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Research Paper

A Variational Principle for a Nonlinear Oscillator Arising in the Microelectromechanical System

Ji-Huan He [1,2,3], Naveed Anjum [1,4,5], Piotr Sebastian Skrzypacz[6]

[1] National Engineering Laboratory for Modern Silk, College of Textile and Engineering, Soochow University, Suzhou, China, Email: hejihuan@suda.edu.cn

[2] School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China

[3] School of Science, Xi'an University of Architecture and Technology, Xi'an, China

[4] School of Mathematical Sciences, Soochow University, Suzhou, China, Email: xsnaveed@yahoo.com

[5] Department of Mathematics, Government College University, Faisalabad, Pakistan

[6] School of Science and Technology, Nazarbayev University, 53 Kabanbay Batyr Ave., Astana, 010000, Kazakhstan, Email: piotr.skrzypacz@nu.edu

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Corresponding author: Ji-Huan He (hejihuan@suda.edu.cn)

Abstract: A nonlinear oscillator arising in the microelectromechanical system is complex and it is difficult to obtain a variational principle. This paper begins with a wrong variational formulation and uses the semi-inverse method to obtain a genuine variational principle. Additionally, this paper gives simple formula for the fast frequency estimation of the nonlinear oscillator. Only simple calculation is needed to have a relatively high accuracy results when compared with the other methods.

Keywords: Variational theory, Euler-Lagrange equation, MEMS oscillator, Amplitude-frequency relationship.

1. Introduction

The microelectromechanical system (MEMS) refers to the high technology devices of small sizes and has become a hot topic in both academic and industrial communities [1-11]. The MEMS are intelligent structures and their systems are of commonly micron or nanometer. Micro electronic technology is the origin of these tiny devices used in vibrators, sensors, switches and so on [3-4,6,9-11]. These systems are modeled by means of Galerikin's method generally and represented by nonlinear mathematical models. Therefore, the approximate solution of these nonlinear models are very important to predict their dynamic behaviour.

Recently Fu et al. [8] studied the following electrically excited MEMS oscillator by employing energy balance method (EBM)

$$u''(a_0+a_2u^2+a_4u^4)+a_1u+a_3u^3+a_5u^5+a_7u^7=0 \tag{1}$$

The parameters a_i (i=0~7) are constants and can be found in Appendix. A detailed derivation of Eq.(1) and the physical understanding of each coefficient are available in Ref.[8]. This paper aims at establishing a variational formulation for Eq.(1) by the semi-inverse method[12-23] and proposed a simple frequency formula based on previous work [24]. The results obtained from the proposed technology not only shows excellent matching with the results those obtained numerically using the Runge-Kutta method of order four (RK4) but also yields better accuracy when compared to other established methods.

2. Variational Principle

Fu et al. obtained the following variational principle[8]

$$J(u)=\int\left\{-\frac{1}{2}u'^2(a_0+a_2u^2+a_4u^4)+\frac{1}{2}a_1u^2+\frac{1}{4}a_3u^4+\frac{1}{6}a_5u^6+\frac{1}{8}a_7u^8\right\}dt \tag{2}$$

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This variational principle is valid only for the case when $a_2=a_4=0$. The stationary condition is

$$\frac{\partial L}{\partial u}-\frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right)=0 \tag{3}$$

where L is the Lagrange function is defined as

$$L=-\frac{1}{2}u'^2(a_0+a_2u^2+a_4u^4)+\frac{1}{2}a_1u^2+\frac{1}{4}a_3u^4+\frac{1}{6}a_5u^6+\frac{1}{8}a_7u^8 \tag{4}$$

It is obvious that

$$\frac{\partial L}{\partial u}=-\frac{1}{2}u'^2(2a_2u+4a_4u^3)+a_1u+a_3u^3+a_5u^5+a_7u^7 \tag{5}$$

and

$$\frac{\partial L}{\partial u'}=-u'(a_0+a_2u^2+a_4u^4) \tag{6}$$

So we have the following Euler-Lagrange equation

$$-\frac{1}{2}u'^2(2a_2u+4a_4u^3)+a_1u+a_3u^3+a_5u^5+a_7u^7+\frac{d}{dt}\left[u'(a_0+a_2u^2+a_4u^4)\right]=0 \tag{7}$$

After simple operation, Eq.(7) becomes

$$u''(a_0+a_2u^2+a_4u^4)+a_1u+a_3u^3+a_5u^5+a_7u^7+u'^2(a_2u+2a_4u^3)=0 \tag{8}$$

It is obvious that Eq.(8) is not equivalent to Eq.(1), so Eq.(2) is not a genuine variational principle. In order to obtain a genuine one, we write Eq.(1) in the form

$$u''+\frac{a_1u+a_3u^3+a_5u^5+a_7u^7}{a_0+a_2u^2+a_4u^4}=0 \tag{9}$$

By the semi-inverse method, its variational formulation is

$$J(u)=\int\left\{-\frac{1}{2}u'^2+F\right\}dt \tag{10}$$

where F is the potential satisfying the following relationship

$$\frac{\partial F}{\partial u}=\frac{a_1u+a_3u^3+a_5u^5+a_7u^7}{a_0+a_2u^2+a_4u^4} \tag{11}$$

Calculating F from Eq.(11) we have

$$\begin{aligned}F=&\frac{a_7}{4a_4}u^4-\left(\frac{a_2a_7}{2a_4^2}-\frac{a_5}{2a_4}\right)u^2-\frac{(a_0a_7-a_2^2a_7+a_2a_5-a_3a_4)}{4a_4^2}\ln(a_4u^4+a_2u^2+a_0)\\&+\frac{3a_0a_2a_4a_7-2a_0a_4^2a_5+4a_1a_4^3-a_2^3a_7+a_2^2a_4a_5-a_2a_3a_4^2}{2a_4^3}\frac{1}{\sqrt{4a_0a_4-a_2^2}}\arctan\left(\frac{2a_4u^2+a_2}{\sqrt{4a_0a_4-a_2^2}}\right)\end{aligned} \tag{12}$$

It is easy to prove that Eq.(10) is a genuine variational principle.

3. Frequency-Amplitude Relationship

Consider the following general nonlinear oscillator

$$u''+f(u)=0\,,\quad u(0)=A,\quad u'(0)=0 \tag{13}$$

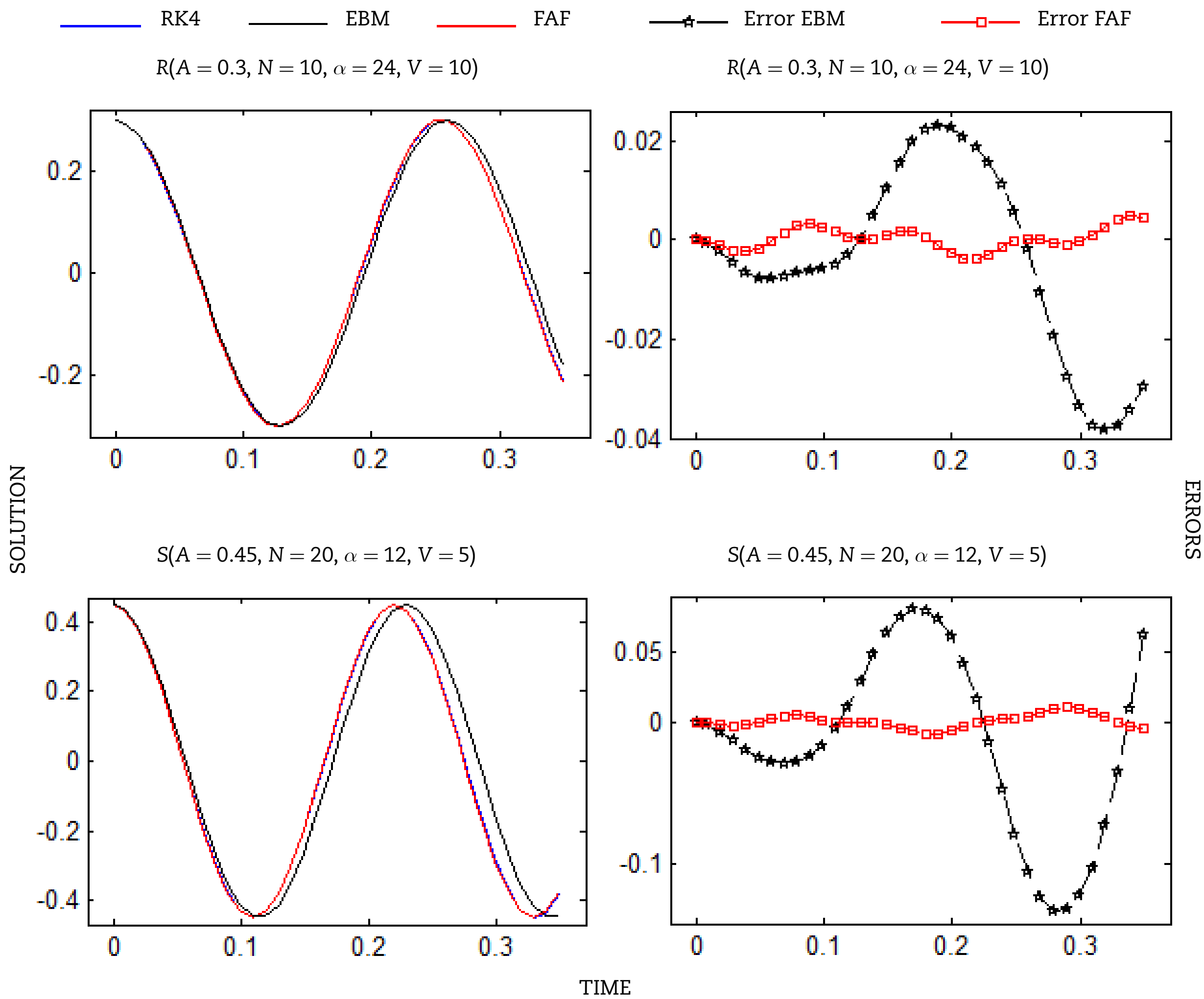

Fig. 1. Comparison of solution and error between FAF and EBM with RK4

Two most simplest frequency-amplitude formulae are listed as follows [24]

$$\omega^2 = \frac{df}{du}(u = A/2) \tag{14}$$

$$\omega^2 = \frac{f(u)}{u}(u = \frac{\sqrt{3}}{2}A) \tag{15}$$

In our problem f(u) is defined as

$$f(u) = \frac{a_1 u + a_3 u^3 + a_5 u^5 + a_7 u^7}{a_0 + a_2 u^2 + a_4 u^4} \tag{16}$$

By Eq.(15), we have

$$\omega^2 = \frac{a_1 + \frac{3}{4}a_3 A^2 + \frac{9}{16}a_5 A^4 + \frac{27}{64}a_7 A^6}{a_0 + \frac{3}{4}a_2 A^2 + \frac{9}{16}a_4 A^4} \tag{17}$$

The approximate solution using frequency-amplitude formulation (FAF) reads

$$u = A\cos\left(\sqrt{\frac{a_1 + \frac{3}{4}a_3 A^2 + \frac{9}{16}a_5 A^4 + \frac{27}{64}a_7 A^6}{a_0 + \frac{3}{4}a_2 A^2 + \frac{9}{16}a_4 A^4}}\, t\right) \tag{18}$$

4. Results and Discussion

Fig. 1 includes the results obtained from the FAF, numerical results using RK4, and those obtained by EBM [8]. Two sets of parameters are considered for the study. The left column shows the solution obtained from numerical results using RK4 (blue line), Fu et. al. [8] (black line), and FAF from Eq. (18) (red line) and this comparison authenticates that the approximate analytical results from the proposed FAF match exceptionally well with the computational results of RK4. We also map error against time for the same parameter values in the right column of Fig. 1. Black stars and red square with solid lines indicate the error of EBM (difference of RK4 solution and EBM solution [8]) and error of the FAF (difference of RK4 solution and FAF solution from Eq. (18)) respectively. Although both errors are small but all right side panels ensure the superiority of the FAF over EBM. It is also noted that as we increase the amplitude value, the error of EBM also increases but the error of the FAF is negligible.

5. Conclusion

In this short paper, we obtained a genuine variational principle for MEMS oscillator, it can be used to study the frequency-amplitude relationship in an energy view, and we will discuss it in a forthcoming paper. Moreover, this paper gives a simple formula for the fast prediction of the frequency of MEMS oscillator and all obtained results are of a relatively high accuracy.

Author Contributions

All authors contribute equally in preparation of this manuscript. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Conflict of Interest

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Nomenclature

A	Amplitide of the oscillator	L	Lagrange function
t	Time	ω	Nonlinear frequency of the oscillator
u	Approximate solution	J	Variational formulation

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APPENDIX

The coefficients $a_j \ (j = 0 \sim 7)$ of Eq. (1) can be determined as follows

$$a_0 = \int_0^1 \xi^2 \, d\eta \tag{A}$$

$$a_1 = \int_0^1 \left(\xi\xi'''' - \mathrm{N}\xi\xi'' - \mathrm{V}^2\xi^2 \right) d\eta \tag{B}$$

$$a_2 = -2\int_0^1 \xi^4 \, d\eta \tag{C}$$

$$a_3 = \int_0^1 \left(-2\xi^3\xi'''' + 2\mathrm{N}\xi^3\xi'' - \alpha\xi\xi'' \int_0^1 \xi'^2 d\eta \right) d\eta \tag{D}$$

$$a_4 = \int_0^1 \xi^6 \, d\eta \tag{E}$$

$$a_5 = \int_0^1 \left(\xi^5\xi'''' - \mathrm{N}\xi^5\xi'' + 2\alpha\xi^3\xi'' \int_0^1 \xi'^2 d\eta \right) d\eta \tag{F}$$

$$a_7 = -\int_0^1 \left(\alpha\xi^5\xi'' \int_0^1 \xi'^2 d\eta \right) d\eta \tag{G}$$

where $\xi(\eta) = 16\eta^2(1-\eta)^2$ uses as trail function and prime $(\bullet')$ represents the partial differentiation w.r.t coordinate variable η.

ORCID iD

Ji-Huan He https://orcid.org/0000-0002-1636-0559
Naveed Anjum https://orcid.org/0000-0001-7810-7877
Piotr Sebastian Skrzypacz https://orcid.org/0000-0001-6422-5469

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