



Journal of Applied and Computational Mechanics



Research Paper

In-Plane Shear-Axial Strain Coupling Formulation for Shear-Deformable Composite Thin-Walled Beams

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Received September 18 2020; Revised November 13 2020; Accepted for publication November 13 2020.

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Abstract. This paper presents an improved description of the in-plane strain coupling in Librescu-type shear-deformable composite thin-walled beams (CTWB). Based on existing descriptions for Euler-type CTWB, an analogous formulation for shear-deformable CTWB is here developed by building, via the Mindlin-Reissner theory and an orthotropic constitutive law of the shell wall, an alternative equation for the in-plane shear force which effectively couples the axial and shear in-plane strains. It is observed that this strain coupling formulation includes some of the transversal (out-of-plane) shear strain terms, thus also functions as a path for transferring transversal shear energy to the in-plane strain field and therefore improves shear-deformability. The performance of the new CTWB model is compared against that of previously available CWTB models (i.e. Euler-type with strain coupling and Timoshenko-type without strain coupling) for several aspect ratios, fibre-orientations and laminate types. Error measures are calculated by comparing several relevant stiffness coefficients and displacement shapes to reference results provided by corresponding 3D shell-based ANSYS finite-element models. Results indicate that for cases involving significant shear energy (i.e. short aspect ratios) and/or in-plane shear-axial strain coupling (i.e. off-axis or asymmetric/unbalanced laminates), the new CTWB model proposed in this work can attain an accuracy level comparable to that associated to more sophisticated models, two to three orders of magnitude larger, at a fraction of the computational cost.

Keywords: Composite Thin-Walled Beam (CTWB); Shear-Deformable; Timoshenko; Shear-Axial Strain Coupling; Isogeometrical Analysis (IGA).

1. Introduction

The increasing complexity of contemporary design endeavours, requiring fast, accurate and recurrent analyses at ever-earlier stages of design, combined with a moderate growth of off-the-shelf computing power, demands new modelling paradigms. This driver has fuelled the development of reduced-order models able to attain precision levels typically associated with more sophisticated (but also much larger) models, while involving a small fraction of the processing burden [1-3]. In structural applications, the development of composite thin-walled beam (CTWB) models represents an active field of research, since this type of reduced-order structural model is able to reproduce, via a simplified mathematical framework, most of the complex behaviour exhibited by slender, flexible and hollow structural components [3-6]. The incorporation of laminated materials and short aspect ratios introduces further complexities in terms of kinematic modelling, material anisotropy, structural coupling and numerical instabilities, which in turn require further abstraction in the mathematical treatment to maintain computational economy and hence the “reduced-order” label [7-23]. Examples of relevant industrial applications of CTWB can be found in aerospace (i.e. fuselages [24]), wind energy (i.e. wind turbine blades [25]), civil engineering (i.e. pipes [26]) and heat storage (i.e. containers [27]).

Beyond the earlier seminal works by Vlasov [7] and Timoshenko [8], a recognizable class of modern CTWB can be traced back to the work of Librescu and co-workers [4], who developed a mathematical framework based on the classical (i.e. Euler-type) beam theory which admits complex geometry and material layout along both the cross section's contour and the beam axis, allowing to capture, via semi-analytical equations, the stiffness coupling occurring both at the cross section level and at the beam structure as a whole. This theory has been subsequently enriched by many researchers, including the following: Lee et al [9] presented a shear-deformable theory for I-shaped CTWB models subject to vertical loads, applicable to arbitrary laminate stacking, fibre-orientation and span-to-height ratio, later extended in [10] to account for flexural-torsional analyses. Cortinez and Piovan [11] developed a theoretical model for the stability analysis of shear-deformable CTWB models with open or closed cross sections, based on a non-linear displacement field and including bending and non-uniform warping. Vo and Lee [12] derived a shear-deformable theory for the coupled vibration and buckling analysis of CTWB with open sections, capable of capturing some degree



of structural coupling due to material anisotropy. This theory was extended by the same authors in [13] to account for geometrical nonlinearities, arbitrary layout and loading, for which a numerical validation was performed on composite box beams. Alsafadie et al [14] introduced a mixed co-rotational formulation for the analysis of nonlinear buckling and post-buckling of 3D elasto-plastic thin-walled beams with generic cross sections and shear-deformability, where warping was accounted for via Bescot-type functions. The Galerkin and perturbation methods were used by Machado et al [15] for the analysis of internal resonances (quadratic and cubic) arising in the non-linear dynamic response of shear-deformable CTWB models subjected to small strains but large rotations, also accounting for bending and warping shear. It was found that the shear effect had a significant impact the stability and amplitude of vibration. Vieira et al [16] presented a higher-order theory in which the displacement field of the CTWB is approximated via a set of products of linearly independent functions, where the terms associated with the cross section are decoupled from those of the beam's axis, further decoupling the equations of motion via a non-linear eigenvalue problem. This model was extended by the same authors in [17], allowing for the hierarchical decoupling of warping modes in CTWB structures, and further extended in [18] to account for linear buckling analyses. Cambrono-Barrientos et al [3] made a case for the importance of relying on reduced-order models for speeding up routine design activities, presenting a 5-DOF (i.e. Degrees of Freedom) beam model capable of handling thin-walled cross sections, shear lag effects, torsion and distortion in the normal stress.

A restriction found in many Librescu-type CTWB models is the uncoupled nature of the in-plane (shear-axial) strain field, a misrepresentation that becomes more evident in cases with considerable axial-bending-torsional structural coupling. Zhang et al [19] developed a formulation for Euler-type CTWB models which successfully captured a more realistic strain coupling behaviour, leading to a radical improvement in accuracy for cases where off-axis fibre-orientations produce a significant amount of structural coupling. The effectiveness of the formulation was demonstrated by comparing the displacement field of the developed CTWB versus that of a 3D shell-based ANSYS model for various fibre-orientations, exhibiting very small deviations. This coupled formulation was subsequently extended in [20] to tackle Euler-type CTWB models with axial curvature, leading to new equations describing the strong structural coupling occurring in curved CTWB models, with the numerical validation yielding equally good results as in the case of straight beams.

This work taps into the aforementioned developments for Euler-type CTWB models, specifically those reported in [19-20], to derive an analogous in-plane coupled strain field for shear-deformable CTWB. The latter feature is derived in a similar fashion as in [19-20], producing a new equation for the internal in-plane shear force which contains significant couplings with other in-plane strain variables. Additionally, it was found that this new equation includes some of the (out-of-plane) shear-deformable terms, thus representing a path for transferring out-of-plane shear energy to the in-plane strain field and further increasing the CTWB's capacity to absorb shear energy. The global equilibrium equations are discretized and solved via an Isogeometrical Analysis (IGA) scheme similar to the one described in [20]. The IGA methodology, originally developed by Hughes, Bazilevs, and co-workers [25-26], allows to construct equivalent numerical equilibrium equations directly from geometry-based basis functions (conventional NURBS-based CAD design tools), resulting in a compact and robust numerical description. The discrete model is validated via several numerical tests of a CTWB with variable aspect ratio, fibre-orientation and laminate type, comparing its stiffness matrix and displaced shape to those yielded by available CTWB (i.e. Euler-type with coupled strain field and Timoshenko-type with uncoupled strain field) and corresponding reference models (3D shell-based finite element ANSYS models). The results obtained show that the novel CTWB yields, for the case of shorter aspect ratios and/or significant structural coupling, a significantly higher accuracy than previously available CTWBs models, with the results being very similar to the ones obtained with the reference model, in spite of model size and computational cost being about three orders-of-magnitude smaller.

This paper proceeds as follows: Section 2 introduces the theoretical framework of shear-deformable CTWB models, emphasizing the inclusion of through-the-thickness (i.e. transversal) shear strain terms in the kinematic field, as well as corresponding shear stiffness coefficients and internal shear forces in the constitutive equations. It also deals with the development of the in-plane strain coupling formulation, leading to a new equation for the internal shear force which exhibits a stronger coupling behaviour and replaces the standard equation. Section 3 provides an in-depth discussion of the various results yielded by several numerical tests, ending with some concluding remarks in Section 4.

2. Mechanics of CTWB Models

The mathematical framework for composite thin-walled beams (CTWB) laid out next is restricted to the following assumptions: 1) linear strain field, 2) the cross sections remain planar and rigid, though not necessarily perpendicular to the bending line (thereby departing from Euler-type theories), and 3) the laminated material behaves according to the Kirchhoff-Love theory (a.k.a. classical laminated theory, or CLT). In the rest of the paper, bold italic notation represents vectors, plain notation represent scalars, and straight bold letters are reserved for matrices.

2.1 Kinematics of rectilinear CTWB

Figures 1 and 2 illustrate the main coordinate systems and the nomenclature used for describing the mechanics of CTWB. The global coordinate system is given by the Cartesian frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$, where $\mathbf{t}$ corresponds to the rectilinear beam's axis (or pole line) and $\mathbf{n}, \mathbf{b}$ describe an orthogonal plane parallel to the planes of the cross sections in the undeformed case. A secondary (or local) coordinate system is given by the Cartesian frame $(\mathbf{t}, \mathbf{s}, \mathbf{h})$, where $\mathbf{s}$ represents the (counter-clockwise positive) tangent vector to the cross section's mid-contour, and $\mathbf{h}$ is a through-the-thickness, outward-positive orthogonal axis. From Figure 1 it can be seen that the $(\mathbf{t}, \mathbf{s}, \mathbf{h})$ coordinate system distinguishes itself from the $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ frame by a rotation around the $\mathbf{t}$ axis, with α measuring the positive rotation of $\mathbf{s}$ with respect to $\mathbf{n}$, and a translation to the mid-contour of the (thin) wall of a cross section of interest. The scalars t, n, b, h represent magnitudes measured along their respective linear axes, while s represents the arc length along the cross section's mid-contour ϑ . The angle β represents the fibre-orientation measured with respect to the $\mathbf{s}$ axis; thus, 0° corresponds to the hoop (i.e. circumferential) direction, while 90° corresponds to the beam's axial direction. Finally, $\mathbf{r}(\mathbf{s})$ and $\mathbf{q}(\mathbf{s})$ are the magnitudes of corresponding planar vectors perpendicular and parallel, to the $\mathbf{s}$ axis, respectively.

Based on the figures above, and the assumptions and nomenclature mentioned, the linear ($\mathbf{u}$) and angular (θ) displacement fields of the pole line of a CTWB can be expressed as:

$$\mathbf{u}(t) = u_t(t) \mathbf{t} + u_n(t) \mathbf{n} + u_b(t) \mathbf{b} \quad (1a)$$

$$\theta(t) = \theta_t(t) \mathbf{t} + \theta_n(t) \mathbf{n} + \theta_b(t) \mathbf{b} \quad (1b)$$

where u_t , u_n , u_b , and θ_t , θ_n , θ_b are scalar linear and angular displacements measured along the $\mathbf{t}, \mathbf{n}, \mathbf{b}$ axes, respectively.



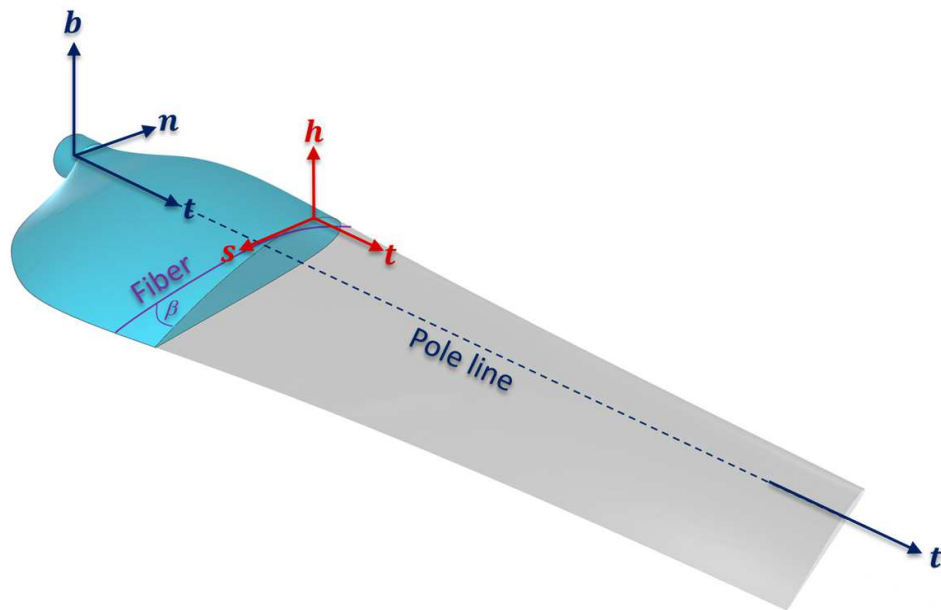


Fig. 1. Coordinate frames and nomenclature of rectilinear composite thin-walled beams (CTWB).

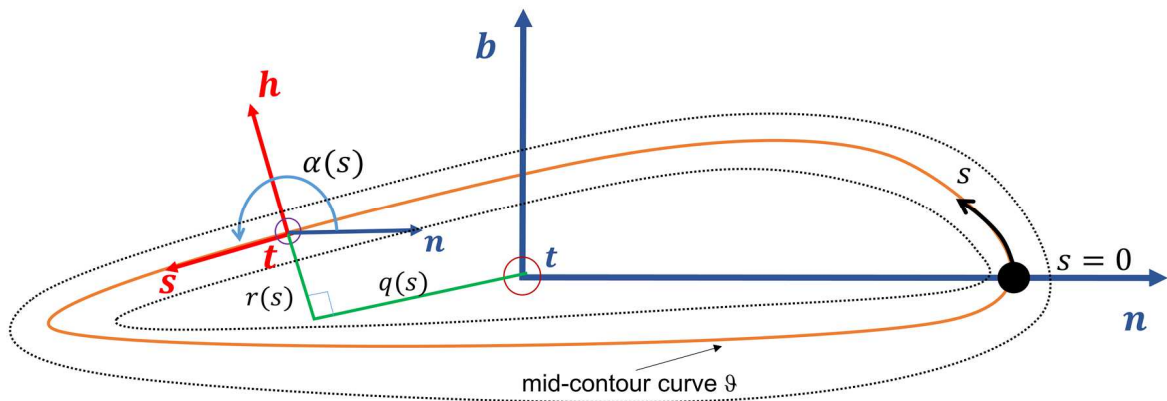


Fig. 2. Coordinate frames and nomenclature of the cross section of a composite thin-walled beam (CTWB).

Restricted by the assumption of a rigid cross section, the displacement field $\mathbf{U}$ at an arbitrary position within the wall can be written as:

$$\mathbf{U}(t, n, b) = U_t(t, n, b) \mathbf{t} + U_n(t, b) \mathbf{n} + U_b(t, n) \mathbf{b} \quad (2a)$$

$$U_n(t, b) = u_n(t) - \phi_t(t) b \quad (2b)$$

$$U_b(t, n) = u_b(t) + \phi_t(t) n \quad (2c)$$

$$U_t(t, n, b) = u_t(t) - \theta_b(t) n + \theta_n(t) b - \theta_t(t) (\omega(s) - q(s) h) \quad (2d)$$

where s is understood as a function of n and b ($s = f(n, b)$), ϕ_t represents the rigid-body angular rotation of the cross-section, and the warping function $\omega(s)$ can be expressed as [4,12]:

$$\omega(s) = \eta(s) - C^{-1} \oint \eta(s') ds', \quad (3)$$

where:

$$\eta(s) = \int_0^s [r(s') - \psi(s')] ds' \quad (4a)$$

$$\psi(s) = \frac{\oint r(s') ds'}{A_{66}(s) * \oint \frac{ds'}{A_{66}(s')}} \quad (4b)$$



$$\frac{d\omega(s)}{ds} = [r(s) - \psi(s)] \quad (4c)$$

C is the perimeter of the mid-contour, and A_{66} is a stiffness coefficient linking the in-plane shear strain with its corresponding stress, later defined in eq.(9). The displacement field of eq.(2) is expressed in the (t, n, b) frame but must be converted to the "local" (i.e. the wallmaterial) frame (t, s, h) to make it compatible with the material's constitutive equations. This can be done by rotating eq.(2) from the (t, n, b) to the (t, s, h) frame, as follows:

$$\begin{Bmatrix} U_s(s, h, t) \\ U_h(s, h, t) \end{Bmatrix} = \begin{bmatrix} \cos \alpha(s) & \sin \alpha(s) \\ \sin \alpha(s) & -\cos \alpha(s) \end{bmatrix} \begin{Bmatrix} U_n(t, b) \\ U_b(t, n) \end{Bmatrix} \quad (5)$$

Equation (2) can then be expressed in the coordinate frame (t, s, h) as:

$$U(s, h, t) = U_t(s, h, t) \mathbf{t} + U_s(s, h, t) \mathbf{s} + U_h(s, h, t) \mathbf{h} \quad (6a)$$

$$U_s(s, h, t) = u_n(t) n'(s, h) + u_b(t) b'(s, h) + \phi_t(t) r(s) + \phi_t(t) h \quad (6b)$$

$$U_h(s, h, t) = u_n(t) b'(s, h) - u_b(t) n'(s, h) - \phi_t(t) q(s) \quad (6c)$$

$$U_t(s, h, t) = u_t(t) - \theta_b(t) \bar{n}(s) + \theta_n(t) \bar{b}(s) - \omega(s) \theta_t'(t) + h(-q(s) \theta_t(t) - \theta_b(t) b'(s, h) - \theta_n(t) n'(s, h)) \quad (6d)$$

where the following geometrical identities were used:

$$\cos \alpha(s) = \bar{n}(s) \quad (7a)$$

$$\sin \alpha(s) = \bar{b}'(s) \quad (7b)$$

$$n(s, h) = \bar{n}(s) + h \sin \alpha(s) \quad (7c)$$

$$b(s, h) = \bar{b}(s) - h \cos \alpha(s) \quad (7d)$$

$$r(s) = \bar{n}(s) \bar{b}'(s) - \bar{b}(s) \bar{n}'(s) \quad (7e)$$

$$q(s) = \bar{n}(s) \bar{n}'(s) + \bar{b}(s) \bar{b}'(s) \quad (7f)$$

$$\bar{n}''(s) = 0 \quad (7g)$$

$$\bar{b}''(s) = 0 \quad (7h)$$

$$q'(s) = 1 \quad (7i)$$

and where variables with an overbar (i.e. $\bar{n}$) represent values measured at the mid-contour, i.e. $n(s, h=0) = \bar{n}(s)$; the $(\cdot)'$ symbol represents differentiation with respect to t . The strain field associated with the displacement field of eq. (6) can be calculated as:

$$\varepsilon_{tt}(s, h, t) = \frac{\partial U_t}{\partial t} \quad (8a)$$

$$\gamma_{st}(s, h, t) = \frac{\partial U_s}{\partial t} + \frac{\partial U_t}{\partial s} \quad (8b)$$

$$\gamma_{ht}(s, h, t) = \frac{\partial U_h}{\partial t} + \frac{\partial U_t}{\partial h} \quad (8c)$$

Equation (8) can be split, according to classical laminated theory (CLT), into mid-contour strains (i.e. at $h=0$) and off-center curvatures (i.e. at $h \neq 0$):

$$\varepsilon_{tt}(s, h, t) = \bar{\varepsilon}_{tt}(s, t) + h k_{tt}(s, t) \quad (9a)$$

$$\gamma_{st}(s, h, t) = \bar{\gamma}_{st}(s, t) + h k_{st}(s, t) \quad (9b)$$

$$\gamma_{ht}(s, h, t) = \bar{\gamma}_{ht}(s, t) \quad (9c)$$



where:

$$\bar{\varepsilon}_{tt}(s, t) = \varepsilon_t^0(t) + \bar{b}(s)K_n(t) - \bar{n}(s)K_b(t) + \omega(s)K_\omega(t) \quad (10a)$$

$$\bar{\gamma}_{st}(s, t) = K_{st}(t) \frac{\psi(s)}{2} + \Gamma_\omega(t) \left(r(s) - \frac{\psi(s)}{2} \right) + \Gamma_{bt}(t) \bar{b}'(s) + \Gamma_{nt}(t) \bar{n}'(s) \quad (10b)$$

$$\bar{\gamma}_{ht}(s, t) = -q(s)\Gamma_\omega(t) - \Gamma_{bt}(t) \bar{n}'(s) + \Gamma_{nt}(t) \bar{b}'(s) \quad (10c)$$

$$k_{tt}(s, t) = -\bar{n}'(s)K_n(t) - \bar{b}'(s)K_b(t) - q(s)K_\omega(t) \quad (10d)$$

$$k_{st}(s, t) = K_{st}(t) \quad (10e)$$

and

$$\varepsilon_t^0(t) = u_t'(t) \quad (11a)$$

$$K_b(t) = \theta_b'(t) \quad (11b)$$

$$K_n(t) = \theta_n'(t) \quad (11c)$$

$$K_{st}(t) = \phi_t'(t) - \theta_t(t) \quad (11d)$$

$$K_\omega(t) = \theta_t'(t) \quad (11e)$$

$$\Gamma_{nt}(t) = -\theta_b(t) + u_n'(t) \quad (11f)$$

$$\Gamma_{bt}(t) = \theta_n(t) + u_b'(t) \quad (11g)$$

$$\Gamma_\omega(t) = \phi_t'(t) + \theta_t(t) \quad (11h)$$

Equation (9) represents the strain field at an arbitrary point t, s, h at the wall, eq. (10) expresses the strains and curvatures at an arbitrary point t, s, h on the cross section's mid-contour (i.e. $h=0$), and Equation (11) represents the strains and curvatures at an arbitrary point t of the pole line. All eq. (9-11) are expressed in the (t, s, h) frame. Shear-deformability is represented by the out-of-plane (i.e. transverse) shear strain $\bar{\gamma}_{ht}$ (see equation (10c)) which can be seen to depend on the angular displacements of the pole line Γ_ω , Γ_{bt} , Γ_{nt} (expressed in eqs. (11f-h)) as well as the geometric variables q , $\bar{b}'$, $\bar{n}'$. It is precisely eq. (10c) which provides the CWTB with a mechanism for absorbing transverse shear energy, which becomes increasingly significant for decreasing aspect ratios.

2.2 System's Hamiltonian

In the case of static problems, considered in the present paper, the variation of the strain energy stored within a given cross section is required to the zero, leading to the following expression:

$$\delta\Gamma = \oint_{s, -h/2}^{h/2} (\sigma_{tt} \delta\varepsilon_{tt} + \sigma_{st} \delta\gamma_{st} + \sigma_{ht} \delta\gamma_{ht} + \sigma_{ss} \delta\varepsilon_{ss}) dh ds = 0. \quad (12)$$

Note that $\sigma_{ss}=0$ by virtue of the assumption of the cross section's non-deformability. By substituting the strain field of eqs. (9-10) into eq. (12), the latter can be expanded in terms of the pole line's strain variables $\delta\varepsilon_t^0$, δK_b , δK_n , δK_{st} , δK_ω , $\delta\Gamma_{bt}$, $\delta\Gamma_{nt}$, Γ_ω , as follows:

$$\begin{aligned} \delta\Gamma = \oint_{s, -h/2}^{h/2} & \left[\delta\varepsilon_t^0(\sigma_{tt}) + \delta K_b(-\bar{n}\sigma_{tt} - h\bar{b}'\sigma_{tt}) + \delta K_n(\bar{b}\sigma_{tt} + h\bar{n}'\sigma_{tt}) + \delta K_{st}\left(\frac{\psi}{2}\sigma_{st} + h\sigma_{st}\right) + \delta K_\omega(\omega\sigma_{tt} - hq\sigma_{tt}) \right. \\ & \left. + \delta\Gamma_{nt}(\bar{n}'\sigma_{st} + \bar{b}'\sigma_{nt}) + \delta\Gamma_{bt}(\bar{b}'\sigma_{st} - \bar{n}'\sigma_{nt}) + \delta\Gamma_\omega\left(r\sigma_{st} - \frac{\psi}{2}\sigma_{st} - q\sigma_{ht}\right) \right] dh ds = 0 \end{aligned} \quad (13)$$

which can in turn be rewritten in terms of the internal forces acting at the cross section

$$\delta\Gamma = (F_t \delta\varepsilon_t^0 + M_b \delta K_b + M_n \delta K_n + M_t \delta K_{st} + M_\omega \delta K_\omega + F_b \delta\Gamma_{bt} + F_n \delta\Gamma_{nt} + T \delta\Gamma_\omega) = 0, \quad (14)$$

where

$$F_t = \oint_{s, -h/2}^{h/2} (\sigma_{tt}) dh ds \quad F_t = \oint F_{tt} ds \quad (15a)$$



$$M_b = \oint \int_{-h/2}^{h/2} (-\bar{n}\sigma_{tt} - h\bar{b}'\sigma_{tt}) dh ds \quad M_b = \oint [-\bar{n}F_{tt} - \bar{b}'M_{tt}] ds \quad (15b)$$

$$M_n = \oint \int_{-h/2}^{h/2} (\bar{b}\sigma_{tt} - h\bar{n}'\sigma_{tt}) dh ds \quad M_n = \oint [\bar{b}F_{tt} - \bar{n}'M_{tt}] ds \quad (15c)$$

$$M_t = \oint \int_{-h/2}^{h/2} \left(\frac{\psi}{2}\sigma_{st} + h\sigma_{st} \right) dh ds \quad M_t = \oint \left[\frac{r}{2}F_{st} + M_{st} \right] ds \quad (15d)$$

$$M_\omega = \oint \int_{-h/2}^{h/2} (\omega\sigma_{tt} - hq\sigma_{tt}) dh ds \quad M_\omega = \oint [\omega_p F_{tt} - qM_{tt}] ds \quad (15e)$$

$$F_b = \oint \int_{-h/2}^{h/2} (\bar{b}'\sigma_{st} - \bar{n}'\sigma_{ht}) dh ds \quad F_b = \oint [\bar{b}'F_{st} - \bar{n}'F_{ht}] ds \quad (15f)$$

$$F_n = \oint \int_{-h/2}^{h/2} (\bar{n}'\sigma_{st} + \bar{b}'\sigma_{ht}) dh ds \quad F_n = \oint [\bar{n}'F_{st} + \bar{b}'F_{ht}] ds \quad (15g)$$

$$T = \oint \int_{-h/2}^{h/2} \left(r\sigma_{st} - \frac{\psi}{2}\sigma_{st} - q\sigma_{ht} \right) dh ds \quad T = \oint \left[\frac{r}{2}F_{st} - qF_{ht} \right] ds. \quad (15h)$$

Here, the expressions in the second column of eq. (15) result from integrating those in the first with respect to the wall-thickness h , eq. (15d,h) in the second column, using the identity $\oint r(s)ds = \oint \psi(s)ds$. F_{tt} , F_{st} , F_{ht} , M_{tt} , M_{st} represent internal forces per-unit-length acting at the cross section, given by the following expressions:

$$F_{tt} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{tt} dh \quad (16a)$$

$$F_{st} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{st} dh \quad (16b)$$

$$F_{ht} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ht} dh \quad (16c)$$

$$M_{tt} = \int_{-\frac{h}{2}}^{\frac{h}{2}} h\sigma_{tt} dh \quad (16d)$$

$$M_{st} = \int_{-\frac{h}{2}}^{\frac{h}{2}} h\sigma_{st} dh \quad (16e)$$

$$F_{ss} = M_{ss} = \sigma_{ss} = 0 \quad (16f)$$

where eq. (16f) also results from the assumption of the cross section's non-deformability.

2.3 Constitutive equations

The constitutive equations relating the strain field of Equation (9) with its corresponding stress field can be expressed as:

$$\begin{Bmatrix} \sigma_{ss} \\ \sigma_{tt} \\ \sigma_{st} \\ \sigma_{ht} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_{ss} \\ \varepsilon_{tt} \\ \gamma_{st} \\ \gamma_{ht} \end{Bmatrix}_k \quad (17)$$

where $\bar{Q}_{ij}$ are material stiffness coefficients of the k th composite layer defined in the **t,s,h** frame. By substituting the strain field of eqs. (9-10) into the right-hand side of eq. (17), and the latter into the right-hand side of eq. (16), the following expression is obtained:

$$\begin{bmatrix} F_{ss} & F_{tt} & F_{st} & M_{ss} & M_{tt} & M_{st} & F_{ht} \end{bmatrix}^T = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_{ss} & \bar{\varepsilon}_{tt} & \bar{\gamma}_{st} & k_{ss} & k_{tt} & k_{st} & \bar{\gamma}_{ht} \end{bmatrix}^T \quad (18)$$

where the coefficients of the **ABD** matrix are given by:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} \bar{Q}_{ij} \times (1, h, h^2) dh, \quad i, j = 1, 2, 5, 6. \quad (19)$$

Enforcing the plane stress conditions enounced in eq. (16f) ($F_{ss}=M_{ss}=0$) in eq. (18), the latter can be rewritten as:



$$\begin{bmatrix} F_{tt} & F_{st} & M_{tt} & M_{st} & F_{ht} \end{bmatrix}^T = [\mathbf{Z}] \begin{bmatrix} \bar{\varepsilon}_{tt} & \bar{\gamma}_{st} & k_{tt} & k_{st} & \bar{\gamma}_{ht} \end{bmatrix}^T \quad (20)$$

where the coefficients of matrix $\mathbf{Z}$ are given in eq. (A1) of the Appendix. Substituting eq. (10) into the right-hand side of eq. (20) and the latter into the right-hand side of eq. (15), the following expression is obtained:

$$\begin{bmatrix} F_t & M_b & M_n & M_t & M_\omega & F_b & F_n & T \end{bmatrix}^T = [\mathbf{E}] \begin{bmatrix} \varepsilon_t^0 & K_b & K_n & K_{st} & K_\omega & \Gamma_{bt} & \Gamma_{nt} & \Gamma_\omega \end{bmatrix}^T \quad (21)$$

where coefficients E_{ij} are given in the first column of Table A1 of the Appendix. Equation (21) relates the strains and curvatures at the pole line with the internal forces at the cross section via matrix $\mathbf{E}$. This matrix is an important and distinctive feature in the development of any CTWB model as it contains semi-analytical expressions for the stiffness components of the cross section, therefore gathering all the fundamental physics governing the CTWB's behaviour.

2.4 In-plane strain coupling

One shortcoming of eq. (9), which can be better visualized in eqs. (4) and (10), is that the in-plane axial (ε_{tt}) and shear (γ_{st}) strains are virtually uncoupled (except for the weak coupling caused by the shear stiffness coefficient A_{66} found in eq. (4b)). This condition yields inaccurate results for off-axis fibre-orientations and/or asymmetric/unbalanced laminates. This shortcoming was alleviated in [19] for the case of Euler-type CTWB models by a procedure which is here reformulated for shear-deformable CTWB, essentially attempting to replace γ_{st} (eq. (10b)) by a counterpart involving other relevant stiffness coefficients and strain variables. With this aim in mind, $\bar{\gamma}_{st}$ and F_{st} are exchanged to opposites sides of eq. (20), yielding:

$$\begin{bmatrix} F_{tt} & M_{tt} & M_{st} & \bar{\gamma}_{st} & F_{ht} \end{bmatrix}^T = [\mathbf{S}] \begin{bmatrix} \bar{\varepsilon}_{tt} & k_{tt} & k_{st} & F_{st} & \bar{\gamma}_{ht} \end{bmatrix}^T \quad (22)$$

where matrix $\mathbf{S}$ is given explicitly in eq. (A2) of the Appendix. The fourth sub-equation of eq. (22) represents a new expression for $\bar{\gamma}_{st}$, written as:

$$\bar{\gamma}_{st} = S_{41}\bar{\varepsilon}_{tt} + S_{42}k_{tt} + S_{43}k_{st} + S_{44}F_{st} \quad (23)$$

Integrating eq. (23) along the entire mid-contour's path s , the following expression is obtained:

$$\oint \bar{\gamma}_{st} ds = \oint (S_{41}\bar{\varepsilon}_{tt} + S_{42}k_{tt} + S_{43}k_{st} + S_{44}F_{st}) ds \quad (24)$$

A similar, but independent expression for eq. (24) can be derived by integrating eq. (10b) along the entire mid-contour's path s , yielding

$$\oint \bar{\gamma}_{st} ds = \oint \left[K_{st}(t) \frac{\psi(s)}{2} + \Gamma_\omega(t) \left(r(s) - \frac{\psi(s)}{2} \right) + \Gamma_{bt}(t) \bar{b}'(s) + \Gamma_{nt}(t) \bar{n}'(s) \right] ds \quad (25)$$

which can be rewritten as

$$\oint \bar{\gamma}_{st} ds = K_{st}(t) \left(\frac{2A_e}{2} \right) ds + \Gamma_\omega(t) \left(2A_e - \frac{2A_e}{2} \right) + \Gamma_{bt}(t) \oint \bar{b}'(s) ds + \Gamma_{nt}(t) \oint \bar{n}'(s) ds \quad (26a)$$

or

$$\oint \bar{\gamma}_{st} ds = K_{st}(t) A_e + \Gamma_\omega(t) A_e + \Gamma_{bt}(t) \oint \bar{b}'(s) ds + \Gamma_{nt}(t) \oint \bar{n}'(s) ds \quad (26b)$$

Where again the relation $\oint r(s) ds = \oint \psi(s) ds$ has been used, and $A_e = \oint r(s) ds / 2$ is the area of the cross section. By equating the right-hand side of eq. (24) and (26b) and solving for F_{st} , the following expression is obtained:

$$F_{st} = \frac{K_{st}(t) A_e + \Gamma_\omega(t) A_e + \Gamma_{bt}(t) \oint \bar{b}'(s) ds + \Gamma_{nt}(t) \oint \bar{n}'(s) ds - \oint (S_{41}\bar{\varepsilon}_{tt} + S_{42}k_{tt} + S_{43}k_{st}) ds}{\oint (S_{44}) ds} \quad (27a)$$

This can be re-written as:

$$F_{st} = \frac{\varepsilon_t^0 (\oint S_{14} ds) + K_b \oint (-\bar{n} S_{14} - \bar{b}' S_{24}) ds + K_n \oint (\bar{b} S_{14} - \bar{n}' S_{24}) ds + K_{st} (A_e + \oint S_{34} ds)}{\oint (S_{44}) ds} + \frac{K_\omega (\oint \omega S_{14} - q S_{24}) ds + \Gamma_{bt} \oint \bar{b} ds + \Gamma_{nt} \oint \bar{n} ds + \Gamma_\omega A_e}{\oint (S_{44}) ds} \quad (27b)$$

Equation (27b) represents an alternate equation for the in-plane internal shear force F_{st} in a shear-deformable CTWB. When substituted into the right-hand side of eq. (23), the latter provides a clear picture of how $\bar{\gamma}_{st}$, the in-plane shear strain, now depends on variables associated with the axial strain field ($\bar{\varepsilon}_{tt}, k_{tt}$). Furthermore, eq. (27b) also involves terms associated with the angular displacements of the pole line ($\Gamma_\omega, \Gamma_{bt}, \Gamma_{nt}$) which are in turn linked to the out-of-plane shear strain $\bar{\gamma}_{ht}$ via eq. (10c). This link gives eq. (23) an additional role, representing a path for transferring out-of-plane shear energy to the in-plane strain field, effectively increasing the CTWB's capability of absorbing shear energy. The effectiveness of this secondary role of the strain coupling formulation is explored in the next section.

$$\begin{bmatrix} F_t & M_b & M_n & M_t & M_\omega & F_b & F_n & T \end{bmatrix}^T = [\mathbf{E}^{\text{mod}}] \begin{bmatrix} \varepsilon_t^0 & K_b & K_n & K_{st} & K_\omega & \Gamma_{bt} & \Gamma_{nt} & \Gamma_\omega \end{bmatrix}^T \quad (28)$$



where the coefficients E_{ij}^{mod} are presented in the second column of Table A1 of the Appendix, along with their standard (i.e. uncoupled) counterparts shown in the first column to allow for an easier comparison. As mentioned earlier, matrix $\mathbf{E}^{mod}$ gathers all the distinctive features of a CTWB, in this case representing the cross-section stiffness of a shear-deformable CTWB including the in-plane strain coupling formulation.

3. Numerical Validation

The CTWB model introduced in this work (featuring shear-deformability and the strain coupling formulation) has been numerically validated against previously available CTWB models missing either of the key features of the new model proposed in this work:

- An Euler-type model including the strain coupling formulation
- A shear-deformable model without the strain coupling formulation.

Additionally, a detailed 3D shell-based ANSYS finite-element model has been built for reference purposes. The accuracy of each of the CTWB models (the one proposed in this work and the two additional CTWB models previously described in literature) has been assessed in terms of the deviation of its results from the ANSYS model. Key results (i.e. the stiffness matrix's coefficients) and displaced shapes (i.e. elastica) are compared among models for several aspect ratios, fibre-orientations and laminate types. The numerical validation has been conducted through three test cases, each designed to highlight a specific feature of the novel CTWB model introduced in this work.

3.1 Test case #1

This case consists of a fixed-fixed cylindrical CTWB subjected to a unitary point load at its mid-length; the dimensions are: 50 mm diameter at its mid-contour, 1 mm wall thickness and a variable length (3 m, 2 m, 1 m, 0.5 m and 0.25 m), where the latter correspond to the following aspect ratios (length/diameter): 60, 40, 20, 10, and 5. The material layup consists of a single layer with material properties $E_{tt}=4.82 \times 10^{10}$ Pa (along-the-fibre), $E_{ss}=1.17 \times 10^{10}$ Pa (transversal-to-the-fibre), $E_{ts}=6.48 \times 10^9$ Pa (through-the-thickness) and $\nu=0.3$ (in-plane Poisson module). The fibre-orientation angle β (see Fig. 1) was varied according to the following values: 0° (i.e. the circumferential direction), 15° , 30° , 45° , 60° , 75° and 90° (i.e. the beam's axial direction). Separate CTWB models were constructed with and without the shear-deformable terms (labelled "Shear" and "Euler", respectively), as well as with and without the strain coupling formulation (labelled "Coupled" and "Uncoupled", respectively), leading to the following three combinations: 1) "Shear-Coupled", 2) "Euler-Coupled", and 3) "Shear-Uncoupled". The fourth combination, "Euler-Uncoupled", was not considered here since it has been extensively validated elsewhere [4, 9-13]).

For each aspect ratio (AR), fibre-orientation (FO) and model type ("Shear-Coupled", "Euler-Coupled", and "Shear-Uncoupled"), a corresponding CTWB model was set up and discretized with an Isogeometrical Analysis (IGA) formulation following the procedure described in [20], resulting in a numerical model (implemented in MATLAB) with a fixed size of 210 Degrees-Of-Freedom (DOF), and solved for the displacement field of the pole line. Corresponding reference models were also constructed via 8-node, 6 DOF-per-node, Shell281-based ANSYS finite element models of variable size depending on the beam's length: 6,872 elements, 20,632 nodes and 123,792 DOF for the shortest beam (0.5 m), and 41,232 elements, 123,792 nodes and 742,752 DOF for the longest beam (3 m). The wall displacement field of the ANSYS models was used to calculate an equivalent field for the beam's pole line, which is not available in this type of models, to allow a direct comparison with its corresponding CTWB model. It was verified that the cross-section's deformation in all reference models was small (less than 2% of the largest axial deformation), so the CTWB's assumption of a non-deformable cross section remained valid throughout all comparisons.

3.1.1 Test case #1: Matrix $\mathbf{E}^{mod}$ vs $\mathbf{E}$

In this subsection, the $\mathbf{E}^{mod}$ (Shear-Coupled) and $\mathbf{E}$ (Shear-Uncoupled) stiffness matrices of test case #1 have been compared for all FO (0° , 15° , 30° , 45° , 60° , 75° and 90°). AR is not considered, since both matrices are independent of the beam's length. The calculated coefficients for the Shear-Coupled (E_{ij}^{mod}) and Shear-Uncoupled (E_{ij}) models are listed in Tables 1 and 2, respectively, for each FO. For this simple test case, most coefficients have the same values in both lists (i.e. $E_{ij}^{mod}=E_{ij}$), and many of these are also zero. A few coefficients are substantially different, however; these coefficient have been marked with bold letters in each list. The following error metric has been used to measure the deviation of E_{ij}^{mod} with respect to E_{ij} for each case where $E_{ij} \neq 0$:

$$\epsilon_{ij} = 100 \left| \frac{E_{ij}^{mod} - E_{ij}}{E_{ij}} \right| \quad (29)$$

The E_{ij} and E_{ij}^{mod} coefficients were calculated with the help of Table A1, and eq. (A2) (which defines the Z_{ij} coefficients) was used to interpret the observed differences. From Fig. 3, the following observations can be made:

- E_{22} shows deviations from 0% (@ 0° FO) up to a maximum of 22.8% (@ 60° FO) and back to 0% (@ 90° FO). From Table A1 and eq. (A2), it can be observed that the first terms in E_{22} and E_{22}^{mod} , respectively, are not that dissimilar, since both depend on the similar stiffness coefficients $Z_{11} \cong S_{11}$ and $Z_{14} \cong S_{14}$ (although Z_{33} and S_{33} can diverge significantly for some fibre-orientations). However, the second term in E_{22}^{mod} , absent in E_{22} and thus the main suspect for the observed deviations, depends on the coefficients S_{14} , S_{24} and S_{44} , all related to the in-plane internal shear force F_{st} . Upon inspection of this second term, the numerator of which involves closed integrals along the mid-contour of variables related to S_{14} and S_{24} , it can be demonstrated that $S_{14}=S_{24}=0$ at $0^\circ/90^\circ$ FO for closed cross sections, and that S_{14} grows stronger as FO values approach the range 45° - 75° . These arguments qualitatively explain why E_{22}^{mod} and E_{22} are equal at $0^\circ/90^\circ$ FO and show a maximum deviation at 60° FO.
- The above discussion holds identically for coefficients E_{33} and E_{33}^{mod} , respectively, as they contain the same stiffness coefficients as in E_{22} and E_{22}^{mod} , respectively, and consequently show an identical behaviour.
- E_{26} depends on Z_{12} and Z_{23} , both modelling the coupling behaviour between F_{tt} , $\bar{\gamma}_{st}$ and F_{st} , k_{tt} , respectively (that is, the axial-shear coupling). E_{26}^{mod} is zero for any FO, in agreement with the fact that its only term depends on $\oint \sin \alpha(s) ds$, which is zero for closed sections. The latter explains the large discrepancies (i.e. 100%) between E_{26} and E_{26}^{mod} evidenced in Fig. 3.
- The above discussion holds true for E_{37} and E_{37}^{mod} as well, as these matrix elements depend on the same stiffness coefficients as in (E_{26} , E_{26}^{mod}) pair, and the term $\oint \cos \alpha(s) ds$ is also zero for closed sections at any FO.
- In the case of the (E_{66} , E_{66}^{mod}) pair deviations range from 50% (@ 0° FO) up to a maximum peak of 61% (@ 60° FO) and back to 50% (@ 90° FO), i.e. the shear-coupled model and the uncoupled model differ substantially at all fibre orientations. From Table A1 and eq. (A2), it can be observed that the first terms in E_{66} and E_{66}^{mod} , respectively, are similar, since both depend on



identical stiffness coefficients $Z_{55}=S_{55}$ (related to the transversal shear strain). However, the second term in E_{66} can be seen to depend on Z_{22} (which is related to the in-plane internal shear force F_{st}), while its counterpart in E_{66}^{mod} depends on the constant value $\oint S_{44} ds$ (also related to F_{st}). This explains why E_{66}^{mod} has the same numerical value (508,940) for any FO, while E_{66} has different values for different FO (within the range 1,017,801-1,306,002), also explaining its relatively large deviations.

- The above discussion holds identically for the (E_{77}, E_{77}^{mod}) pair, since they contain the same stiffness coefficients as in (E_{66}, E_{66}^{mod}) , respectively. This is evidenced in Fig. 3.

Based on the above comparisons, it can be concluded that the $\mathbf{E}^{mod}$ (Shear-Coupled) and $\mathbf{E}$ (Shear-Uncoupled) matrices are similar at $0^\circ/90^\circ$ FO, where the in-plane strain coupling has little impact, and increasingly diverge as FO departs from $0^\circ/90^\circ$, mainly due to stiffness terms related to the in-plane shear force. Maximum deviations are observed in the vicinity of 60° , where the in-plane strain coupling has maximum strength.

3.1.2 Test case #1: Displacement shapes

The quality of a given CTWB model is here measured by the degree of similarity of its displaced shape (i.e. its elastical response) to its reference's counterpart, as quantified by the Fréchet Distance (FD) among both curves. The FD is a measure of similarity between two curves which accounts for the location and ordering of the points conforming each curve. It computes the maximum distance a point on the first curve must travel as this curve is being continuously deformed into the second curve. Given two sequences of points $\pi \equiv p_1, \dots, p_n$ and $\sigma \equiv q_1, \dots, q_m$ and a monotone path $(i,j) \equiv \pi(n,m)$, the FD between the two sequences is the distance

$$d_F(\pi, \sigma) = \min_{(i,j) \in \Pi(n,m)} \ell(i,j) = \min_{(i,j) \in \Pi(n,m)} \max_t p_{i(t)} - q_{j(t)} \quad (30)$$

In this work, the FD is expressed as a percentage of the reference's maximum displacement (in this case obtained at the mid-span, where the load is located), where 0% indicates a perfect match (i.e. two identical curves) while 100% indicates that the FD between both curves is the same as the reference's maximum displacement. Figures 4 to 8 display, respectively for each AR (60, 40, 20, 10 and 5), the Fréchet Distance associated with each type of CTWB (Shear-Coupled, Euler-Coupled, and Shear-Uncoupled), represented by different curves in any given plot, and for each FO (0° , 15° , 30° , 45° , 60° , 75° and 90° , measured in the horizontal axis). Apart from the main graph, each figure contains two subplots showing the displaced shapes of all models for 60° and 90° FO, respectively, where maximum errors occur in the majority of cases. The horizontal axes of all the insets represent the corresponding beam length, while the vertical axes are scaled to represent a unity maximum displacement of the corresponding reference model. The following discussion analyses the causes of the similarities and differences of each model.

Table 1. Coefficients E_{ij}^{mod} ("Shear-Coupled") for test case #1 and each fibre orientation. Highlighted in bold are those coefficients that differ from their E_{ij} counterpart shown in Table 2.

E_{ij}^{mod}	Fibre Orientation						
	0°	15°	30°	45°	60°	75°	90°
(1,1)	1,837,696	1,895,426	2,159,208	2,950,219	4,663,195	6,691,618	7,539,267
(1,4)	0	1,380	4,291	10,032	16,288	13,716	0
(1,8)	0	1,380	4,291	10,032	16,288	13,716	0
(2,2)	574	589	646	791	1,125	1,793	2,356
(2,6)	0	0	0	0	0	0	0
(3,3)	574	589	646	791	1,125	1,793	2,357
(3,7)	0	0	0	0	0	0	0
(4,4)	159	170	201	241	249	197	159
(4,8)	159	170	201	241	249	197	159
(6,6)	508,940	508,940	508,940	508,940	508,940	508,940	508,940
(7,7)	508,861	508,861	508,861	508,861	508,861	508,861	508,861
(8,8)	159	170	201	241	249	197	159
(1,2),(1,3),(1,5),(1,6),(1,7),(2,3),(2,4),(2,5),(2,7),(2,8),(3,4),(3,5),(3,6),(3,8),(4,5),(4,6),(4,7),(5,5),(5,6),(5,7),(5,8),(6,7),(6,8),(7,8)=0							

Table 2. Coefficients E_{ij} ("Shear-Uncoupled") for each fibre-orientation. Only those coefficients that differ from their E_{ij}^{mod} counterpart (see Table 1) are shown.

E_{ij}	Fiber Orientation						
	0°	15°	30°	45°	60°	75°	90°
(2,2)	574	592	675	922	1,457	2,091	2,356
(2,6)	0	1,380	4,290	10,031	16,287	13,715	0
(3,3)	574	592	675	922	1,458	2,092	2,357
(3,7)	0	1,380	4,291	10,033	16,289	13,717	0
(6,6)	1,017,801	1,052,081	1,151,247	1,279,083	1,306,002	1,138,824	1,017,801
(7,7)	1,017,801	1,052,086	1,151,268	1,279,123	1,306,047	1,138,842	1,017,801



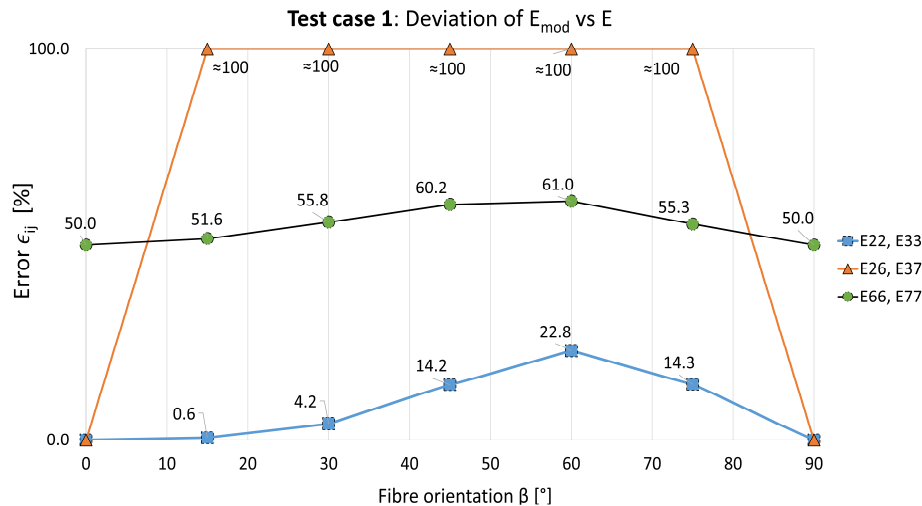


Fig. 3. Deviation ϵ_{ij} of E_{ij}^{mod} with respect to E_{ij} for all fibre-orientations (test case #1). Only coefficients with a non-zero deviation are shown in this figure.

- Performance at large AR and 0°/90° FO:** At large AR (i.e. 60-40), shear-deformability has little impact, while for FO aligned with a principal direction (i.e. circumferential (0°) or axial (90°)) the in-plane strain coupling has a weak effect; for those fibre orientations all three CTWB models are expected to perform similarly. This can be confirmed in Fig. 4 (AR=60), for which the Fréchet distance of all three CTWB lies between 0.61% (Euler-Coupled @ 0° FO) and 2.05% (Euler-Coupled @ 90° FO), while the corresponding bounds for AR=40 (see Fig. 5) are 0.72% (Shear-Coupled @ 0° FO) and 4.4% (Euler-Coupled @ 90° FO). These low FD measures confirm the trends observed in the subplots of Fig. 4 and 5 @ 90° FO, showing that in these cases the displacement shapes among all models are virtually undistinguishable. The comparatively larger (although still small) deviations at 90° FO (versus 0°) are explained by the lack of fibres in the circumferential direction, which produces slightly higher deformations in the cross section of the reference model. This invalidates to a small degree the assumption of the cross section's non-deformability in all three CTWB models and hence leads to slightly higher deviations as compared to 0° FO (this argument holds for any AR).
- Performance of the strain coupling formulation:** At large AR (i.e. 60-40), but now within the range 45°-75° FO, for which strain coupling is significant, it is possible to isolate the impact of the strain coupling formulation. At AR=60 and FO=60° (Fig. 4), the Shear-Uncoupled model yields its maximum peak FD value of 10.69%, whereas the Shear-Coupled and Euler-Coupled only have maximum FD values of 0.42% and 0.62%, respectively. For AR=40 (Fig. 5), the corresponding measures are 10.83%, 1.39%, and 0.94%, respectively. These FD values indicate an improvement in accuracy of about one order of magnitude due to the strain coupling formulation in both the shearable and Euler-type theories. The aforementioned trends can be confirmed in the subplots of Fig. 4 and 5 at 60° FO, where only the Shear-Uncoupled displacement shape exhibits an appreciable deviation at the beam mid-length. For decreasing AR, the differences between the three models within the range 45°-75° FO become less dissimilar, indicating that shear-deformability increasingly takes predominance over the strain coupling behaviour. For example, at AR=10 the Shear-Uncoupled significantly outperforms the Euler-Coupled at any FO (a reversal of the trend observed at larger AR), the former yielding between 50%-100% of the latter's FD measures. The displacement shapes shown in the subplots of Fig. 7 (AR=10) @ 60° FO indicate that, in spite of somewhat similar FD measures, the Shear-Coupled model (FD = 12.6%) exhibits greater flexibility than the reference model (i.e. a higher displacement peak), while the Shear-Uncoupled (15.5%) and the Euler-Coupled (20.4%) yield a stiffer response (i.e. lower peak displacement than its reference). The corresponding inset in Fig. 8 (AR=5 @ 60° FO) shows that the latter trend strengthens even more, although the inaccuracies exhibited by the Euler-Coupled and the Shear-Uncoupled @ 90° FO (FD of 38% and 77%, respectively) render them essentially unusable.
- Performance of the shear-deformable terms:** At smaller AR (i.e. 10-5) and 0°/90° FO, it is possible to isolate the impact of the shear-deformable terms. The Euler-Coupled, Shear-Uncoupled and Shear-Coupled yield the following relevant FD measures for AR=10: 16.7%, 8.2% and 0.8% @ 0° FO, and 45.5%, 21.7% and 2.4% @ 90° FO, while for AR=5 the corresponding measures are 44.9%, 22.0% and 1.7% @ 0° FO, and 77.2%, 38.5% and 2.2% @ 90° FO. These values (and their corresponding displacement shapes shown in the insets) evidence the superiority of both shear models over the Euler model, and thus the effectiveness of the shear-deformable terms. A somewhat counterintuitive result is the significant difference between both Shear models at 0°/90° FO, which grows with decreasing AR, as in principle the "Coupled/Uncouple" labels should be less relevant at these FO. While the aforementioned is true, the reader is reminded that the in-plane strain coupling formulation of eq. (27b) includes some (out-of-plane) shear-deformable terms, thus also acting as a path for transferring out-of-plane shear energy to the in-plane strain field, which increases the CTWB's capability of absorbing shear energy and improves the description of shear-deformability. This effect is demonstrated in Fig. 4 to 8, representing a secondary but significant role of the strain coupling formulation.
- Model sizes:** The ANSYS-based reference models had sizes between about 1.2 million and 750 thousand DOFs, whereas the CTWB models had a fixed size of 210 DOF. This represents a difference of about two to three orders-of-magnitude in size, making the novel CTWB model introduced in this work an attractive alternative for cases where recursive calculations are performed (i.e. multiple "what-if" scenarios, fluid-structure interaction, etc.).

From the above discussions it can be concluded that, for arbitrary AR and FO, the new (Shear-Coupled) model significantly outperforms the other two models. For typical AR between 60-20, it exhibits deviations bounded by 0.45% (@ 60 AR, 60° FO) and 4% (@ 20 AR, 60° FO), while for shorter AR it yields much lower deviations than its Euler-type counterpart.



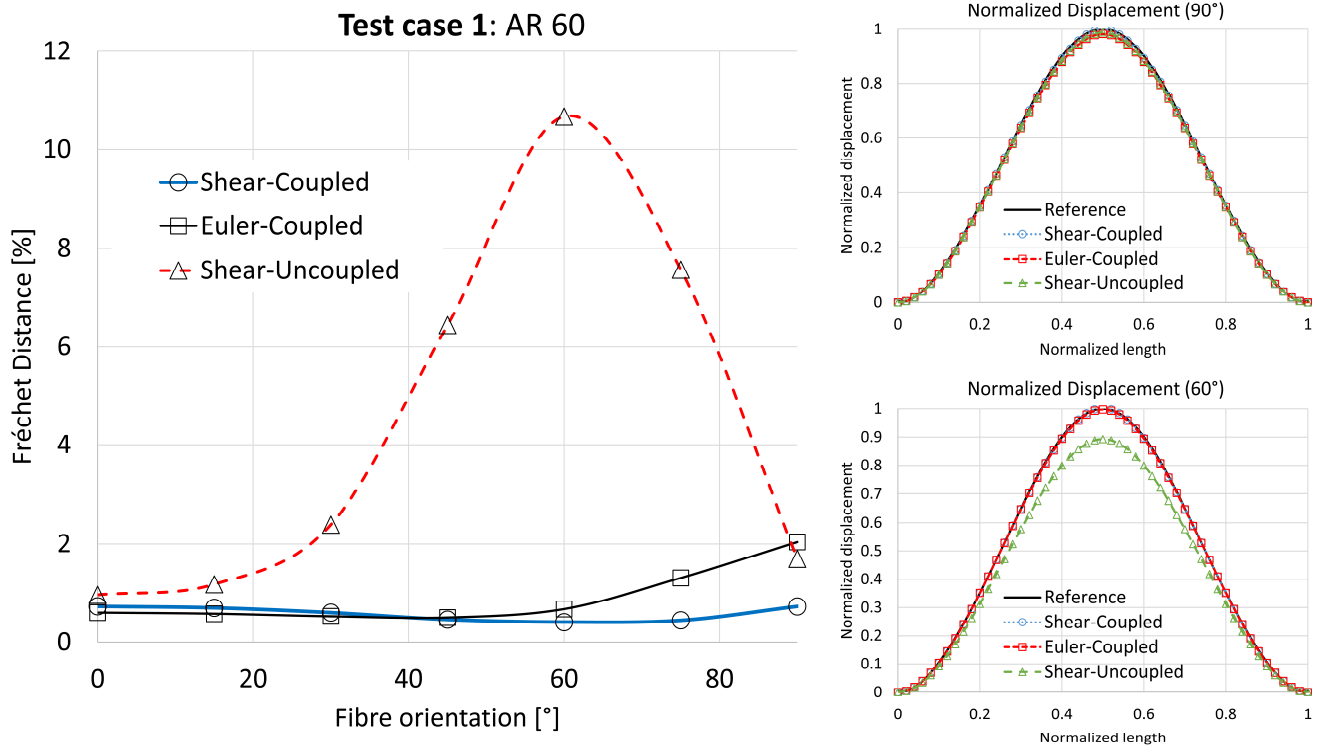


Fig. 4. Fréchet Distances of CTWB models for AR=60 and all fibre-orientations (test case #1). The two insets show the displacement curves of all models at fibre-orientations of 60° and 90°, respectively.

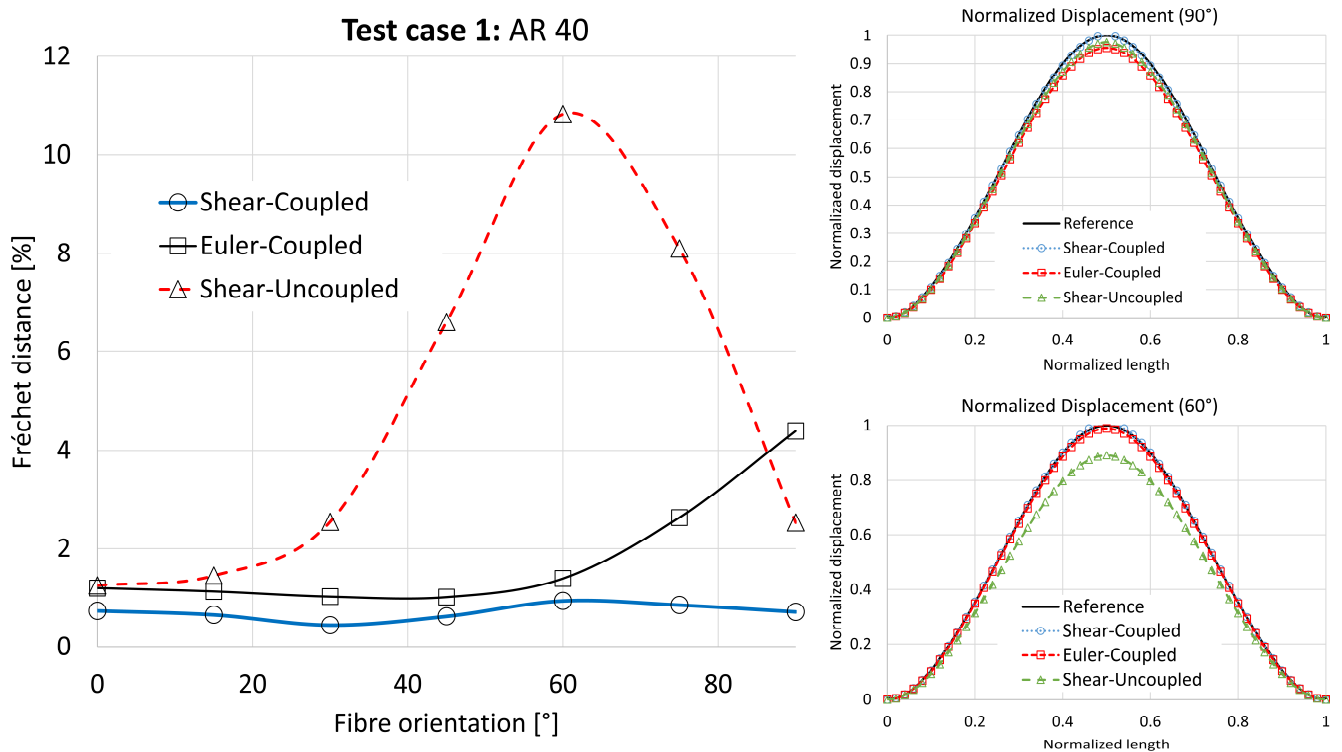


Fig. 5. Fréchet Distances of CTWB models for AR=40 and all fibre-orientations (test case #1). The two insets show the displacement curves of all models at fibre-orientations of 60° and 90°, respectively.



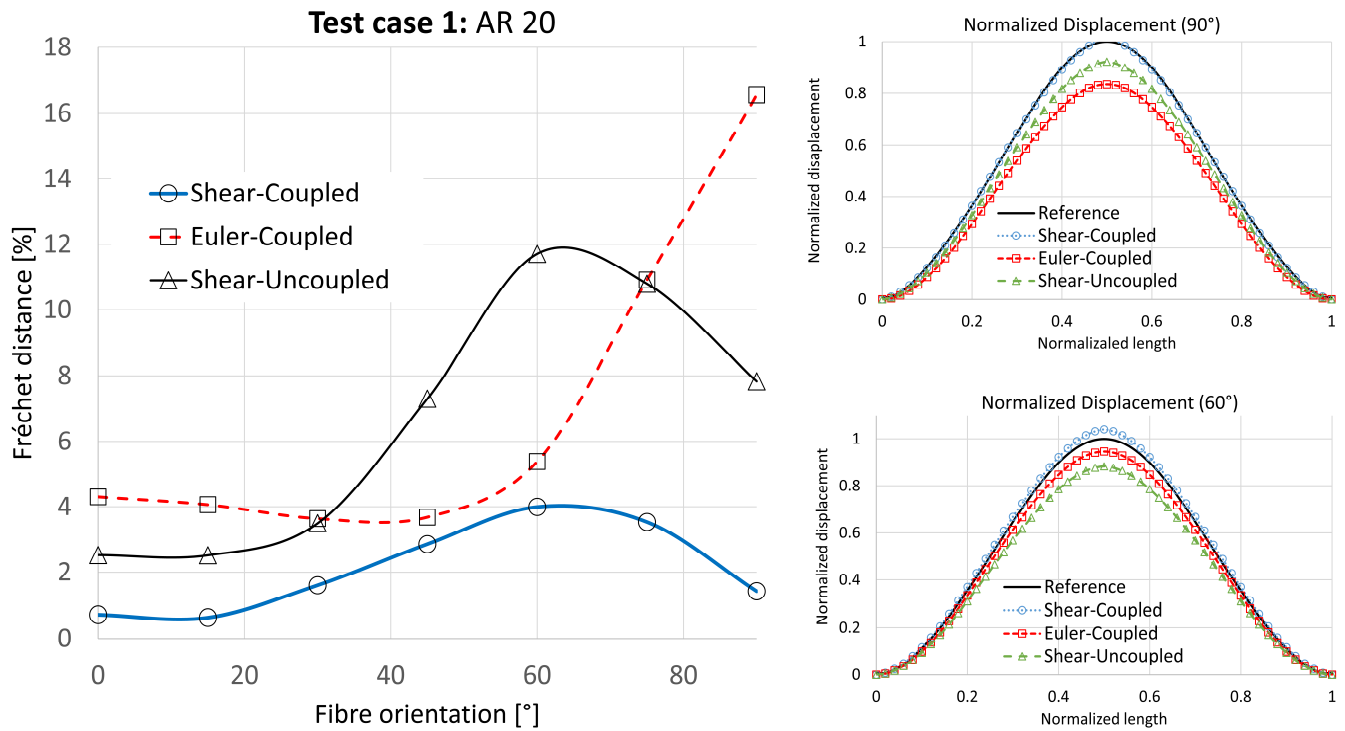


Fig. 6. Fréchet Distances of CTWB models for AR=20 and all fibre-orientations (test case #1). The two insets show the displacement curves of all models at fibre-orientations of 60° and 90°, respectively.

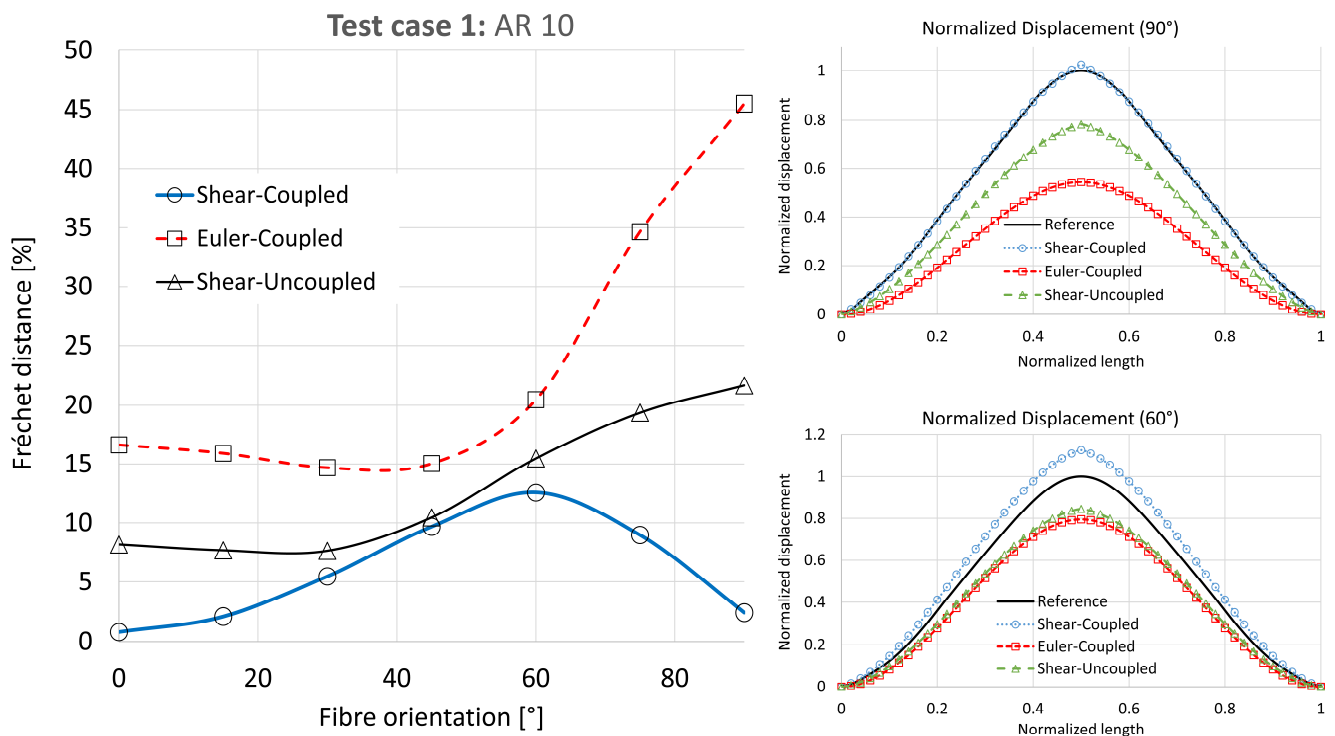


Fig. 7. Fréchet Distances of CTWB models for AR=10 and all fibre-orientations (test case #1). The two insets show the displacement curves of all models at fibre-orientations of 60° and 90°, respectively.



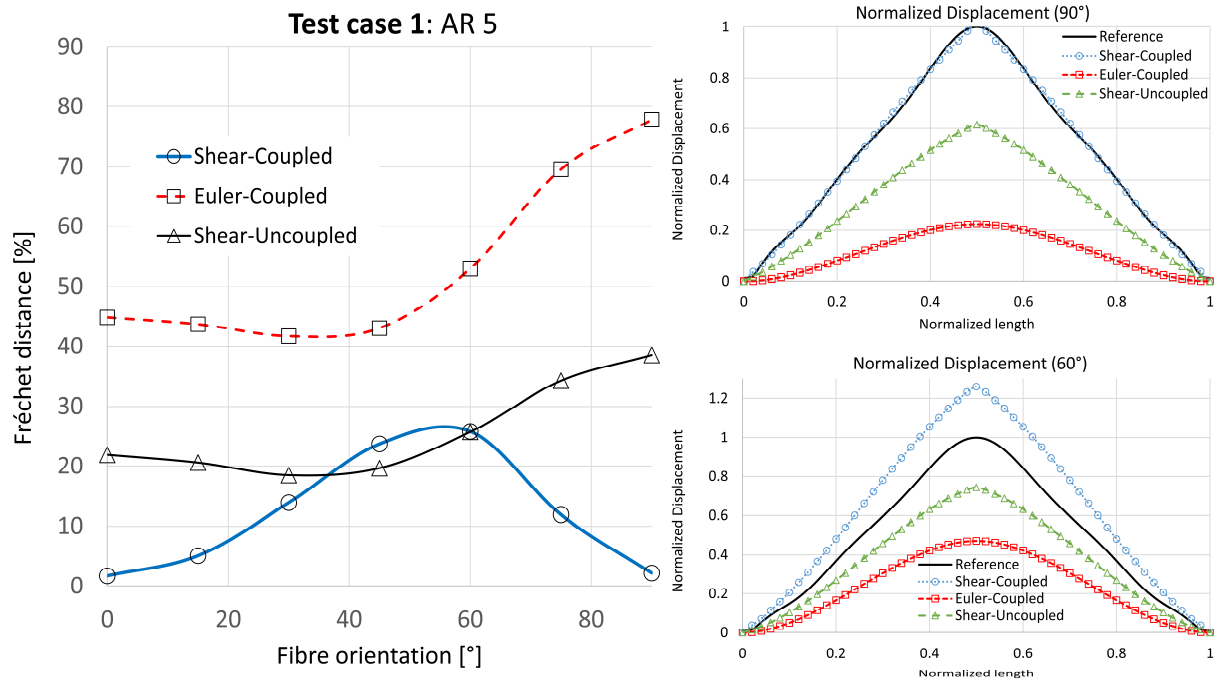


Fig. 8. Fréchet Distances of CTWB models for AR=5 and all fibre-orientations (test case #1). The two insets show the displacement curves of all models at fibre-orientations of 60° and 90°, respectively.

3.2 Test case #2

The strain coupling formulation represented by eq. (23) allows Librescu-type CTWB models (typically featuring an uncoupled strain field) to analyse cases where significant coupling exist between the in-plane shear strain ($\bar{\gamma}_{st}$) and any/all of the remaining strain measures ($\bar{\varepsilon}_{tt}, k_{tt}, k_{st}$). This scenario often occurs in off-axis laminates, leading to significant axial-bending-torsional structural coupling. The range of anisotropic couplings that can be represented by eq. (23) can be better visualized by recasting it in terms of the A_{ij} and B_{ij} coefficients contained within the **ABD** matrix defined in eq. (24)

$$\bar{\gamma}_{st} = \frac{A_{11}}{A_{11}A_{66} - A_{16}^2} \left(\left(\frac{A_{12}A_{16}}{A_{11}} - A_{26} \right) \bar{\varepsilon}_{tt} + \left(\frac{B_{12}A_{16}}{A_{11}} - B_{26} \right) k_{tt} + \left(\frac{B_{16}A_{16}}{A_{11}} - B_{66} \right) k_{st} + F_{st} \right) \quad (31a)$$

$$\bar{\gamma}_{st}(s, t) = C_0 (C_1 \bar{\varepsilon}_{tt} + C_2 k_{tt} + C_3 k_{st} + C_4 F_{st}) \quad (31b)$$

where the A_{ij} and B_{ij} coefficients in eq. (31a) represent the material anisotropy distribution around the cross-section's mid-contour, while the coefficients C_1, C_2, C_3, C_4 in eq. (31b) can be visualized as weights defining the strength of the individual coupling between $\bar{\gamma}_{st}$ and each of the remaining strain ($\bar{\varepsilon}_{tt}, k_{tt}, k_{st}$) and force (F_{st}) variables. Table 3 illustrates the different mechanical couplings between forces and strains associated to each A_{ij} or B_{ij} coefficient, as inferred from eq. (24) or eq. (31).

Table 3. Coupling associated with each A_{ij}, B_{ij} coefficient

Coefficient	Coupling description
A_{11}	links F_{ss} (hoop force) with $\bar{\varepsilon}_{ss}$ (hoop strain)
	links F_{ss} (hoop force) with $\bar{\varepsilon}_{tt}$ (axial strain)
A_{12}	links F_{tt} (axial force) with $\bar{\varepsilon}_{ss}$ (hoop strain)
	links F_{ss} (hoop force) with $\bar{\gamma}_{st}$ (in-plane shear strain)
A_{16}	links F_{st} (in-plane shear force) with $\bar{\varepsilon}_{ss}$ (hoop strain)
	links F_{tt} (axial force) with $\bar{\gamma}_{st}$ (in-plane shear strain)
A_{26}	links F_{st} (in-plane shear force) with $\bar{\varepsilon}_{tt}$ (axial strain)
A_{66}	links F_{st} (in-plane shear force) with $\bar{\gamma}_{st}$ (in-plane shear strain)
	links M_{tt} (torsional moment around the $t-t$ plane) with $\bar{\varepsilon}_{tt}$ (axial strain)
B_{12}	links F_{ss} (hoop force) with k_{tt} (axial curvature)
	links M_{ss} (bending moment in the $s-s$ plane) with $\bar{\gamma}_{st}$ (in-plane shear strain)
B_{16}	links F_{ss} (hoop force) with k_{st} (in-plane shear curvature)
	links M_{tt} (torsional moment in the $t-t$ plane) with k_{st} (in-plane shear curvature)
B_{26}	links F_{tt} (axial force) with k_{st} (in-plane shear curvature)
	links M_{st} (bending moment in the $s-t$ plane) with k_{st} (in-plane shear curvature).
B_{66}	links F_{st} (in-plane shear force) with k_{st} (in-plane shear curvature)



Table 4. Strength and type of the strain coupling associated to each type of laminate. A_{ij} , B_{ij} are given in GPa-mm and GPa-mm², respectively.

Laminate Code	Laminate Type	ABD Matrix			C1	C2	C3	C4
[0°/0°/0°/0°/0°/0°]	Symmetrical / Balanced	A=	2454	178.7	0	0	0	0
			178.7	595.7	0			
			0	0	322.7			
		B=	0	0	0			
			0	0	0			
			0	0	0			
[60°/60°/60°/60°/60°/60°]	Symmetrical / Unbalanced	A=	797.5	441.5	250.6	415	0	0
			441.5	1727	554			
			250.6	554	585.5			
		B=	0	0	0			
			0	0	0			
			0	0	0			
[60°/45°/0°/0°/45°/60°]	Symmetrical / Unbalanced	A=	1475	383.1	238.4	278	0	0
			383.1	1166	339.5			
			238.4	339.5	527.1			
		B=	0	0	0			
			0	0	0			
			0	0	0			
[60°/45°/0°/0°/-45°/-60°]	Anti-symmetrical / Balanced	A=	1475	383.1	0	0	5760	0
			383.1	1166	0			
			0	0	527.1			
		B=	0	0	3661			
			0	0	5760			
			3661	5760	0			
[60°/45°/0°/0°/-45°/60°/90°]	Asymmetric / Unbalanced	A=	1575	412.9	83.54	163	2649	728
			412.9	1575	184.7			
			83.54	184.7	580.9			
		B=	3650	848.2	2275			
			848.2	5347	2694			
			2275	2694	848.2			

As seen in eq. (31b), the coefficients C_1 , C_2 , C_3 , C_4 get activated/deactivated depending on the existence of certain A_{ij} or B_{ij} coefficients, which in turn depends on the type of laminate used in the CTWB material. Table 4 shows the numerical value of the coefficients C_1 , C_2 , C_3 , C_4 corresponding to four different laminates: symmetrical / balanced, symmetrical / unbalanced, anti-symmetrical / balanced and asymmetrical / unbalanced. Each laminate has a total thickness of 50 mm conformed of several laminas, each at a definite FO and with material properties as defined for the test case #1. The following qualitative conclusions can be drawn for each laminate:

- Symmetrical / Balanced on-axis 0° (unidirectional):** For this type of laminate all B_{ij} coefficients are zero, $B_{ij}=0$, as well as $A_{16}=A_{26}=0$; therefore, there are no couplings between $\bar{\gamma}_{st}$ and any of the other strain measures. Therefore, this type of laminate can be safely modelled without implementing a mechanical strain coupling formulation.
- Symmetrical / Unbalanced off-axis 60° (unidirectional):** As before, all of the B_{ij} coefficients remain zero but the matrix **A** becomes fully populated, thus leading to a strong coupling between $\bar{\gamma}_{st}$ and $\bar{\epsilon}_{tt}$. In this case, there is a need of a strain coupling formulation for properly modelling this type of laminate.
- Symmetrical / Unbalanced off-axis 45°-60°:** As for the symmetrical / unbalanced laminate above, this laminate has a fully populated A matrix and vanishing B_{ij} coefficients. Therefore, a coupling can be observed only between $\bar{\gamma}_{st}$ and $\bar{\epsilon}_{tt}$, although with a strength of about 67% of the previous case.
- Anti-symmetrical / Balanced off-axis 45°-60°:** A very strong mechanical coupling can be observed only between $\bar{\gamma}_{st}$ and k_{tt} due to the fact that the $A_{16}=A_{26}=B_{66}=0$, but B_{16} and B_{26} are non-zero. This type of laminate is used, for example, in functionally graded materials to minimize thermal deformations; this type of materials will greatly benefit from the strain coupling formulation for high accuracy analyses.
- Asymmetrical / Unbalanced off-axis 45°-60°-90°:** This type of laminate represents a general situation in which no particular order and/or symmetry between laminas is observed. Due to fully populated A and B matrices, couplings are produced between $\bar{\gamma}_{st}$ and all of the remaining strain measures ($\bar{\epsilon}_{tt}, k_{tt}, k_{st}$), each with a distinct strength, although the coupling with k_{tt} is still dominant.

The above results justify the development of the strain coupling formulation as an essential tool for the analysis of most types of laminates. Except for the on-axis symmetrical / balanced laminate, a standard but still restrictive application, all other types imply some form of strain coupling.

3.3 Test case #3

In this section, Shear-Coupled, Euler-Coupled, and Shear-Uncoupled CTWB models having a asymmetric/unbalanced laminate shown in Table 4 ([60°/45°/0°/0°/-45°/60°/90°]) and a non-symmetrical cross section (shown in Fig. 9) are analyzed for several aspect ratios (40, 35, 30, 25, 20, 15 and 10) and the same materials properties used in test case #1. It was demonstrated in test case #2 that an asymmetrical/unbalanced laminate represents a worst-case coupling scenario in terms of the materials definition. Geometrically speaking, a non-symmetrical cross-section also represents a worst-case coupling scenario as it leads to a fully populated matrix E_{ij} , thus increasing the force-strain coupling behaviour of a CTWB.



Test case 3: Non-symmetrical elliptical cross section

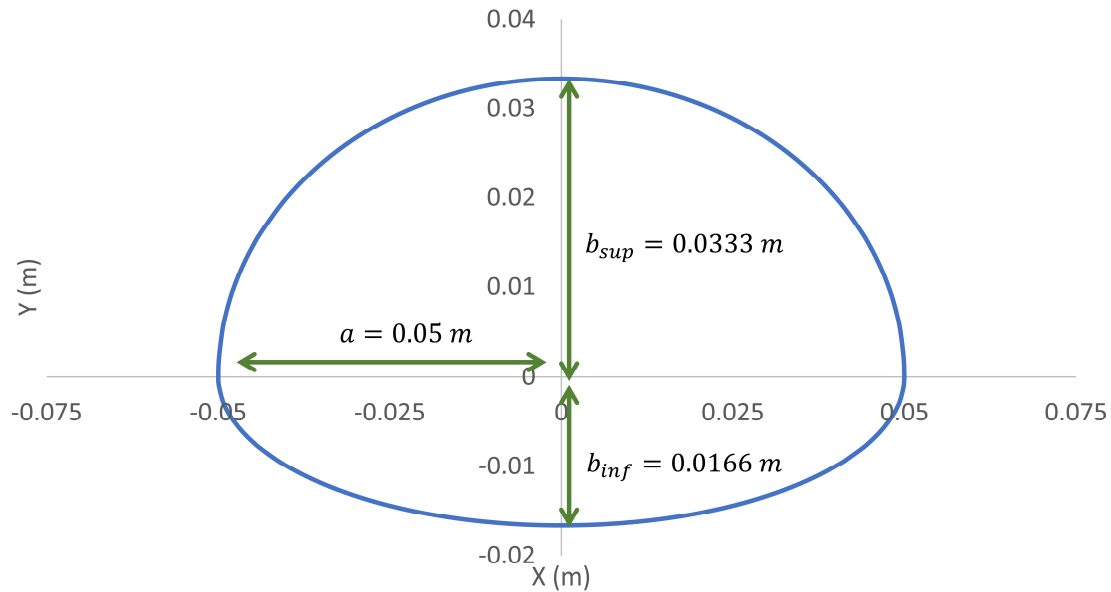


Fig. 9. Cross-section of the CTWB of test case #3. Units are given in mm.

Test case 3: Evolution of model accuracy with aspect ratio

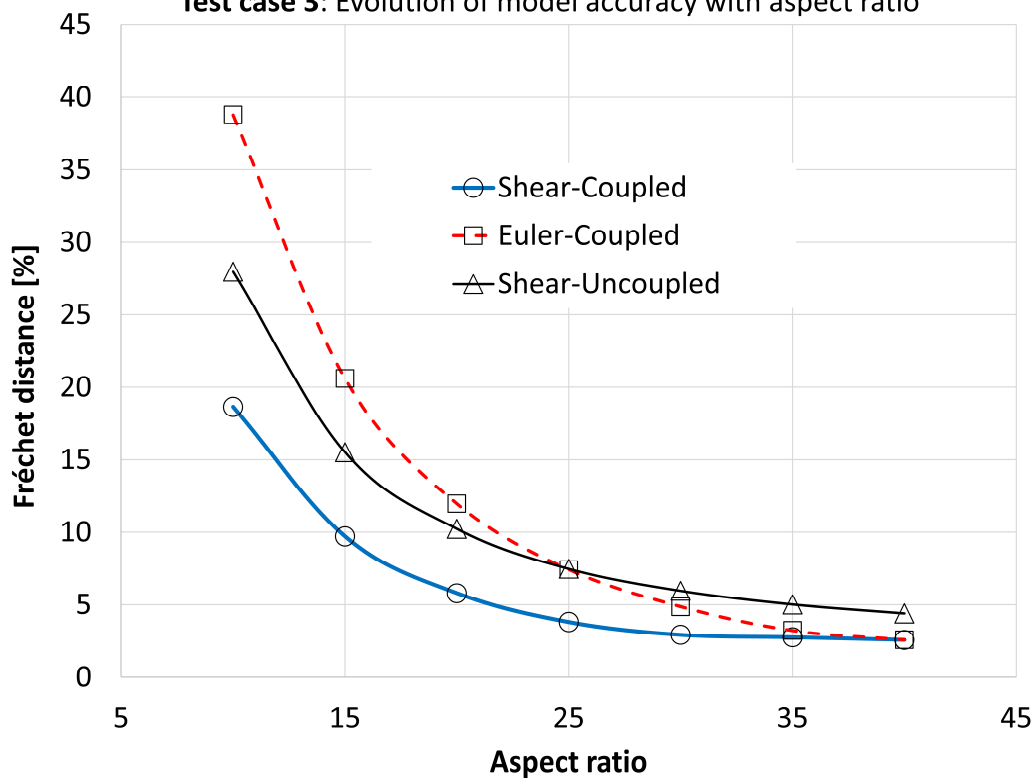


Fig. 10. Fréchet Distances of CTWB models for all aspect ratios (test case #3).

Figure 10 shows the Fréchet Distance (FD) of the Shear-Coupled, Euler-Coupled and Shear-Uncoupled models for various AR, while Fig. 11 show the displacement shapes of all models for each AR. For increasing AR, a converging trend towards lower FD values can be observed, measuring 2.6%, 2.6% and 4.4%, respectively, for AR=40. These results indicate that any of the three CTWB models can reproduce the mechanical behaviour of longer beams fairly demonstrating that shear and Euler theories essentially converge in this case. In-plane shear-strain coupling provides a large relative improvement in the accuracy of mode shape prediction (from 4.4% to 2.6%), but the error is already small to start with.

As expected, the discrepancy in the displacement shapes, as measures by the Fréchet distance, increases towards smaller aspect ratios for all models. However, particularly for smaller AR values the discrepancy becomes quite substantial. While the Euler-coupled model performs very well at large AR's, its performance at smaller aspect ratios is actually worse than the one of the Euler-uncoupled model, a fact that was already observed in test case #1. The Shear-Coupled model, introduced in this work, effectively corrects this shortcoming and provides a consistently improved performances at all aspect ratios.



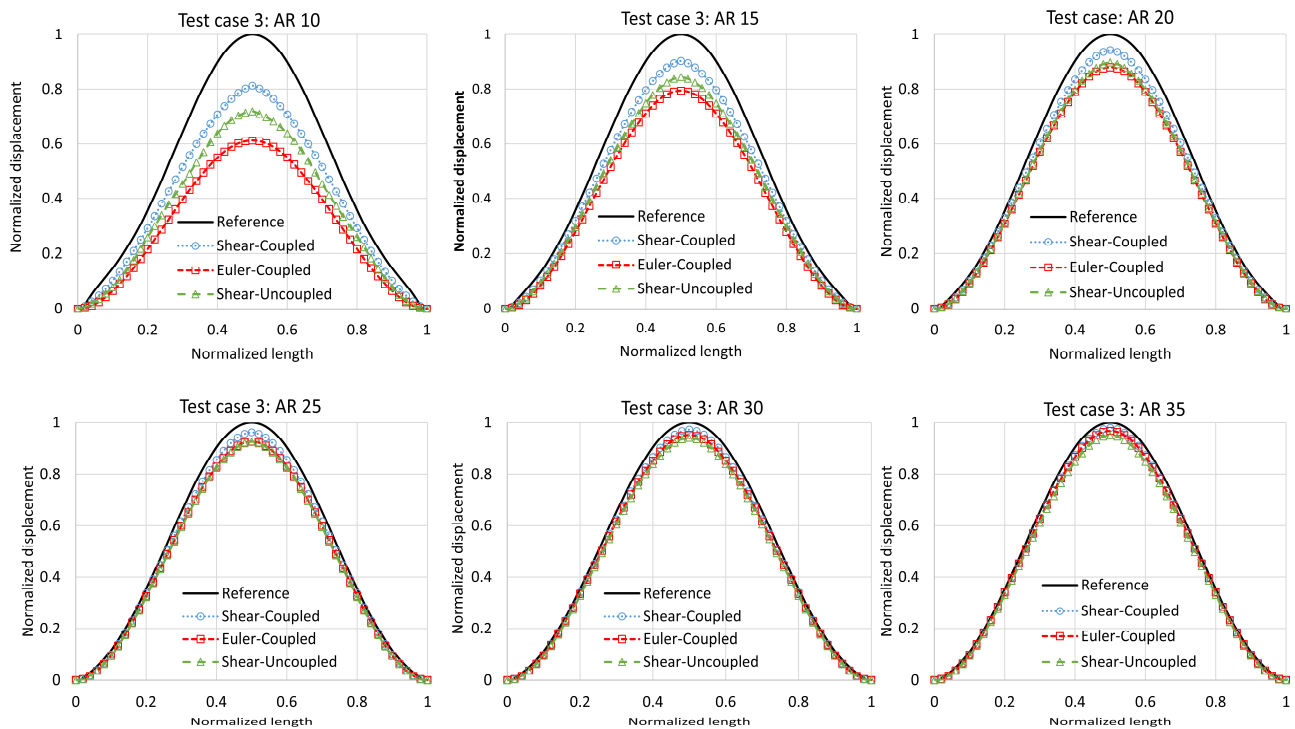


Fig. 11. Displacement curves of all models at aspect ratios 10, 15, 20, 25, 30, and 35, for test case 3.

4. Conclusion

In this paper a novel Librescu-type shear-deformable composite thin-walled beam (CTWB) model is introduced, featuring an in-plane strain coupling field. Based on prior work by Zangh & Wang [19] for Euler-type CTWB, an analogous formulation for shear-deformable CTWB models has been developed, leading to significant improvements in the modelling accuracy in cases where off-axis fibre-orientations and/or asymmetric/unbalanced layups produce axial-bending-torsional structural coupling. This strain coupling formulation provides primarily a coupling between the in-plane shear and axial strains through an alternative equation for the in-plane internal shear force, improving upon the original formulation by Librescu and others. This alternate equation also involves strain measures related to the out-of-plane shear strain field, thus providing a path for transferring out-of-plane shear energy to the in-plane strain field. The latter fact represents a secondary role of the strain coupling formulation which further improves the description of shear-deformability by increasing the total absorption of shear energy.

The global equilibrium equations were discretized and solved via an Isogeometrical Analysis (IGA) scheme similar to the one described in [20], building equivalent numerical equilibrium equations directly from geometry-based basis functions (conventional NURBS-based CAD design tools), resulting in a compact and robust numerical description. Numerical validation was provided via three numerical test cases, each designed to highlight specific features of the novel CTWB model introduced in this work, compared to previously available CTWB (i.e. Euler-type coupled and Timoshenko-type uncoupled) for various aspect ratios, fibre orientations and laminate types. In all cases, reference measures were provided by detailed 3D shell-based ANSYS models. The differences in performance among models were explained by a few relevant stiffness coefficients, which contained (or lacked) certain parameters related to the in-plane shear force.

The overall quality of a given CTWB was measured by the Fréchet distance between its displacement shape (i.e. the elastica) and the one of the corresponding reference model, finding that the inclusion of the strain coupling formulation consistently provides more accurate solutions for arbitrary aspect ratios, fibre-orientations and laminate types. For non-critical conditions (i.e. large aspect ratios, fibre-orientations nearby 0° and symmetric/balanced on-axis laminates), the increment in accuracy is relatively small; in these cases, it is safe to use either of the models. However, for a number of critical conditions (small aspect ratios, fibre-orientations nearby $60^\circ/90^\circ$ and asymmetric/unbalanced off-axis laminates) the increase in accuracy can reach one-to-two orders of magnitude. Evidence of the interaction between the strain coupling formulation and shear-deformability was found for fibre orientations close to 90° at any aspect ratio, where the increase in shear energy absorption caused significantly lower deviations (i.e. by a factor of 3 to 10) than the uncoupled formulation.

In view of the above results, it can be concluded that the in-plane strain coupling for shear-deformable CTWB presented in this work represents a significant improvement in accuracy for this type of models, with results with are very similar to complex finite-element models, obtained with a fraction of the size and hence the computational cost.

Author Contributions

Authors 1, 2 and 4 designed the work plan and steps of the project. Authors 1 and 2 developed the analytical expressions in detail. Author 3 coded the equations of the model and simulated the cases studies. Authors 1, 2 and 4 reviewed and analysed the results and prepared the final figures. Authors 1 and 4 wrote most of the paper.

Acknowledgments

AD is grateful for a PhD tuition scholarship from Tecnológico de Monterrey and a stipend for living expenses from CONACYT (Mexico). HE, DC and OP acknowledge support from the Energy and Climate Change group at the School of Engineering and Sciences at Tecnológico de Monterrey.



Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

Funding

The authors received no financial support for the research, authorship, and publication of this article.

Nomenclature

a	Major semi-axis of the elliptical beam [m]	β	Fibre-orientation measured with respect to the s axis
b_{sup}	Minor semi-axis of the elliptical beam (top part) [m]	u_i, u_n, u_b	Displacement of beam the pole line
b_{inf}	Minor semi-axis of the elliptical beam (bottom part) [m]	$\theta_i, \theta_n, \theta_b$	Angular displacement of beam the pole line
ABD	ABD matrix from CLT theory	$\omega(s)$	Warping function
AR	Aspect Ratio	U_i, U_n, U_b	Wall displacement in (t,n,b) frame
FD	Fréchet Distance	U_b, U_s, U_h	Wall displacement in (t,s,h) frame
FO	Fiber Orientation	$\bar{\epsilon}_{ss}$	Hoop strain
F_{ss}	Hoop force	$\bar{\epsilon}_{tt}$	Axial strain
F_{tt}	Axial force	$\bar{\gamma}_{st}$	In-plane shear strain
F_{st}	In-plane shear force	k_{tt}	Axial curvature
M_{tt}	Torsional moment around the t-t plane	k_{st}	In-plane shear curvature
M_{ss}	Bending moment in the s-s plane	$\Gamma_\omega, \Gamma_{bs}, \Gamma_{nt}$	Timoshenko shear terms
M_{st}	Bending moment in the s-t plane		
$\bar{Q}_{ij}$	Material stiffness coefficients of the k_{th} composite layer		
(t,n,b)	Cartesian frame		
(t,s,h)	Wall thickness frame		

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Appendix

Z and **S** matrices:

$$[Z] = \begin{bmatrix} A_{22} - \frac{A_{12}^2}{A_{11}} & A_{26} - \frac{A_{12}A_{16}}{A_{11}} & B_{22} - \frac{A_{12}B_{12}}{A_{11}} & B_{26} - \frac{A_{12}B_{16}}{A_{11}} & 0 \\ & A_{66} - \frac{A_{16}^2}{A_{11}} & B_{26} - \frac{A_{16}B_{12}}{A_{11}} & B_{66} - \frac{A_{16}B_{16}}{A_{11}} & 0 \\ & & D_{22} - \frac{B_{12}^2}{A_{11}} & D_{22} - \frac{B_{12}^2}{A_{11}} & 0 \\ & & & D_{66} - \frac{B_{16}^2}{A_{11}} & 0 \\ & & & & A_{55} \end{bmatrix} \quad (A1)$$

$$[S] = \begin{bmatrix} Z_{11} - \frac{Z_{12}^2}{Z_{22}} & Z_{13} - \frac{Z_{12}Z_{13}}{Z_{22}} & Z_{14} - \frac{Z_{12}Z_{24}}{Z_{22}} & \frac{Z_{12}}{Z_{22}} & 0 \\ S_{12} & Z_{33} - \frac{Z_{23}^2}{Z_{22}} & Z_{34} - \frac{Z_{23}Z_{24}}{Z_{22}} & \frac{Z_{23}}{Z_{22}} & 0 \\ S_{13} & S_{23} & Z_{44} - \frac{Z_{24}^2}{Z_{22}} & \frac{Z_{24}}{Z_{22}} & 0 \\ -S_{14} & -S_{24} & -S_{34} & \frac{1}{Z_{22}} & 0 \\ 0 & 0 & 0 & 0 & Z_{55} \end{bmatrix} \quad (A2)$$

Table A1. Analytical expressions for the coefficients of the **E** and **E^{mod}** matrices.

(i,j) term	Matrix E (Standard or “Shear-Uncoupled”)	Matrix E^{mod} (modified or “Shear-Coupled”)
(1,1)	$\oint Z_{11} ds$	$\oint S_{11} ds + \frac{(\oint S_{14} ds)^2}{\oint S_{44} ds}$
(1,2)	$\oint [-\bar{n}(s)Z_{11} - \sin \alpha(s)Z_{13}] ds$	$\oint [-\bar{n}(s)S_{11} - \sin \alpha(s)S_{12}] ds + \frac{\oint S_{14} ds \oint (-\bar{n}(s)S_{14} - \sin \alpha(s)S_{24}) ds}{\oint S_{44} ds}$
(1,3)	$\oint [\bar{b}(s)Z_{11} - \cos \alpha(s)Z_{13}] ds$	$\oint [\bar{b}(s)S_{11} - \cos \alpha(s)S_{12}] ds + \frac{\oint S_{14} ds \oint (\bar{b}(s)S_{14} - \cos \alpha(s)S_{24}) ds}{\oint S_{44} ds}$
(1,4)	$\oint \left[\frac{\psi(s)Z_{12}}{2} + Z_{14} \right] ds$	$\oint S_{13} ds + \frac{\oint S_{14} ds (A_e + \oint S_{34} ds)}{\oint S_{44} ds}$
(1,5)	$\oint [\omega_p(s)Z_{11} - q(s)Z_{13}] ds$	$\oint (\omega_p(s)S_{11} - q(s)S_{12}) ds + \frac{\oint S_{14} ds \oint (\omega_p(s)S_{14} - q(s)S_{24}) ds}{\oint S_{44} ds}$
(1,6)	$\oint \sin \alpha(s) Z_{12} ds$	$\frac{\oint S_{14} ds \oint \sin \alpha(s) ds}{\oint S_{44} ds}$
(1,7)	$\oint \cos \alpha(s) Z_{12} ds$	$\frac{\oint S_{14} ds \oint \cos \alpha(s) ds}{\oint S_{44} ds}$
(1,8)	$\oint \left[r(s) - \frac{\psi(s)}{2} \right] Z_{12} ds$	$\frac{A_e \oint S_{14} ds}{\oint S_{44} ds}$
(2,2)	$\oint [\bar{n}^2(s)Z_{11} + 2\bar{n}(s)\sin \alpha(s)Z_{13} + \sin^2 \alpha(s)Z_{33}] ds$	$\oint [\bar{n}^2(s)S_{11} + 2\bar{n}\sin \alpha(s)S_{12} + \sin^2 \alpha(s)S_{22}] ds + \frac{[\oint (-\bar{n}(s)S_{14} - \sin \alpha(s)S_{24}) ds]^2}{\oint S_{44} ds}$
(2,3)	$\oint [-\bar{b}(s)\bar{n}(s)Z_{11} + \bar{n}(s)\cos \alpha(s)Z_{13}] ds$ $+ \oint [-\bar{b}(s)\sin \alpha(s)Z_{13} + \cos \alpha(s)\sin \alpha(s)Z_{33}] ds$	$\oint [-\bar{b}(s)\bar{n}(s)S_{11} + \bar{n}(s)\cos \alpha(s)S_{12}] ds$ $+ \oint [-\bar{b}(s)\sin \alpha(s)S_{12} + \cos \alpha(s)\sin \alpha(s)S_{22}] ds$ $+ \frac{\oint (\bar{b}(s)S_{14} - \cos \alpha(s)S_{24}) ds \oint (-\bar{n}(s)S_{14} - \sin \alpha(s)S_{24}) ds}{\oint S_{44} ds}$
(2,4)	$\oint \left[-\frac{\bar{n}(s)\psi(s)Z_{12}}{2} - \bar{n}(s)Z_{14} \right] ds + \oint \left[-\frac{\psi(s)\sin \alpha(s)Z_{23}}{2} - \sin \alpha(s)Z_{34} \right] ds$	$\oint [-\bar{n}(s)S_{13} - \sin \alpha(s)S_{23}] ds + \frac{\oint (-\bar{n}(s)S_{14} - \sin \alpha(s)S_{24}) ds (\oint S_{34} ds + A_e)}{\oint S_{44} ds}$
(2,5)	$\oint [-\bar{n}(s)\omega_p(s)Z_{11} + \bar{n}(s)q(s)Z_{13}] ds + \oint [-\omega_p(s)\sin \alpha(s)Z_{13} + q(s)\sin \alpha(s)Z_{33}] ds$	$\oint [-\bar{n}(s)\omega_p(s)S_{11} + \bar{n}(s)q(s)S_{12}] ds + \oint [-\omega_p(s)\sin \alpha(s)S_{12} + q(s)\sin \alpha(s)S_{22}] ds$ $+ \frac{\oint (\omega_p(s)S_{14} - q(s)S_{24}) ds \oint (-\bar{n}(s)S_{14} - \sin \alpha(s)S_{24}) ds}{\oint S_{44} ds}$



$$\begin{aligned}
(2,6) \quad & \oint [-\bar{n}(s) \sin \alpha(s) Z_{12} - \sin^2 \alpha(s) Z_{23}] ds & \frac{\oint \sin \alpha(s) ds \oint (-\bar{n}(s) S_{14} - \sin \alpha(s) S_{24}) ds}{\oint S_{44} ds} \\
(2,7) \quad & \oint [-\bar{n}(s) \cos \alpha(s) Z_{12} - \cos \alpha(s) \sin \alpha(s) Z_{23}] ds & \frac{\oint \cos \alpha(s) ds \oint (-\bar{n}(s) S_{14} - \sin \alpha(s) S_{24}) ds}{\oint S_{44} ds} \\
(2,8) \quad & \oint \left[-\bar{n}(s) \left(r(s) - \frac{\psi(s)}{2} \right) Z_{12} \right] ds + \oint \left[-\left(r(s) - \frac{\psi(s)}{2} \right) \sin \alpha(s) Z_{23} \right] ds & \frac{A_e \oint (-\bar{n}(s) S_{14} - \sin \alpha(s) S_{24}) ds}{\oint S_{44} ds} \\
(3,3) \quad & \oint [\bar{b}^2(s) Z_{11} - 2\bar{b}(s) \cos \alpha(s) Z_{13} + \cos^2 \alpha(s) Z_{33}] ds & \oint [\bar{b}^2(s) S_{11} - 2\bar{b}(s) \cos \alpha(s) S_{12} + \cos^2 \alpha(s) S_{22}] ds + \frac{[\oint (\bar{b}(s) S_{14} - \cos \alpha(s) S_{24}) ds]^2}{\oint S_{44} ds} \\
(3,4) \quad & \oint \left[\frac{\bar{b}(s) \psi(s) Z_{12}}{2} + \bar{b}(s) Z_{14} \right] ds + \oint \left[-\frac{\psi(s) \cos \alpha(s) Z_{23}}{2} - \cos \alpha(s) Z_{34} \right] ds & \oint [\bar{b}(s) S_{13} - \cos \alpha(s) S_{23}] ds + \frac{\oint (\bar{b}(s) S_{14} - \cos \alpha(s) S_{24}) ds (\oint S_{34} ds + A_e)}{\oint S_{44} ds} \\
(3,5) \quad & \oint [\bar{b}(s) \omega_p(s) Z_{11} - \bar{b}(s) q(s) Z_{13}] ds + \oint [-\omega_p(s) \cos \alpha(s) Z_{13} + q(s) \cos \alpha(s) Z_{33}] ds & \oint [\bar{b}(s) \omega_p(s) S_{11} - \bar{b}(s) q(s) S_{12}] ds \\
& & + \oint [-\omega_p(s) \cos \alpha(s) S_{12} + q(s) \cos \alpha(s) S_{22}] ds \\
& & + \frac{\oint (\omega_p(s) S_{14} - q(s) S_{24}) ds \oint (\bar{b}(s) S_{14} - \cos \alpha(s) S_{24}) ds}{\oint S_{44} ds} \\
(3,6) \quad & \oint [\bar{b}(s) \sin \alpha(s) Z_{12} - \cos \alpha(s) \sin \alpha(s) Z_{23}] ds & \frac{\oint \sin \alpha(s) ds \oint (\bar{b}(s) S_{14} - \cos \alpha(s) S_{24}) ds}{\oint S_{44} ds} \\
(3,7) \quad & \oint [\bar{b}(s) \cos \alpha(s) Z_{12} - \cos^2 \alpha(s) Z_{23}] ds & \frac{\oint \cos \alpha(s) ds \oint (\bar{b}(s) S_{14} - \cos \alpha(s) S_{24}) ds}{\oint S_{44} ds} \\
(3,8) \quad & \oint \left[\bar{b}(s) \left(r(s) - \frac{\psi(s)}{2} \right) Z_{12} \right] ds + \oint \left[-\left(r(s) - \frac{\psi(s)}{2} \right) \cos \alpha(s) Z_{23} \right] ds & \frac{A_e \oint (\bar{b}(s) S_{14} - \cos \alpha(s) S_{24}) ds}{\oint S_{44} ds} \\
(4,4) \quad & \oint \left[\frac{\psi^2(s) Z_{22}}{4} + \psi(s) Z_{24} + Z_{44} \right] ds & \oint S_{33} ds + \frac{(A_e + \oint S_{34} ds)^2}{\oint S_{44} ds} \\
(4,5) \quad & \oint \left[\frac{\psi(s) \omega_p(s) Z_{12}}{2} + \omega_p(s) Z_{14} \right] ds + \oint \left[-\frac{q(s) \psi(s) Z_{23}}{2} - q(s) Z_{34} \right] ds & \oint [\omega_p(s) S_{13} - q(s) S_{23}] ds + \frac{\oint (\omega_p(s) S_{14} - q(s) S_{24}) ds (\oint S_{34} ds + A_e)}{\oint S_{44} ds} \\
(4,6) \quad & \oint \left[\frac{\psi(s) \sin \alpha(s) Z_{22}}{2} + \sin \alpha(s) Z_{24} \right] ds & \frac{\oint \sin \alpha(s) ds (\oint S_{34} ds + A_e)}{\oint S_{44} ds} \\
(4,7) \quad & \oint \left[\frac{\psi(s) \cos \alpha(s) Z_{22}}{2} + \cos \alpha(s) Z_{24} \right] ds & \frac{\oint \cos \alpha(s) ds (\oint S_{34} ds + A_e)}{\oint S_{44} ds} \\
(4,8) \quad & \oint \left[\frac{\psi(s)}{2} \left(r(s) - \frac{\psi(s)}{2} \right) Z_{22} \right] ds + \oint \left[\left(r(s) - \frac{\psi(s)}{2} \right) Z_{24} \right] ds & \frac{A_e (\oint S_{34} ds + A_e)}{\oint S_{44} ds} \\
(5,5) \quad & \oint [\omega_p^2(s) Z_{11} - 2q(s) \omega_p(s) Z_{13} + q^2(s) Z_{33}] ds & \oint [\omega_p^2(s) S_{11} - 2q(s) \omega_p(s) S_{12} + q^2(s) S_{22}] ds + \frac{(\oint (\omega_p(s) S_{14} - q(s) S_{24}) ds)^2}{\oint S_{44} ds} \\
(5,6) \quad & \oint [\omega_p(s) \sin \alpha(s) Z_{12} - q(s) \sin \alpha(s) Z_{23}] ds & \frac{\oint \sin \alpha(s) ds \oint (\omega_p(s) S_{14} - q(s) S_{24}) ds}{\oint S_{44} ds} \\
(5,7) \quad & \oint [\omega_p(s) \cos \alpha(s) Z_{12} - q(s) \cos \alpha(s) Z_{23}] ds & \frac{\oint \cos \alpha(s) ds \oint (\omega_p(s) S_{14} - q(s) S_{24}) ds}{\oint S_{44} ds} \\
(5,8) \quad & \oint \left[\omega_p(s) \left(r(s) - \frac{\psi(s)}{2} \right) Z_{12} \right] ds + \oint \left[-q(s) \left(r(s) - \frac{\psi(s)}{2} \right) Z_{23} \right] ds & \frac{A_e \oint (\omega_p(s) S_{14} - q(s) S_{24}) ds}{\oint S_{44} ds} \\
(6,6) \quad & \oint [\sin^2 \alpha(s) Z_{22} + \cos^2 \alpha(s) Z_{55}] ds & \oint \cos^2 \alpha(s) S_{55} ds + \frac{(\oint \sin \alpha(s) ds)^2}{\oint S_{44} ds} \\
(6,7) \quad & \oint [\cos \alpha(s) \sin \alpha(s) Z_{22} - \cos \alpha(s) \sin \alpha(s) Z_{55}] ds & \oint -\sin \alpha(s) \cos \alpha(s) S_{55} ds + \frac{\oint \cos \alpha(s) ds \oint \sin \alpha(s) ds}{\oint S_{44} ds} \\
(6,8) \quad & \oint \left[\sin \alpha(s) \left(r(s) - \frac{\psi(s)}{2} \right) Z_{22} + q(s) \cos \alpha(s) Z_{55} \right] ds & \oint q(s) \cos \alpha(s) S_{55} ds + \frac{A_e \oint \sin \alpha(s) ds}{\oint S_{44} ds} \\
(7,7) \quad & \oint [\cos^2 \alpha(s) Z_{22} + \sin^2 \alpha(s) Z_{55}] ds & \oint \sin^2 \alpha(s) S_{55} ds + \frac{(\oint \cos \alpha(s) ds)^2}{\oint S_{44} ds}
\end{aligned}$$



$$\begin{aligned}
 (7,8) \quad & \oint \left[\cos \alpha(s) \left(r(s) - \frac{\psi(s)}{2} \right) Z_{22} \right] ds + \oint [-q(s) \sin \alpha(s) Z_{55}] ds & \oint -q(s) \sin \alpha(s) S_{55} ds + \frac{A_e \oint \cos \alpha(s) ds}{\oint S_{44} ds} \\
 (8,8) \quad & \oint \left[\left(r(s) - \frac{\psi(s)}{2} \right)^2 Z_{22} + q^2(s) Z_{55} \right] ds & \oint q^2(s) S_{55} ds + \frac{A_e^2}{\oint S_{44} ds}
 \end{aligned}$$

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How to cite this article: Elizalde H., Cardenas D., Delgado-Gutierrez A., Probst O. In-Plane Shear-Axial Strain Coupling Formulation for Shear-Deformable Composite Thin-Walled Beams, *J. Appl. Comput. Mech.*, 7(2), 2021, 450–469. <https://doi.org/10.22055/JACM.2020.35066.2556>

