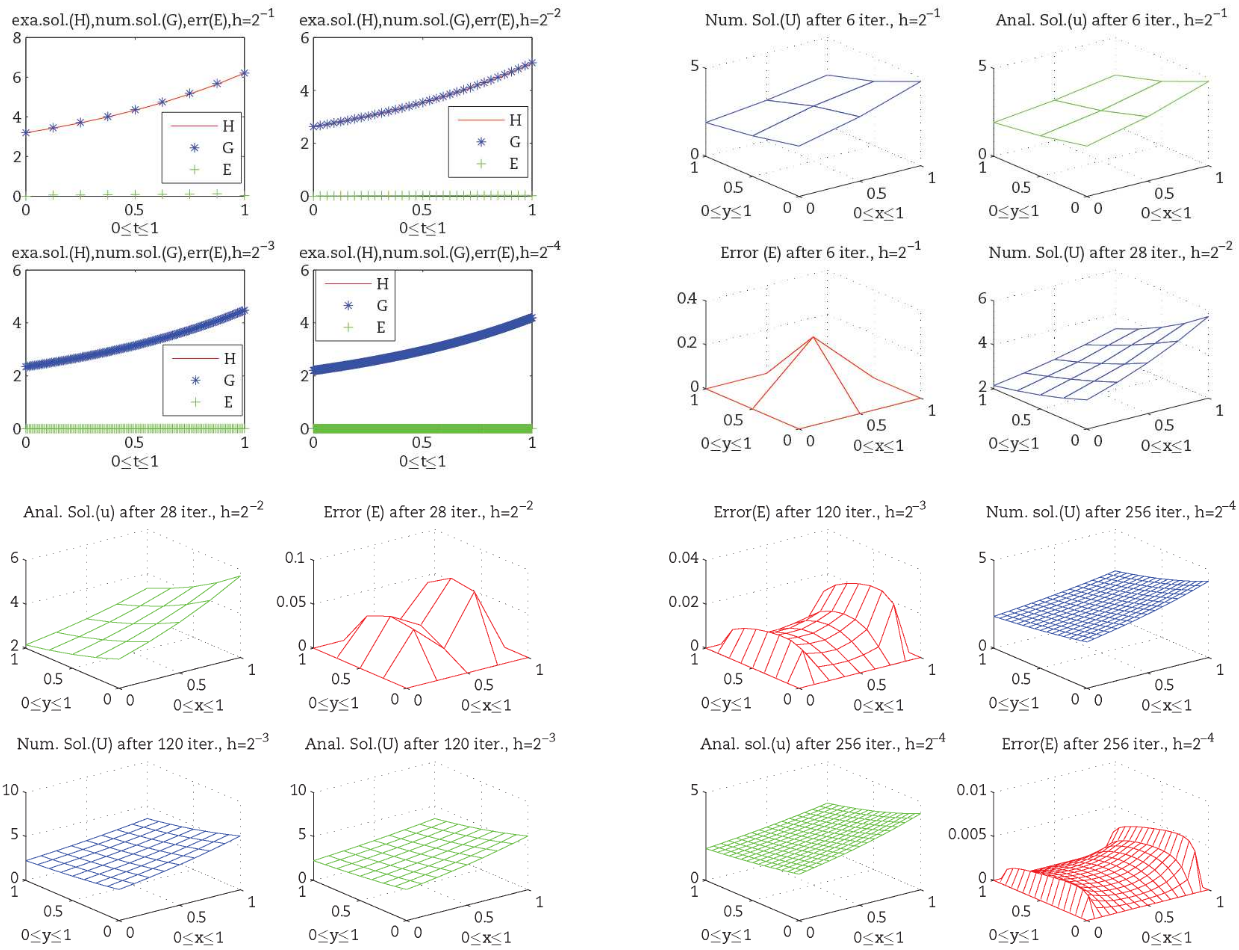

Fig. 2. (Exact sol. (green), Numerical sol. (blue), Error (red)), $\bar{u} = 1 + \exp\left(t + \sqrt{2}x/2 - \sqrt{2}y/2\right)$.

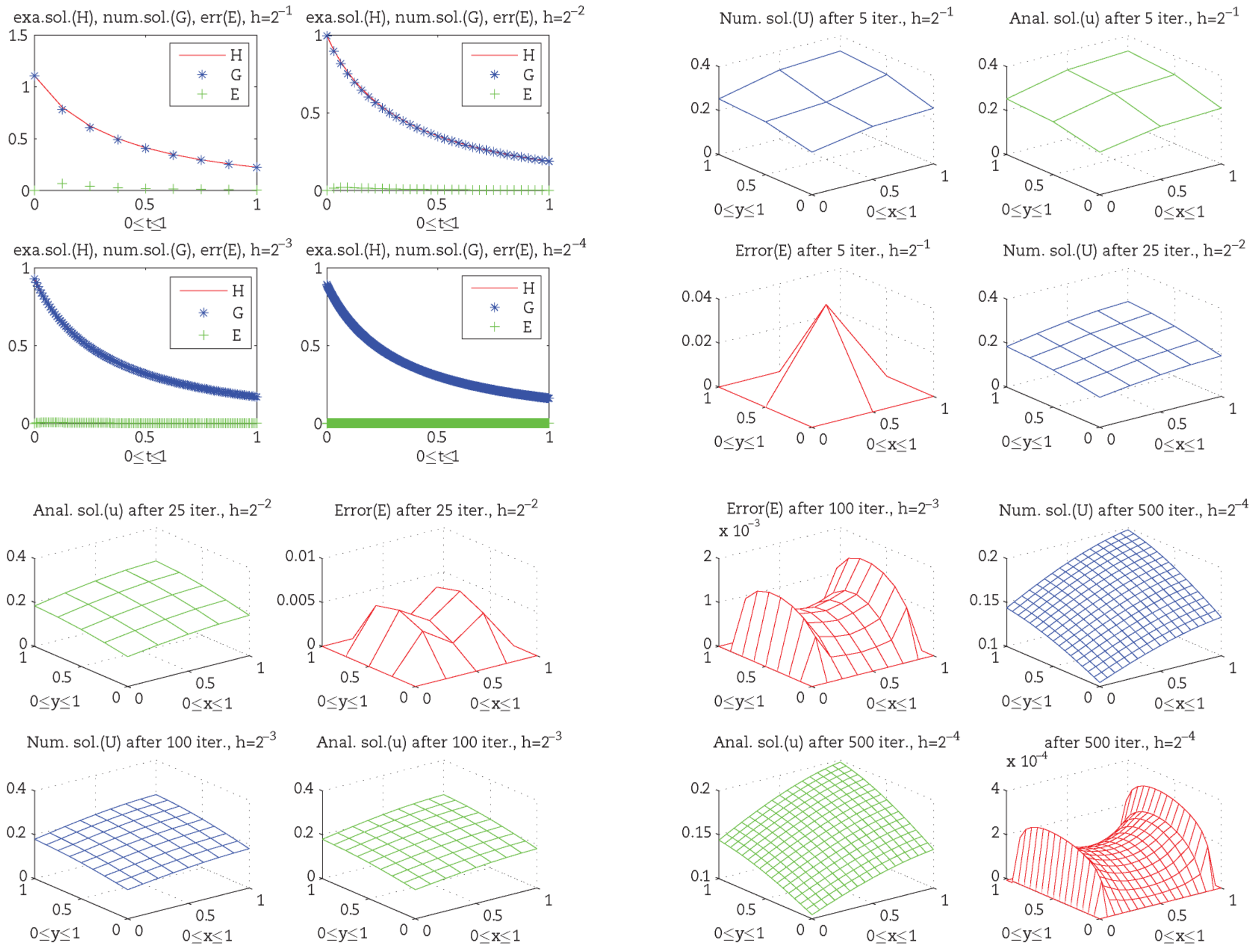

Fig. 3. $a = 1$, $\mu = 0.8$ and $\overline{u} = \exp\left(-a(x - \mu t - 0.5)^2 / (1 + 4t) - a(y - \mu t - 0.5)^2 / (1 + 4t)\right) / (1 + 4t)$

6. General conclusion and future works

In this paper we have developed a robust three-level explicit time-split MacCormack scheme for solving the two-dimensional unsteady convection-diffusion equation (1)-(3), and we have analyzed the stability (Theorem 1), error estimates (Theorem 2) and the convergence rate (**Tables 1-9**) of the proposed approach. The analysis has suggested that our method is stable, consistent, second-order accuracy in time and fourth-order convergent in space under the time step restriction (41). This convergence rate is confirmed by a wide set of numerical examples (Figures 2-4 and **Tables 1-9**). Numerical evidences also indicate that the new algorithm is: (a) more efficient and effective than a large class of numerical schemes applied to problem (1)-(3), and (b) robust tools for the integration of general systems of hyperbolic/parabolic PDEs. For problem with high Reynolds number flows where the viscous region becomes too thin, MacCormack developed a hybrid version of his scheme, so called MacCormack rapid solver method [37]. The new hybrid scheme is an explicit-implicit method which can be proved to be from 10 to 100 faster than a time-split MacCormack algorithm (for example, the readers can consult [54], p. 632). Our future investigations will consider the three-level time-split MacCormack for solving two-dimensional evolutionary advection-diffusion equations with variable coefficients.

Acknowledgment

The author would like to thank both editor and anonymous referees for valuable comments which have helped to greatly improve the quality of this paper. This work has been partially supported by the deanship of scientific research of Al-Imam Muhammad Ibn Saud Islamic University (IMSIU) under the Grant No. 331203.

Conflict of interest

The author declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

This work has been partially supported by the deanship of scientific research of Al-Imam Muhammad Ibn Saud Islamic University (IMSIU) under the Grant No. 331203.

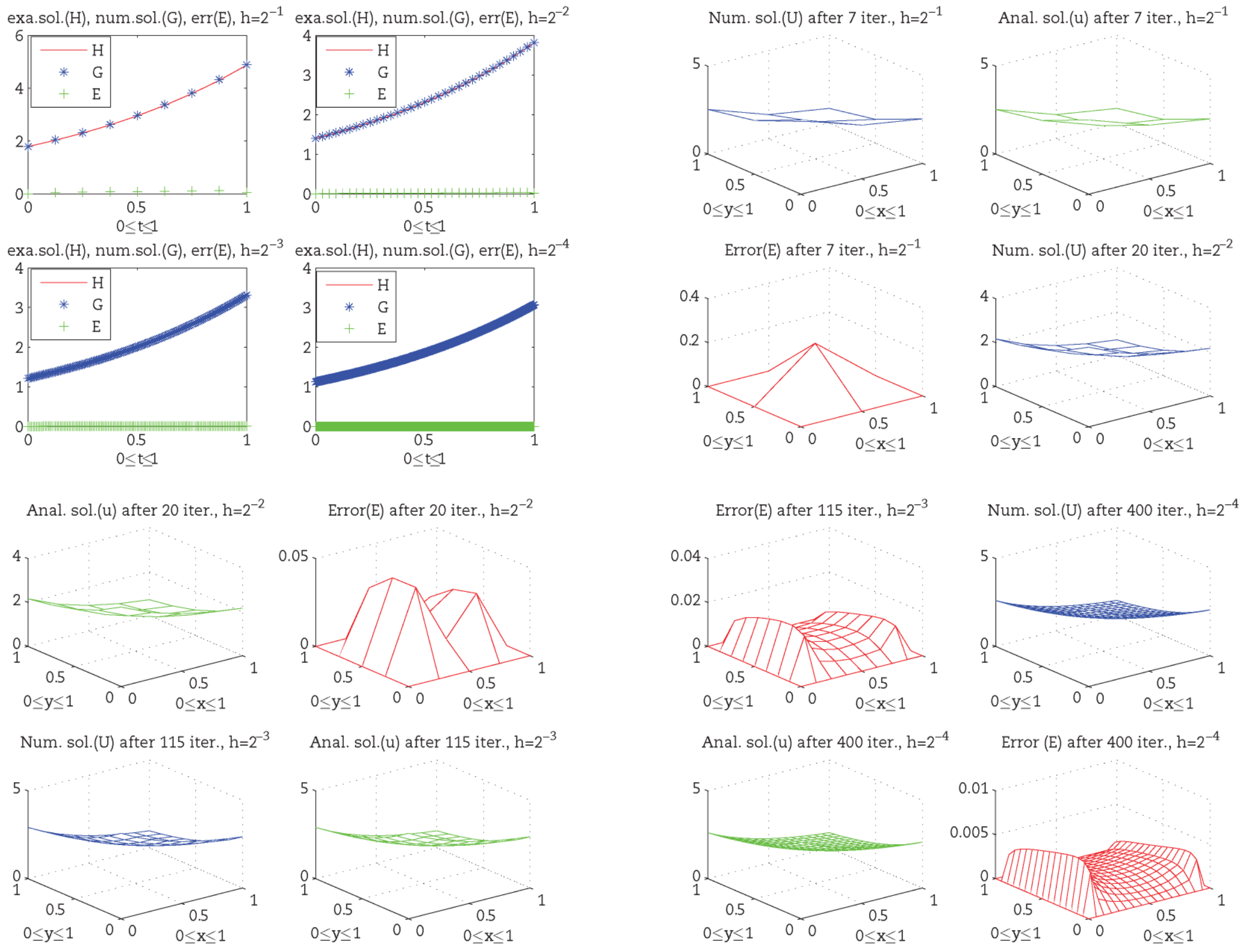

Fig. 4. $a=1$, $\mu=-1$ and $\bar{u}(x,y,t)=\exp(t)\left[\exp\left((\mu-\sqrt{5})x/2\right)+\exp\left((\mu-\sqrt{5})y/2\right)\right]$

Nomenclature

μ	Convective term	$L^2(\Omega)$, $L^2(0,T;L^2(\Omega))$	Spaces of square integrable functions
(u,u)	Velocity of the fluid in region $\Omega=(0,1)\times(0,1)$	$\lVert\cdot\rVert_{L^2}$ $(\cdot,\cdot)$, $\langle\cdot,\cdot\rangle_x$, $\langle\cdot,\cdot\rangle_y$	Usual norm and scalar products defined on $L^2(\Omega)$
T	Final time	a	Diffusive coefficient
x,y	Space-variables	$\partial\Omega$	Boundary of Ω
φ	Boundary condition	t	Time-variable
$\partial u/\partial t$	Partial derivative of u with respect to t	u_0	Initial condition
$\partial u/\partial x$, $\partial u/\partial y$	Partial derivative of u with respect to x and y	Δx, Δy	Space steps
$A_j(u)$	One-dimensional differential operator	$t^{*}, t^{**}, \bar{t}^{*}, \bar{\bar{t}}^{*}$	Predicted times
Δt_x , Δt_y	Time step in the direction of x and y	$u^{*}, u^{**}, \bar{u}^{*}, \bar{\bar{u}}^{*}$	Predicted values
$L_x(\Delta t_x)$, $L_y(\Delta t_y)$	One-dimensional difference operators		

References

[1] Yu, C. P., Liu, C. S., Taulbee, D. B., Simultaneous diffusion and sedimentation of aerosols in a horizontal cylinder, *Journal of Aerosol Science*, 8(5), 1977, 309-316.
[2] Menni, Y., Ghazvini, M., Ameur, H. *et al.* Combination of baffling technique and high-thermal conductivity fluids to enhance the overall performances of solar channels, *Engineering with Computers,* 2020, https://doi.org/10.1007/s00366-020-01165-x.
[3] Menni, Y., Chamkha, A., J., Lorenzini G., Benyoucef, B., Computational fluid dynamics based numerical simulation of thermal and thermo-hydraulic performance of a solar air heater channel having various ribs on absorber plates, *Mathematical Modelling of Engineering Problems*, 6(2), 2019, 170-174.
[4] Menni, Y., Azzi, A., Chamkha, A., J., Computational thermal analysis of turbulent forced-convection flow in an air channel with a flat rectangular fin and downstream v-shaped baffle, *Heat Transfer Research*, 50(18), 2019, 1781-1818.
[5] Menni, Y., Azzi, A., Didi, F., Harmand, S., Computational fluid dynamical analysis of new obstacle design and its impact on the heat transfer enhancement in a specific type of air flow geometry, *Computational Thermal Sciences: An International Journal*, 10(5), 2018, 421-447.

[6] Sheikholeslami, M., Jafaryar, M., Sheremet, M., A., Shafee, A., Babazadeh, H., Nanomaterial thermal performance within a pipe in presence of turbulator, *Appllied Nanoscience*, 10, 2020, 3421–3430.
[7] Menni, Y., Ameur, H., Inc, M., Improvement of the performance of solar channels by using vortex generators and hydrogen fluid, *Journal of Thermal Analysis and Calorimetry*, 2020, https://doi.org/10.1007/s10973-020-10239-3.
[8] Sheikholeslami, M., Farshad, S., A., Shafee, A., Babazadeh, H., Numerical modeling for nanomaterial behavior in a solar unit analyzing entropy generation, *Journal of the Taiwan Institute of Chemical Engineers*, 112, 2020, 271-285.
[9] Menni, Y., Ghazvini, M., Ameur, H., Ahmadi, H. M., Sharifur, M., Sadeghzadeh, M., Numerical calculations of the thermal-aerodynamic characteristics in a solar duct with multiple V-baffles, *Engineering Applications of Computational Fluid Mechanics*, 14(1), 2020, 1173-1197.
[10] Dehghan, Y. M, On the numerical solution of the one-dimensional convection-diffusion equation, *Mathematic Problems in Engineering*, 500(1), 2005, 61-74.
[11] Dehghan, Y. M., Weighted finite difference techniques for the one-dimensional advection-diffusion equation, *Applied Mathematics and Computation*, 147(2), 2004, 307-319.
[12] Isenberg, J., Guttfinger, C., Heat transfer to a draining film, *International Journal of Heat and Mass Transfer*, 16(2), 1973, 505-512.
[13] P. C. Chatwin, C. M. Allen., Mathematical models of dispersion in rivers and estuaries, *Annual Review of Fluid Mechanics*, 17(1), 1985, 119-149.
[14] Kumar, N., Unsteady flows against dispersion infinite porous media, *Journal of Hydraulics*, 63(3), 1983, 345-358.
[15] Chaudhry, M. H., Cass, D. E., Edinger. J. E., Modelling of unsteady-flow water temperatures, *Journal of Hydraulic Engineering*, 109(5), 1983, 657-669.
[16] Menni, Y., Ghazvini, M., Ameur, H. *et al.,* Combination of baffling technique and high-thermal conductivity fluids to enhance the overall performances of solar channels, *Engineering with Computers*, 2020, https://doi.org/10.1007/s00366-020-01165-x.
[17] Gupta, M. M., Manohar, R. P., Stephenson, J. W., A single cell high order scheme for the convection-diffusion equation with variable coefficients, *International Journal for Numerical Methods in Fluids*, 4(7), 1984, 641-651.
[18] Karaa, S., Zhang, J., High order ADI method for solving unsteady convection-diffusion problems, *Journal of Computational Physics*, 198(1), 2004, 1-9.
[19] Noye, B. J., Tan, H. H., Finite difference methods for solving the two-dimensional advection-diffusion equation, *International Journal for Numerical Methods in Fluids*, 9(1), 1989, 75-98.
[20] Kalita, J. C., Dalal, D. C., Dass, A. K., A class of high order compact schemes for the unsteady two-dimensional convection-diffusion equation with variable convection coefficients, *International Journal for Numerical Methods in Fluids*, 38(12), 2002, 1111-1131.
[21] Man, C., Tsai, C. W., A high order predictor-corrector scheme for two-dimensional advection-diffusion equation, *International Journal for Numerical Methods in Fluids*, 56(4), 2008, 401-418.
[22] Kadalbajoo, M. K., Arora, P., Taylor-Galerkin B-splines finite element method for the one-dimensional advection-diffusion equation, *Numerical Methods for Partial Differential Equations*, 26(5), 2010, 1206-1223.
[23] Ngondiep, E., Asymptotic growth of the spectral radii of collocation matrices approximating elliptic boundary problems, *International Journal of Applied Mathematics and Computer Science*, 4, 2012, 199-219.
[24] Ngondiep, E., A two-level factored Crank-Nicolson method for two-dimensional nonstationary advection-diffusion equation with time dependent dispersion coefficients and source terms, *Advances in Applied Mathematics and Mechanics*, 2020.
[25] Ngondiep, E., A novel three-level time-split MacCormack scheme for two-dimensional evolutionary linear convection-diffusion-equation with source term, *International Journal of Computer Mathematics*, 2020, DOI: 10.1080/00207160.2020.1726896.
[26] Li, J., Chen, Y., Pepper, D., Radial basis function method for 1d and 2d groundwater contaminant transport modeling, *Computational Mechanics*, 32(1-2), 2003, 10-15.
[27] Saad, K., Gomez-Aguilar, M.J.F., Analysis of reaction-diffusion system via a new fractional derivative with non-singular kernel, *Physica A: Statistical Mechanics and its Applications*, 509, 2018, 703-716.
[28] Ngondiep, E., *A novel three-level time-split MacCormack method for solving two-dimensional viscous coupled Burgers equations*, preprint available online from http://arxiv.org/abs/1906.01544, 2019.
[29] Ngondiep, E., An efficient three-level explicit time-split approach for solving two-dimensional heat conduction equation, *Applied Mathematics & Information Sciences*, 14(6), 2020, 1-18.
[30] Ngondiep, E., Kerdid, N., Abaoud, M. A. M., Aldayel, I., A., I., A three-level time-split MacCormack method for two-dimensional nonlinear reaction-diffusion equations, *International Journal for Numerical Methods in Fluids*, 2020, https://doi.org/10.1002/fld.4844.
[31] Ngondiep, E., *A three-level explicit time-split MacCormack method for 2D nonlinear reaction-diffusion equations*, preprint available online from http://arxiv.org/abs/1903.10877, 2019.
[32] MacCormack, R. W., Paullay, A. J., The effect of viscosity in hypervelocity impact cratering, *AIAA*, 1969, 69-354.
[33] MacCormack, R. W., A numerical method for solving the equations of compressible viscous-flows, *AIAA*, 1981, 81-0110.
[34] Lax, P. D., Wendroff, B., Systems of conservation laws, *Communications on Pure and Applied Mathematics*, 13, 1960, 217-237.
[35] Namio, F. T., Ngondiep, E., Ntchantcho, R., Ntonga, J. C., Mathematical models of complete shallow water equations with source terms, stability analysis of Lax-Wendroff scheme, *Journal of Theoretical and Computational Science*, 2(4), 2015, 1000132.
[36] Ngondiep, E., Alqahtani, T. R., Ntonga, J. C., *Stability analysis and convergence rate of MacCormack scheme for complete shallow water equations with source terms*, preprint available online from http://arxiv.org/abs/1903.11104, 2019.
[37] MacCormack, R. W., An efficient numerical method for solving the time-dependent compressible Navier-Stokes equations at high Reynolds numbers, *NASA TM*, 1976, 73-129.
[38] Ngondiep, E., Long time unconditional stability of a two-level hybrid method for nonstationary incompressible Navier-Stokes equations, *Journal of Computational and Applied Mathematics*, 345, 2019, 501-514.
[39] Ngondiep, E., Long time stability and convergence rate of MacCormack rapid solver method for nonstationary Stokes-Darcy Problem, *Journal of Computational and Applied Mathematics*, 75, 2018, 3663-3684.
[40] Ngondiep, E., Stability analysis of MacCormack rapid solver method for evolutionary Stokes-Darcy problem, *Journal of Computational and Applied Mathematics*, 345, 2019, 269-285.
[41] Ngondiep, E., *Error estimate of MacCormack rapid solver method for 2D incompressible Navier-Stokes problems*, preprint available online from http://arxiv.org/abs/1903.10857, 2019.
[42] Ngondiep, E., *An efficient explicit approach for predicting the Covid-19 spreading with undetected infectious: The case of Cameroon*, preprint available online from http://arxiv.org/abs/2005.11279, 2020.
[43] Holly, F. M., Usseglio-Polatera, J., Dispersion simulation in two-dimensional tidal flow, *Journal of Hydraulic Engineering*, 101(7), 1984, 905-926.
[44] Chen, Y., Falconer, R. A., Advection-diffusion modelling using modified QUICK scheme, *International Journal for Numerical Methods in Fluids*, 15(10), 1992, 1171-1196.
[45] Ahmad, Z., Kothyari, U. C., Time-line cubic spline interpolation scheme for solution of advection-diffusion equation, *Computers and Fluids*, 30(6), 2001, 737-752.
[46] Dag, I., Irk, D., Tombul, M., Least-squares finite element method for the advection-diffusion equation, *Applied Mathematics and Computation*, 173(1), 2006, 554-565.
[47] Nazir, T., Abbas, M., Izani, A., Abd, A., The numerical solution of advection-diffusion problems using new cubic trigonometric B-splines approach, *Applied Mathematical Modeling*, 40(7), 2016, 4586-4611.
[48] Verma, P., Prasad, K. S. H., Ojha, C. S. P., MacCormack scheme based numerical solution of advection-dispersion equation, *ISH Journal of Hydraulic Engineering*, 12(1), 2006, 27-38.
[49] Tian, Z. F., Ge, Y. B., A fourth order compact ADI method for solving two-dimensional unsteady convection-diffusion problems, *Journal of Computational and Applied Mathematics*, 198(1), 2007, 268-286.
[50] Anderson, F. A., Pletcher, R. H., Tannehill, J. C., *Computational fluid mechanics and Heat Transfer*, Second Edition, Taylor and Francis, New York, 1997.
[51] MacCormack, R. W., Paullay, A. J., Computational efficiency achieved by time splitting of finite difference operators, *AIAA*, 1972, 72-154.
[52] Ngondiep, E., An efficient three-level explicit time-split scheme for solving two-dimensional unsteady nonlinear coupled Burgers' equations, *International Journal for Numerical Methods in Fluids*, 92, 2020, 266-284.
[53] Chinchapatnam, P. P., Djidjeli, K., Nair, P. B., Unsymmetric and symmetric meshless schemes for the unsteady convection-diffusion equation, *Computer Methods in Applied Mechanics and Engineering*, 195(19); (2006) 2432-2453.
[54] MacCormack, R. W., Baldwin, B. S., A numerical method for solving the Navier-Stokes equations with applications to Shock-Boundary Layer Interactions, *AIAA*, 1975, 75-101.

ORCID iD

Eric Ngondiep https://orcid.org/0000-0002-7487-3301

How to cite this article: Ngondiep E. A Robust Three-Level Time-Split MacCormack Scheme for Solving Two-Dimensional Unsteady Convection-Diffusion Equation, *J. Appl. Comput. Mech.*, 7(2), 2021, 559–577.
https://doi.org/10.22055/JACM.2020.35224.2601