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Research Paper

Analytical Simulation for Transient Natural Convection in a Horizontal Cylindrical Concentric Annulus

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Abstract. In this study, a new scheme is suggested to find the analytical approximating solutions for a two-dimensional transient natural convection in a horizontal cylindrical concentric annulus bounded by two isothermal surfaces. The new methodology depends on combining the algorithms of Yang transform and the homotopy perturbation methods. Analytical solutions for the core, the outer layer and the inner layer at small times are found by a new method. Also, the effect of Grashof number, Prandtl number and radius proportion on the heat transfer and the flow of fluid (air) at different values was studied. Moreover, the study calculates the mean of the Nusselt number along with the effect of the Grashof number and radius proportion on it as parameters which acts as clues for heat transfer calculations of the natural convection for the annulus. The results, obtained by using the new method, prove that it is efficient and has high exactness compared to the other methods, used to find the analytical approximate solution for the transient natural convection in a horizontal cylindrical concentric annulus. The convergence of the new method was also discussed theoretically by referring to some theorems, and experientially by a verification of the solutions resulting from the simulations of the convergent condition. Furthermore, the graphs of the new solutions show the veracity, utility and exigency of the new method, and come in line with solutions offered by previous studies.

Keywords: Yang transform, Homotopy perturbation method, Natural convection, Cylinder annulus, Convergence analysis.

1. Introduction

The production of economic cooling and heating systems with high performances is a permanent anxiety in industry. The natural convection is one of the phenomena that happen in the production and fulfill the performance objectives required for those systems. In the last years, the problem of the natural convection heat transfer in a horizontal cylindrical concentric annulus has received a great deal of attention from many scientists and researchers. This great interest is due to its technological applications, like, refrigeration of the electronic constituent, thermal storage systems, nuclear reactors and the solar collectors, etc [4, 5, 17]. To predict the good performance of such systems and the importance of natural convection in their construction, many scientists and researchers in the fields of applied mathematics, physics and engineering have attempted to find solutions to the problem of the natural convection heat transfer in a horizontal cylindrical concentric annulus using various analytical and numerical methods. For example, Crawford and Lemlich [1] are the first scientists to find the numerical solution of the natural convection problem at the Prandtl number 0.7 and the diameter ratios are 2, 8 and 57 by using the Gauss-Seidel iterative method. Yang and Kong [2] used the smoothed particle hydrodynamics (SPH) method to find the numerical solution of the natural convection in a horizontal concentric annulus. Four different natural convection cases are detected, which are: the stable case with one plume (SP1), the unstable case with one plume (UP1), the stable case with multiple plumes (SPM), and unstable case with multiple plumes (UPM). The two one-plume cases are noted for all Prandtl numbers, whilst the two multiple-plume cases are noted only for $Pr = 0.1$ and 0.01 . The numerical results reveal that the vortex interactions resulted in exception flow states at $Pr = 0.1$, that is, the flow is unstable at $Ra = 10^4$ but stable at $Ra = 10^5$, whilst the flow is usually unstable at the high Rayleigh number but stable at the low Rayleigh number. A comparison with other methods described in the literature was made to validate this method and a good agreement was obtained. Pop et. al. [3] applied a matched asymptotic expansions method to find the analytical solution for transient natural convection in a horizontal concentric porous annulus. The solutions are obtained for the regions (core, the inner layer and the outer layer) to tiny times, and the present short time solution is found to be significantly different from the steady-state solution. Kuehn and Goldstein [4] obtained experimental and theoretical-numerical solutions for natural convection in the annulus between horizontal concentric cylinders. The experimental solutions were found by using a Mach-Zehnder to locate temperature dispensation and local heat-transfer coefficients, and the problem was solved numerically by using the finite difference method. The results of the numerical and experimental solutions are compared under the same conditions and a good agreement was obtained. Tsui and Tremblay [5] solved numerically the



problem of transient natural convection heat transfer between two horizontal isothermal cylinders by using the alternating direction implicit (ADI) method for the vorticity and energy equations and successive over-relaxation (SOR) method for the stream function equation. Their results are summarized by three Nusselt numbers vs. Grashof number curves with the diameter ratio as a parameter. Sano and Kuribayashi [6] applied the method of matched asymptotic expansions to find the analytical solution for the transient natural convection around a horizontal circular cylinder. They aimed to fill a gap in the past works by considering the displacement effect. Hassan and Al-Lateef [7] utilized a numerical method to find the solution for the two-dimensional transient natural convection heat transfer from the isothermal horizontal cylindrical annuli by using the alternating direction implicit (ADI) method, which offers solutions to both vorticity and energy equations. The results are recapitulated by Nusselt number vs. Grashof number curves with the diameter ratios and Prandtl as a parameter. Abdulateef [8] founded the numerical solution for the transient two-dimensional natural convection in the horizontal isothermal cylindrical annuli. From the results obtained, he noted that the diameter ratio and Grashof number have a lot of influence on the flow and heat transfer characteristics whereas the Prandtl number has no significant effect. Mack and Bishop [9] employed an analytical method to find the solutions for natural convection between two horizontal concentric cylinders. The solutions are acquired from the first three terms in the power series of the Rayleigh number. Yang et. al. [10] used the lattice Boltzmann method to emulate the stability and transitions of natural convection in a horizontal annulus at the Prandtl number, which varies from 0.1 to 0.7 and the Rayleigh number that ranges between 10^3 to 5.0×10^5 . The results show that the critical Rayleigh number increases with the decrease of the aspect ratio and with the increase in the Prandtl number. Bhowmik et. al. [11] analyzed the power-on transient natural convection heat transfer around a horizontal cylinder, experimentally heated in the air. The results prove that the transient heat transfer around the cylinder is highly affected by the position of thermocouples. Further, the test rig working condition is verified by comparing the results with those available in the literature. Khudhayer et. al. [12] applied the finite element method to solve the natural convection in a porous horizontal cylindrical annulus. The test of the mesh is made to ensure that the mesh size does not impact the results. The results are presented in terms of the average Nusselt number (Nu), isothermal lines and streamlines. Mohamad et. al. [13] analyzed numerically the unsteady state and the natural convection in the annular cylinders. The major aim of their study was to establish correlations for the rate of the heat transfer as a function of time and other controlling parameters. Mahmood et. al. [14] studied numerically the natural convection heat transfer between two isothermal concentric vertical cylinders inserted into a porous medium. The results showed that the average Nusselt number was a function of the varied Rayleigh number and non-dimensional parameter of ratio. Additionally, the results showed an increase in the natural convection as the increase in the heat flux leads to an improvement in the heat transfer process and the average Nusselt number increases when the Rayleigh number and radius ratio increase. Zubkov and Narygin [15] used the control volume method and the SIMPLER algorithm to find a solution to the natural convection of a viscous fluid, flowing between two concentric cylinders. They found out that the kinetic energy of the cylinder depends largely on the radius and the thermal conductivity. Besides, they discovered that the radius does not depend on the volume of the area atop which the temperature is maintained, or on the site of these areas. Wei et. al. [16] applied a thermal immersed boundary-lattice Boltzmann method to study bifurcation and find dual solutions to the natural convection in a horizontal annulus. The results showed the existence of three convection patterns in a horizontal annulus with a rotating inner cylinder which affects the heat transfer in different ways, and the linear speed locates the ratio of each convection. Medebber et. al. [17] utilized a finite volume method to solve the free convection in a partially vertical cylinder annulus. The solutions were obtained for Rayleigh numbers of 10^3 to 10^4 , height ratios 0.5 and Prandtl numbers 7. The impact of physical and geometrical parameters on the velocity fields, average Nusselt and isotherms has been numerically studied. Touzani et. al. [18] studied numerically the natural convection in a horizontal annulus with two heating blocks. They observed that the heat flux and transfer rate calculated per region (top, midst and down) show that heat transfer is more important for the annulus upper region, and they found that the existence of the block contributes to the overall heat transfer improvement. Hu et al. [19] utilized the immersed boundary-lattice Boltzmann method to calculate the natural convection in a concentric horizontal annulus that has a constant flux wall. They concentrate on the effects of the Prandtl and Rayleigh numbers on the three types of the steady-state solutions of flow patterns, which were discussed. Yuan et. al. [20] analyzed the natural convection in horizontal concentric annuli of various inner forms and examined influence of the variation in the inner form on the heat transfer and flow properties. The results of their model prove that the radiation plays an important role in the overall heat transfer behavior in the natural convection at high temperature levels. Mehrizi et. al. [21] used the lattice Boltzmann method to study the impact of nanoparticles on natural convection heat transfer in the horizontal annulus. The influence of nanoparticle volume that enhances the heat transfer at various Rayleigh numbers and the influence of horizontal, diagonal and vertical eccentricities at different locations at $Ra = 10^4$ were examined. The results in the form of local and average Nusselt numbers, isotherms and streamlines were reported. These results demonstrated that the maximum stream functions and the Nusselt number increase with the increase in the part solid volume fraction. Kumar [22] studied the natural convection of gases in a horizontal annulus numerically. The results of velocity, heat transfer and temperature were offered for diameter ratios from 1.2-10 and a wide range of Rayleigh numbers that extends from conduction to the convection-dominated steady flow system.

Despite the advantages of the methods mentioned above in simulating the transient natural convection heat transfer problems, they come with a few disadvantages. For example, some of these methods require high iterations, a great deal of time, limitations and effort to obtain an accurate solution for the problem under consideration.

What was presented above reflects the importance of studying the problem of the natural convection by researchers in different mediums and treating it with different simulation methods, in addition to confirming that it is still an open problem to study. It is also possible to describe multiple phenomena in the natural convection, and this is what motivates researchers to continue to study it and update the results of their studies. Based on this information and to the best of our knowledge, the merging process between analytical (exact) solution methods and approximate solution methods may alleviate or reduce many of the limitations that accompany tackling each method alone. This reason motivates us to merge the two methods in this study. One of which is an analytical called the Yang transform (YT), which is a new integral transform proposed in 2016, by Xiao-Jun Yang [23]. It was first applied to the heat transfer equation in the steady-state. Note that this method is precise and efficacious in finding the exact solutions to linear differential equations, and it is used by many researchers to solve different problems [23, 24]. Secondly, the homotopy perturbation method is semi-analytical (approximate) method, and depends on a small perturbation parameter. This method offers a new form to the homotopy perturbation method (HPM). It was discovered in (2010) by Aminikhah and Hemmatnezhad [25], while they were trying to find an analytical approximate solution to partial and ordinary differential equations. In this method, the solution is assumed to be an infinite series that converges readily with the exact solution. Many researchers used this method to solve various equations [26,27] and they noted that it's a powerful, effective, easy and accurate



tool to solve linear and nonlinear differential equations, compared to the standard homotopy perturbation method. This combination results in a new method named as the Yang transform- homotopy perturbation method (YTHPM).

So, this work offers many innovative ideas; firstly, it builds and presents a sophisticated new method (YTHPM) to handle the two-dimensional transient natural convection in a horizontal cylindrical concentric annulus bounded by two isothermal surfaces analytically. Also, we divided the problem into the core region, outer boundary layer (adjacent to the outer cylinder) and the inner boundary layer (adjacent to the inner cylinder) and found the analytical solution for each region. As such, we arrived at new analytical solutions. Secondly, this work introduced some theorems to analyze and proofs to validate the convergence of the new method (YTHPM) theoretically and experientially (simulation). Thirdly, the new method can be as an improvement to the HPM [25]. Moreover, the advantage of the new method is that all the results are obtained through the 1st iteration. Furthermore, the numerical results, obtained by using the new method, prove the efficiency, effectiveness, and high accuracy of this method and also its results agree well with those reported by other researchers in the field.

2. The YTHPM Algorithm

The basic idea of YTHPM depends on the algorithms of YT and HPM, which will be discussed in this section.

For clarifying the fundamental ideas of these methods (YT, HPM & YTHPM), we will contemplate the general nonlinear equation in the form of differential operators as:

$$A_1(v) - q(\vartheta) = 0 \quad , \quad \vartheta \in \Omega_1 \quad (1)$$

together with the conditions of the boundary:

$$B_1\left(v, \frac{\partial v}{\partial n}\right) = 0 \quad , \quad \vartheta \in \Gamma_1 \quad (2)$$

where B_1 is the operator of the boundary; $q(\vartheta)$ is a function which is known and Γ_1 is the boundary of the domain Ω_1 . A_1 is the differential operator which can be divided into two parts N_1 nonlinear and L_1 , R_1 are the linear operators. So, Equation (1) can be written as follows;

$$L_1(v) + R_1(v) + N_1(v) - q(\vartheta) = 0 \quad (3)$$

Now, to illustrate the algorithm of **YT**, which is defined in relation to the function $g(t)$ and denoted by $Y\{g(t)\}$ or $T(s)$ as follows:

$$Y\{g(t)\} = T(s) = \int_0^\infty e^{-\frac{t}{s}} g(t) dt \quad (4a)$$

If we substitute, $x = t / s$, then equation (4a) becomes:

$$Y\{g(t)\} = T(s) = s \int_0^\infty e^{-x} g(sx) dx \quad (4b)$$

Then, the application of YT for the linear parts of Equation (3) with initial condition $v(0)$ and if we assume that $L_1 = \partial / \partial t$, and take the Yang transform for both sides of Equation (3), we have:

$$Y(L_1(v) + R_1(v)) = Y(q(\vartheta)) ,$$

From the derivative property of YT [23], we get:

$$\frac{1}{s} Y(v) - v(0) = Y(q(\vartheta) - R_1(v)) ,$$

The rearrangement of the above equation yields:

$$Y(v) = s \left(Y\{q(\vartheta) - R_1(v)\} + v(0) \right) , \quad (5a)$$

By taking the inverse Yang transform for both sides of Equation (5a), then we have the solution in the form:

$$v = Y^{-1} \left[s \left(Y\{q(\vartheta) - R_1(v)\} + v(0) \right) \right] , \quad (5b)$$

Now, the demonstration of the algorithm of **HPM** [25,26] is as follows:

By the technique of the homotopy, we build a homotopy $u(\vartheta, \beta): \Omega_1 \times [0, 1] \rightarrow \mathbb{R}$, which achieves:



$$H_1(u, \beta) = (1 - \beta)[L_1(u) - v_0^*] + \beta[A_1(u) - q(\vartheta)] = 0, \quad \beta \in [0, 1], \vartheta \in \Omega_1, \quad (6)$$

or

$$H_1(u, \beta) = L_1(u) - v_0^* + \beta[v_0^* + N_1(u) + R_1(u) - q(\vartheta)] = 0, \quad (7)$$

where $\beta \in [0, 1]$ is an embedding parameter, and v_0^* is an initial solution of Equation (3). Clearly, from Equations (6) and (7), we have:

$$H_1(u, 0) = L_1(u) - v_0^* = 0, \quad (8)$$

$$H_1(u, 1) = L_1(u) + N_1(u) + R_1(u) - q(\vartheta) = 0. \quad (9)$$

Let's suppose the solutions of Equations (6) and (7) as a force chain in β to be as follows:

$$u = \sum_{n=0}^{\infty} \beta^n u_n \quad (10)$$

Now, we rewrite Equation (7) in the following form:

$$L_1(u) - v_0^* + \beta[N_1(u) + R_1(u) - q(\vartheta) + v_0^*] = 0 \quad (11)$$

By taking L_1^{-1} to both sides of Equation (11), we get:

$$u = L_1^{-1}(v_0^*) + \beta[L_1^{-1}(q(\vartheta)) - L_1^{-1}(N_1(u) + R_1(u)) - L_1^{-1}(v_0^*)] \quad (12)$$

Let's postulate the initial approximation of Equation (3) as follows:

$$v_0^* = \sum_{n=0}^{\infty} a_n p_n \quad (13)$$

where $a_0, a_1, a_2, \dots$ are the coefficients which are unknown and $p_0, p_1, p_2, \dots$ are special functions rely on the problem. Through putting Equations (10) and (13) into Equation (12), we obtain:

$$\sum_{n=0}^{\infty} \beta^n u_n = L_1^{-1}\left(\sum_{n=0}^{\infty} a_n p_n\right) + \beta\left[L_1^{-1}(q(\vartheta)) - L_1^{-1}\left(N_1\left(\sum_{n=0}^{\infty} \beta^n u_n\right) + R_1\left(\sum_{n=0}^{\infty} \beta^n u_n\right)\right) - L_1^{-1}\left(\sum_{n=0}^{\infty} a_n p_n\right)\right] \quad (14)$$

Comparing the coefficients which have the same powers of β leads to:

$$\begin{aligned} \beta^0 : u_0 &= L_1^{-1}\left(\sum_{n=0}^{\infty} a_n p_n\right) \\ \beta^1 : u_1 &= L_1^{-1}(q(\vartheta)) - L_1^{-1}\left(\sum_{n=0}^{\infty} a_n p_n\right) - L_1^{-1}(N_1(u_0) + R_1(u_0)) \\ \beta^2 : u_2 &= -L_1^{-1}(N_1(u_0, u_1) + R_1(u_0, u_1)) \\ &\vdots \\ \beta^j : u_j &= -L_1^{-1}(N_1(u_0, u_1, \dots, u_{j-1}) + R_1(u_0, u_1, \dots, u_{j-1})) \\ &\vdots \end{aligned} \quad (15)$$

If we assume that $u_1 = 0$, then Equation (15) results in $u_2 = u_3 = \dots = 0$. Then, the exact solution can be found as follows:

$$v = u_0 = L^{-1}\left(\sum_{n=0}^{\infty} a_n p_n\right) \quad (16)$$

Therefore, the fundamental notion of the new technique (**YTHPM**) for Equation (3) is summarized in the following steps:

Step1: By the HPM, we have:

$$L_1(u) - v_0^* + \beta[v_0^* + N_1(u) + R_1(u) - q(\vartheta)] = 0 \quad (17)$$



Step2: Taking the Yang transform for both sides of Equation (17), we get:

$$Y(L_1(u) - v_0^* + \beta(v_0^*) + \beta[N_1(u) + R_1(u) - q(\vartheta)]) = 0 \quad (18)$$

Step3: If we postulate that $L_1 = \partial / \partial t$, then using the differential property of Yang transform, we obtain:

$$\frac{1}{s} Y(u) - u(0) - Y(v_0^*) + Y(\beta(v_0^*)) + Y(\beta[N_1(u) + R_1(u) - q(\vartheta)]) = 0 \quad (19)$$

$$Y(u) = s u(0) + s Y(v_0^*) - s Y(\beta(v_0^*)) - s Y(\beta[N_1(u) + R_1(u) - q(\vartheta)]) \quad (20)$$

Step4: When we take the Yang inverse to both sides for Equation (20), we get:

$$u = Y^{-1}(s u(0)) + Y^{-1}(s Y(v_0^*)) - Y^{-1}(s Y(\beta(v_0^*))) - Y^{-1}[s Y(\beta[N_1(u) + R_1(u) - q(\vartheta)])] \quad (21)$$

Step5: Via the NHPM, let's suppose that $u = \sum_{n=0}^{\infty} \beta^n u_n$, $v_0^* = \sum_{n=0}^{\infty} a_n p_n$, $u(0) = v(0)$, then, Equation (21) becomes:

$$\sum_{n=0}^{\infty} \beta^n u_n = Y^{-1}(s u(0)) + Y^{-1}\left[s Y\left(\sum_{n=0}^{\infty} a_n p_n\right)\right] - Y^{-1}\left[s Y\left(\beta\left(\sum_{n=0}^{\infty} a_n p_n\right)\right)\right] - Y^{-1}\left[s Y\left(\beta\left[N_1\left(\sum_{n=0}^{\infty} \beta^n u_n\right) + R_1\left(\sum_{n=0}^{\infty} \beta^n u_n\right) - q(\vartheta)\right]\right)\right] \quad (22)$$

Step6: By equalizing the terms that have the same power of β , we have:

$$\begin{aligned} \beta^0 : u_0 &= Y^{-1}(s u(0)) + Y^{-1}\left[s Y\left(\sum_{n=0}^{\infty} a_n p_n\right)\right] \\ \beta^1 : u_1 &= -Y^{-1}\left[s Y\left(\sum_{n=0}^{\infty} a_n p_n\right)\right] - Y^{-1}[s Y(N_1(u_0) + R_1(u_0) - q(\vartheta))] \\ \beta^2 : u_2 &= -Y^{-1}[s Y(N_1(u_0, u_1) + R_1(u_0, u_1))] \\ &\vdots \end{aligned} \quad (23)$$

Step7: The analytical approximate solution can be found by putting $\beta = 1$,

$$v = \lim_{\beta \rightarrow 1} u = u_0 + u_1 + u_2 + \dots$$

3. Governing Equations

Let's consider Newtonian fluid in two concentric horizontal cylinders bounded by two isothermal surfaces. Then we assume that the fluid is viscous and incompressible, as well as the movement of the fluid and the temperature distribution are two-dimensional, and the fricative heating is slight, as shown in Figure (1). Therefore, the governing equations can be written in the Boussinesq approximation as follows [5]:

$$\frac{\partial \ddot{u}}{\partial \ddot{x}} + \frac{\partial \ddot{v}}{\partial \ddot{y}} = 0 \quad (24a)$$

$$\frac{\partial \ddot{u}}{\partial \ddot{t}} + \ddot{u} \frac{\partial \ddot{u}}{\partial \ddot{x}} + \ddot{v} \frac{\partial \ddot{u}}{\partial \ddot{y}} = \left(-\frac{1}{\rho}\right) \frac{\partial \ddot{P}}{\partial \ddot{x}} + v \left(\frac{\partial^2 \ddot{u}}{\partial \ddot{x}^2} + \frac{\partial^2 \ddot{u}}{\partial \ddot{y}^2}\right) \quad (24b)$$

$$\frac{\partial \ddot{v}}{\partial \ddot{t}} + \ddot{u} \frac{\partial \ddot{v}}{\partial \ddot{x}} + \ddot{v} \frac{\partial \ddot{v}}{\partial \ddot{y}} = \left(-\frac{1}{\rho}\right) \frac{\partial \ddot{P}}{\partial \ddot{y}} + \ddot{g} \ddot{\alpha} (\ddot{T} - \ddot{T}_0) + v \left(\frac{\partial^2 \ddot{v}}{\partial \ddot{x}^2} + \frac{\partial^2 \ddot{v}}{\partial \ddot{y}^2}\right) \quad (24c)$$

$$\frac{\partial \ddot{T}}{\partial \ddot{t}} + \ddot{u} \frac{\partial \ddot{T}}{\partial \ddot{x}} + \ddot{v} \frac{\partial \ddot{T}}{\partial \ddot{y}} = \ddot{k} \left(\frac{\partial^2 \ddot{T}}{\partial \ddot{x}^2} + \frac{\partial^2 \ddot{T}}{\partial \ddot{y}^2}\right) \quad (24d)$$

where; $\ddot{u}$, $\ddot{v}$ are the velocity components in $\ddot{x}$, $\ddot{y}$ directions; $\ddot{T}$ is temperature, v kinematic viscosity, ρ density, $\ddot{\alpha}$ thermic dilation coefficient, $\ddot{g}$ gravitational acceleration and $\ddot{k}$ thermal diffusivity.



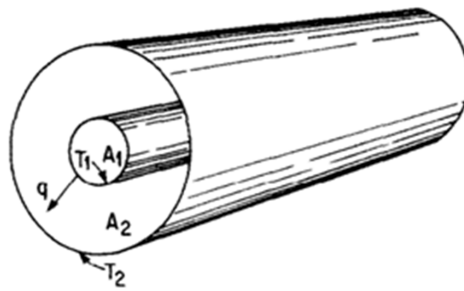


Fig. 1. The transient natural convection in an air-filled annulus bounded by two isothermal surfaces

where $T_1 = T_i$, $T_2 = T_o$ are the inner and outer cylinder temperatures; $A_1 = R_i$, $A_2 = R_o$ are the proportion of inner and outer radii to the hiatus and q is the heat flux.

Now to facilitate the solution, we can write these equations in the form of the vorticity-stream formulation. This formula can be obtained by differentiation Equation (24b) with respect to $\tilde{y}$ and Equation (24c) to $\tilde{x}$. Then Equation (24b) is subtracted from Equation (24c) and by relying on the definition of vorticity ($\eta = v_x - u_y$), the pressure is eliminated. Also, we alternated the Cartesian coordinates system to the polar coordinates system. And by introducing the dimensionless set as the following:

$$u = \frac{\tilde{u}\tilde{L}}{v}, \quad v = \frac{\tilde{v}\tilde{L}}{v}, \quad \Psi = \frac{\tilde{\Psi}}{v}, \quad r = \frac{\tilde{r}}{\tilde{L}}, \quad t = \frac{\tilde{t}v}{\tilde{L}^2}, \quad \eta = \frac{\tilde{\eta}}{v}, \quad T = \frac{\tilde{T} - \tilde{T}_o}{\tilde{T}_h - \tilde{T}_c}, \quad \nabla^2 = \tilde{L}^2 \tilde{\nabla}^2$$

Therefore; the dimensionless equations in the stream- vorticity formula in the polar coordinate take on the following form:

$$\eta_t + u\eta_r + \frac{v}{r}\eta_\theta = Gr \left(\cos(\theta)T_r - \frac{\sin(\theta)}{r}T_\theta \right) + \nabla^2 \eta \quad (25a)$$

$$T_t + uT_r + \frac{v}{r}T_\theta = \frac{1}{Pr} \nabla^2 T \quad (25b)$$

$$\eta = -\nabla^2 \Psi \quad (25c)$$

$$v = -\Psi_r, u = \frac{1}{r}\Psi_\theta \quad (25d)$$

with initial and boundary conditions respectively as:

Everywhere at $t = 0 \rightarrow \eta = \Psi = T = 0$,

$$\left. \begin{aligned} \Psi = \Psi_r = \frac{1}{r}\Psi_\theta = 0, \quad \forall r = R_i, r = R_o \\ r = R_i \rightarrow T = 1 \\ r = R_o \rightarrow T = 0 \end{aligned} \right\} \quad (26)$$

where R_i is the proportion of inner radius to the hiatus, R_o the proportion of outer radius to the hiatus, $\tilde{L}$ annulus hiatus, Gr Grashof number, Pr Prandtl number, r the dimensionless radial coordinate, t time, T temperature, u radial velocity, v tangent velocity, θ polar coordinate, Ψ stream function, η vorticity function, while the subscripts h, c, o refer to spicy, cool and ambient, respectively and i, o inner, outer, respectively.

4. Application of YTHPM

Now, to apply the algorithm of YTHPM to the governing Equations (25a-c), first, we combine Equations (25 c, d) and Equations (25 a, b) to obtain the following equations:

$$\left. \begin{aligned} (-\nabla^2 \Psi)_t = Gr \left(\cos(\theta)T_r - \frac{\sin(\theta)}{r}T_\theta \right) + \nabla^2 (-\nabla^2 \Psi) + \frac{1}{r} \left(\Psi_\theta \frac{\partial}{\partial r} (-\nabla^2 \Psi) + \Psi_r \frac{\partial}{\partial \theta} (-\nabla^2 \Psi) \right) \\ T_t = \frac{1}{Pr} \nabla^2 T - \frac{1}{r} (\Psi_\theta T_r + \Psi_r T_\theta) \end{aligned} \right\} \quad (27)$$

Then, the basic steps of the new technique for Equations (27) are illustrated as follows:



Step 1: By using HPM, we obtain:

$$(-\nabla^2 \Psi)_t - \Psi_0^* + \beta \Psi_0^* - \beta \left[Gr \left(\cos(\theta) T_r - \frac{\sin(\theta)}{r} T_\theta \right) + \nabla^2 (-\nabla^2 \Psi) \right] + \frac{1}{r} \left[\Psi_\theta \frac{\partial}{\partial r} (-\nabla^2 \Psi) + \Psi_r \frac{\partial}{\partial \theta} (-\nabla^2 \Psi) \right] = 0$$

$$T_t - T_0^* + \beta T_0^* - \beta \left[\frac{1}{Pr} \nabla^2 T - \frac{1}{r} (\Psi_\theta T_r + \Psi_r T_\theta) \right] = 0$$

Step 2: Taking the Yang transform for both sides of the above equation in step (1), we have:

$$\frac{1}{s} \tilde{\Psi} - \Psi(0) - Y(\Psi_0^*) + Y(\beta \Psi_0^*) - Y \left[\beta \left[Gr \left(\cos(\theta) T_r - \frac{\sin(\theta)}{r} T_\theta \right) + \nabla^2 (-\nabla^2 \Psi) \right] + \frac{1}{r} \left[\Psi_\theta \frac{\partial}{\partial r} (-\nabla^2 \Psi) + \Psi_r \frac{\partial}{\partial \theta} (-\nabla^2 \Psi) \right] \right] = 0$$

$$\frac{1}{s} \tilde{T} - T(0) - Y(T_0^*) + Y(\beta T_0^*) - Y \left[\beta \left[\frac{1}{Pr} \nabla^2 T - \frac{1}{r} (\Psi_\theta T_r + \Psi_r T_\theta) \right] \right] = 0$$

where $\Psi_0^* = \Psi(0)$ and $T_0^* = T(0)$ are the initial conditions. The rearrangement of the above equations yields:

$$\left. \begin{aligned} \tilde{\Psi} &= s\Psi(0) + sY(\Psi_0^*) - sY(\beta \Psi_0^*) \\ &+ sY \left[\beta \left[Gr \left(\cos(\theta) T_r - \frac{\sin(\theta)}{r} T_\theta \right) + \nabla^2 (-\nabla^2 \Psi) + \frac{1}{r} \left(\Psi_\theta \frac{\partial}{\partial r} (-\nabla^2 \Psi) + \Psi_r \frac{\partial}{\partial \theta} (-\nabla^2 \Psi) \right) \right] \right] \\ \tilde{T} &= sT(0) + sY(T_0^*) - sY(\beta T_0^*) + sY \left[\beta \left[\frac{1}{Pr} \nabla^2 T - \frac{1}{r} (\Psi_\theta T_r + \Psi_r T_\theta) \right] \right] \end{aligned} \right\} \quad (28)$$

Step 3: Taking the inverse YT for both sides of Equation (28), we get:

$$\begin{aligned} \Psi &= Y^{-1}(s\Psi(0)) + Y^{-1}(sY(\Psi_0^*)) - Y^{-1}(sY(\beta \Psi_0^*)) \\ &+ Y^{-1} \left[sY \left[\beta \left[Gr \left(\cos(\theta) T_r - \frac{\sin(\theta)}{r} T_\theta \right) + \nabla^2 (-\nabla^2 \Psi) + \frac{1}{r} \left(\Psi_\theta \frac{\partial}{\partial r} (-\nabla^2 \Psi) + \Psi_r \frac{\partial}{\partial \theta} (-\nabla^2 \Psi) \right) \right] \right] \right] \\ T &= Y^{-1}(sT(0)) + Y^{-1}(sY(T_0^*)) - Y^{-1}(sY(\beta T_0^*)) + Y^{-1} \left[sY \left[\beta \left[\frac{1}{Pr} \nabla^2 T - \frac{1}{r} (\Psi_\theta T_r + \Psi_r T_\theta) \right] \right] \right] \end{aligned}$$

Step 4: From the assumption of the HPM that puts $\Psi = \Psi_0 + \beta \Psi_1 + \beta^2 \Psi_2 + \dots$, and $T = T_0 + \beta T_1 + \beta^2 T_2 + \dots$, we have:

$$\begin{aligned} \Psi_0 + \beta \Psi_1 + \beta^2 \Psi_2 + \dots &= Y^{-1}(s\Psi(0)) + Y^{-1}(sY(\Psi_0^*)) - Y^{-1}(sY(\beta \Psi_0^*)) \\ &+ Y^{-1} \left[sY \left[\beta \left[Gr \left(\cos(\theta) (T_0 + \beta T_1 + \beta^2 T_2 + \dots)_r - \frac{\sin(\theta)}{r} (T_0 + \beta T_1 + \beta^2 T_2 + \dots)_\theta \right) \right. \right. \right. \\ &\quad \left. \left. + \nabla^2 (-\nabla^2 (\Psi_0 + \beta \Psi_1 + \beta^2 \Psi_2 + \dots)) \right. \right. \\ &\quad \left. \left. + \frac{1}{r} \left((\Psi_0 + \beta \Psi_1 + \beta^2 \Psi_2 + \dots)_\theta \frac{\partial}{\partial r} (-\nabla^2 (\Psi_0 + \beta \Psi_1 + \beta^2 \Psi_2 + \dots)) + \right. \right. \right. \\ &\quad \left. \left. \left. (\Psi_0 + \beta \Psi_1 + \beta^2 \Psi_2 + \dots)_r \frac{\partial}{\partial \theta} (-\nabla^2 (\Psi_0 + \beta \Psi_1 + \beta^2 \Psi_2 + \dots)) \right) \right] \right] \right] \\ T_0 + \beta T_1 + \beta^2 T_2 + \dots &= Y^{-1}(sT(0)) + Y^{-1}(sY(T_0^*)) - Y^{-1}(sY(\beta T_0^*)) \\ &+ Y^{-1} \left[sY \left[\beta \left[\frac{1}{Pr} \nabla^2 (T_0 + \beta T_1 + \beta^2 T_2 + \dots) \right. \right. \right. \\ &\quad \left. \left. - \frac{1}{r} ((\Psi_0 + \beta \Psi_1 + \beta^2 \Psi_2 + \dots)_\theta (T_0 + \beta T_1 + \beta^2 T_2 + \dots)_r + \right. \right. \\ &\quad \left. \left. (\Psi_0 + \beta \Psi_1 + \beta^2 \Psi_2 + \dots)_r (T_0 + \beta T_1 + \beta^2 T_2 + \dots)_\theta) \right] \right] \right] \end{aligned}$$

Step5: By equalizing the terms that have the same power of β , we possess:



$$\begin{aligned}
\beta^0 : & \begin{cases} \Psi_0 = Y^{-1}(s \Psi(0)) + Y^{-1}(s Y(\Psi_0^*)) \\ T_0 = Y^{-1}(s T(0)) + Y^{-1}(s Y(T_0^*)) \end{cases} \\
\beta^1 : & \begin{cases} \Psi_1 = -Y^{-1}(s Y(\Psi_0^*)) + Y^{-1} \left(s Y \left(\text{Gr} \left(\cos(\theta)(T_0)_r - \frac{\sin(\theta)}{r}(T_0)_\theta \right) + \nabla^2(-\nabla^2(\Psi_0)) \right. \right. \\ \left. \left. + \frac{1}{r} \left((\Psi_0)_\theta \frac{\partial}{\partial r}(-\nabla^2(\Psi_0)) + (\Psi_0)_r \frac{\partial}{\partial \theta}(-\nabla^2(\Psi_0)) \right) \right) \right) \\ T_1 = -Y^{-1}(s Y(T_0^*)) + Y^{-1} \left(s Y \left(\frac{1}{\text{Pr}} \nabla^2(T_0) - \frac{1}{r} ((\Psi_0)_\theta (T_0)_r + (\Psi_0)_r (T_0)_\theta) \right) \right) \end{cases} \\
\beta^2 : & \begin{cases} \Psi_2 = Y^{-1} \left(s Y \left(\text{Gr} \left(\cos(\theta)(T_1)_r - \frac{\sin(\theta)}{r}(T_1)_\theta \right) + \nabla^2(-\nabla^2(\Psi_1)) \right. \right. \\ \left. \left. + \frac{1}{r} \left((\Psi_1)_\theta \frac{\partial}{\partial r}(-\nabla^2(\Psi_0)) + (\Psi_0)_\theta \frac{\partial}{\partial r}(-\nabla^2(\Psi_1)) + (\Psi_0)_r \frac{\partial}{\partial \theta}(-\nabla^2(\Psi_1)) + (\Psi_1)_r \frac{\partial}{\partial \theta}(-\nabla^2(\Psi_0)) \right) \right) \right) \\ T_1 = Y^{-1} \left(s Y \left(\frac{1}{\text{Pr}} \nabla^2(T_1) - \frac{1}{r} ((\Psi_0)_\theta (T_1)_r + (\Psi_1)_\theta (T_0)_r + (\Psi_0)_r (T_1)_\theta + (\Psi_1)_r (T_0)_\theta) \right) \right) \end{cases} \\
& \vdots
\end{aligned}$$

Step6: The analytical approximate solution can be acquired by putting $\beta = 1$,

$$\begin{aligned}
\Psi &= \lim_{\beta \rightarrow 1} \Psi = \Psi_0 + \Psi_1 + \Psi_2 + \dots \\
T &= \lim_{\beta \rightarrow 1} T = T_0 + T_1 + T_2 + \dots
\end{aligned}$$

Now, to find the analytical solution for Equations (27), YTHPM is used and as follows:

Firstly we divided the problem into inner and outer and core regions. Then, from the initial and boundary conditions (26) in addition to applying the steps of YTHPM that are illustrated above, the analytical solutions for each regions are as follows:

$$\begin{aligned}
\Psi_0^o &= 2 T_o \sqrt{t} \sin(\theta) \left(\zeta \operatorname{erfc}(\zeta) - \frac{1}{\sqrt{\pi}} e^{-\zeta^2} + \frac{1}{\sqrt{\pi}} \right) + t \left(2 T_o \sqrt{t} \sin(\theta) \left(\zeta \operatorname{erfc}(\zeta) - \frac{1}{\sqrt{\pi}} e^{-\zeta^2} + \frac{1}{\sqrt{\pi}} \right) \right), \\
T_0^o &= T_o \operatorname{erfc}(\zeta) + t (T_o \operatorname{erfc}(\zeta)), \\
\Psi_0^i &= 2 T_i \sqrt{t} \sin(\theta) \left(-\xi \operatorname{erfc}(\xi) + \frac{1}{\sqrt{\pi}} e^{-\xi^2} - \frac{1}{\sqrt{\pi}} \right) + t \left(2 T_i \sqrt{t} \sin(\theta) \left(-\xi \operatorname{erfc}(\xi) + \frac{1}{\sqrt{\pi}} e^{-\xi^2} - \frac{1}{\sqrt{\pi}} \right) \right), \\
T_0^i &= T_i \operatorname{erfc}(\xi) + t (T_i \operatorname{erfc}(\xi)), \\
\Psi_0^c &= \frac{2\sqrt{t}}{(R^2 - 1)} \sin(\theta) ((T_i + R T_o) r - R (T_i + R T_o) r^{-1}) + t \left(\frac{2\sqrt{t}}{(R^2 - 1)} \sin(\theta) ((T_i + R T_o) r - R (T_i + R T_o) r^{-1}) \right), \\
T_0^c &= 0
\end{aligned}$$

where; $\zeta = (R - r) / 2\sqrt{t}$ and $\xi = (r - 1) / 2\sqrt{t}$. Ψ_0^o , Ψ_0^i , Ψ_0^c and T_0^o , T_0^i , T_0^c are the initial conditions in the inner, outer and core regions respectively. T_o , T_i are the outer, inner cylinder temperatures respectively, and R is the radius proportion. Then;

$$\begin{aligned}
\Psi_1^o &= \frac{1}{4\sqrt{\pi} r^4 t^{\frac{3}{2}}} \left(T_o \left[24 t^3 \sin(\theta) + r^6 e^{-\zeta^2} \sin(\theta) - 24 t^3 e^{-\zeta^2} \sin(\theta) - 2 R r^5 e^{-\zeta^2} \sin(\theta) - 6 r^4 t e^{-\zeta^2} \sin(\theta) \right] \right. \\
&+ \frac{1}{4\sqrt{\pi} r^4 t^{\frac{3}{2}}} \left(T_o \left[R^2 r^4 e^{-\zeta^2} \sin(\theta) - 12 r^2 t^2 e^{-\zeta^2} \sin(\theta) + 4 G r t^2 r^4 e^{-\zeta^2} \cos(\theta) + \right] \right) \\
&+ \frac{1}{4\sqrt{\pi} r^4 t^{\frac{3}{2}}} \left(T_o \left[12\sqrt{\pi} t^{\frac{5}{2}} R \operatorname{erfc}(\zeta) \sin(\theta) + 4 R t r^3 e^{-\zeta^2} \sin(\theta) \right] \right) \\
&- t \left(2 T_o \sqrt{t} \sin(\theta) \left(\zeta \operatorname{erfc}(\zeta) - \frac{1}{\sqrt{\pi}} e^{-\zeta^2} + \frac{1}{\sqrt{\pi}} \right) \right) + \dots \\
T_1^o &= -t (T_o \operatorname{erfc}(\zeta)) + \frac{T_o \sqrt{t} e^{-\zeta^2} (2R - 2r)}{6\sqrt{\pi} \text{Pr}} - \frac{3 T_o e^{-\zeta^2} (2R - 2r)}{2\pi \text{Pr} \sqrt{t}} + \frac{T_o t^{\frac{3}{2}} e^{-\zeta^2}}{120 r \text{Pr}} + \frac{2 T_o \sqrt{t} e^{-\zeta^2}}{3 r \sqrt{\pi} \text{Pr}} \\
&- \frac{1}{4\pi r^5 t^2} \left(T_o^2 e^{-\zeta^2} \left[r^6 e^{-\zeta^2} \cos(\theta) - 2 R r^5 e^{-\zeta^2} \cos(\theta) + R^2 r^4 e^{-\zeta^2} \cos(\theta) - 4 G r t^2 r^4 e^{-\zeta^2} \sin(\theta) \right] \right) \\
&- \frac{1}{4\pi r^5 t^2} \left(T_o^2 e^{-\zeta^2} \left[12\sqrt{\pi} t^{\frac{5}{2}} R \operatorname{erfc}(\zeta) \cos(\theta) \right] \right) + \dots
\end{aligned}$$



$$\begin{aligned}
\Psi_1^i &= \frac{1}{4\sqrt{\pi}r^4t^{\frac{3}{2}}} \left(T_i \left[r^4 e^{-\zeta^2} (\sin(\theta) - 6t \sin(\theta)) - 2r^5 e^{-\zeta^2} \sin(\theta) + r^6 e^{-\zeta^2} \sin(\theta) + 4tr^3 e^{-\zeta^2} \sin(\theta) \right] \right. \\
&\quad \left. + \frac{1}{4\sqrt{\pi}r^4t^{\frac{3}{2}}} \left(T_o \left[-12\sqrt{\pi}t^{\frac{5}{2}} \operatorname{erfc}(\xi) \sin(\theta) - 12r^2 t^2 e^{-\zeta^2} \sin(\theta) + 4Gr t^2 r^4 e^{-\zeta^2} \cos(\theta) \right] \right) \right. \\
&\quad \left. - t \left(2T_i \sqrt{t} \sin(\theta) \left[-\xi \operatorname{erfc}(\xi) + \frac{1}{\sqrt{\pi}} e^{-\zeta^2} - \frac{1}{\sqrt{\pi}} \right] \right) + \dots \right. \\
T_1^i &= -t(T_i \operatorname{erfc}(\xi)) + \frac{T_i \sqrt{t} e^{-\zeta^2} (2r-2)}{6\sqrt{\pi} \operatorname{Pr}} - \frac{3T_i e^{-\zeta^2} (2r-2)}{2\pi \operatorname{Pr} \sqrt{t}} - \frac{T_i t^{\frac{3}{2}} e^{-\zeta^2}}{120r \operatorname{Pr}} - \frac{2T_i \sqrt{t} e^{-\zeta^2}}{3r\sqrt{\pi} \operatorname{Pr}} \\
&\quad - \frac{1}{4\pi r^5 t^2} \left(T_i^2 e^{-\zeta^2} \left[r^4 e^{-\zeta^2} (\cos(\theta) - 6t \cos(\theta)) + r^6 e^{-\zeta^2} \cos(\theta) - 4tr^3 e^{-\zeta^2} \cos(\theta) - 4Gr t^2 r^4 e^{-\zeta^2} \sin(\theta) \right] \right) \\
&\quad - \frac{1}{4\pi r^5 t^2} \left(T_i^2 e^{-\zeta^2} \left[-12\sqrt{\pi}t^{\frac{5}{2}} \operatorname{erfc}(\xi) \cos(\theta) \right] \right) + \dots \\
\Psi_1^c &= 0, \quad T_1^c = 0 \\
&\vdots
\end{aligned}$$

Then the solutions can be found as:

$$\Psi = (\Psi_0^o + \Psi_1^o + \Psi_0^i + \Psi_1^i + \Psi_0^c + \Psi_1^c + \dots) \quad \text{and} \quad T = (T_0^o + T_1^o + T_0^i + T_1^i + T_0^c + T_1^c + \dots) \quad (29)$$

5. Results and Discussion

Figures (1) and (2), explain the comparison of results for a stream (left) and isotherm (right) between; (a) Tsui and Tremblay [5], (b) Hassan and Al-Lateef [7] and (c) the analytical solutions of YTHPM at $\operatorname{Pr} = 0.71$, $R = 2$ and $Gr = 10000, 38800$, respectively.

From Figures (2,3), we can observe the effect of the Grashof number on the lines of the stream and isotherm when it increases. The pattern of flow appears as kidney-shaped and the distribution of temperature is misshaped. Also, the results show a good agreement with previous studies in the refs.[5,7] for the same values of Gr , Pr and the radius proportion R .

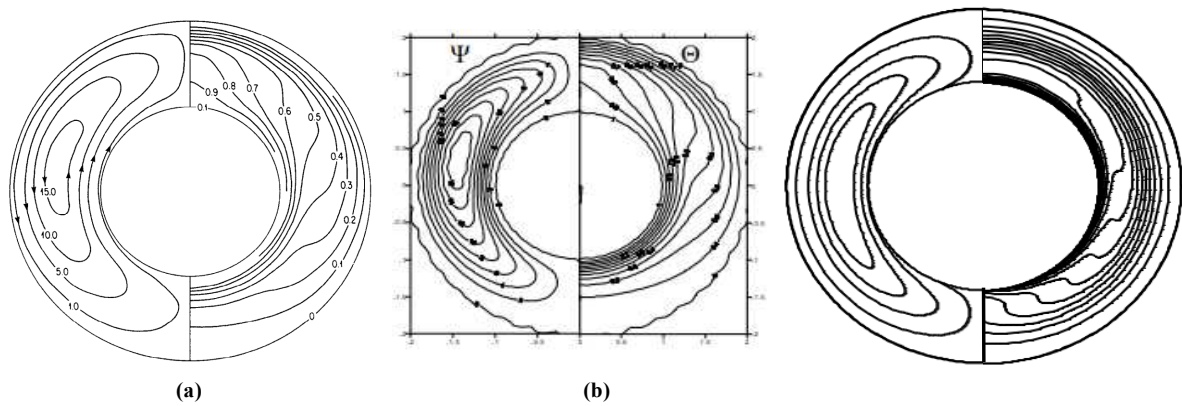


Fig. 2. Comparison between (a) [5], (b) [7] and (c) YTHPM for stream (left) and isotherm (right) for $Gr = 10000$, $\operatorname{Pr} = 0.71$ and $R = 2$.

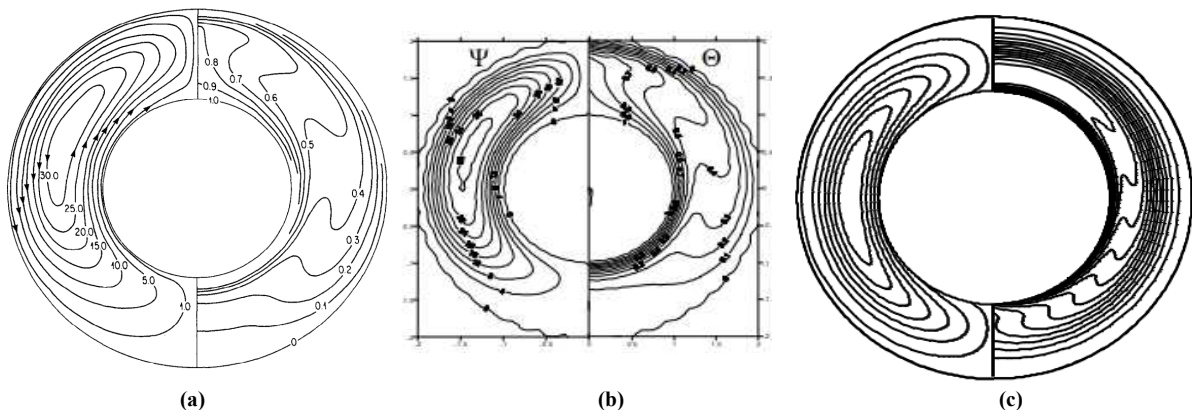


Fig. 3. Comparison between (a) [5], (b) [7] and (c) YTHPM for stream (left) and isotherm (right) for $Gr = 38800$, $\operatorname{Pr} = 0.71$ and $R = 2$.

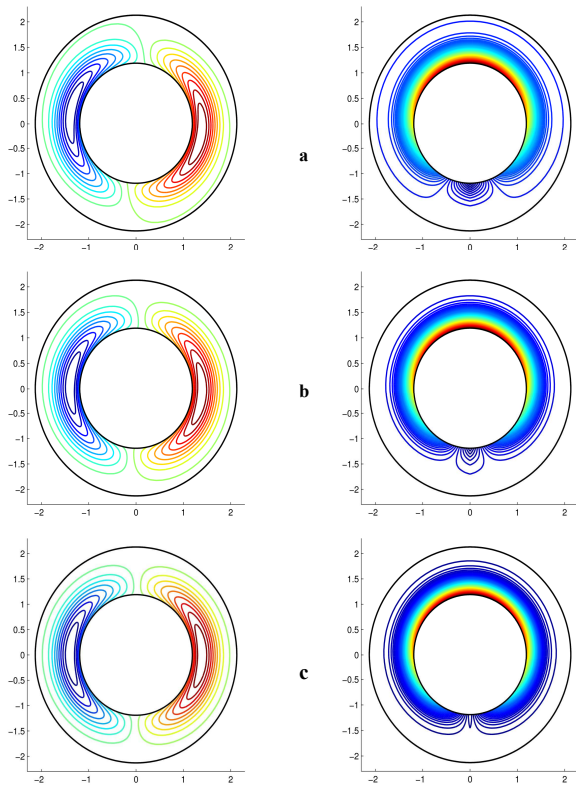
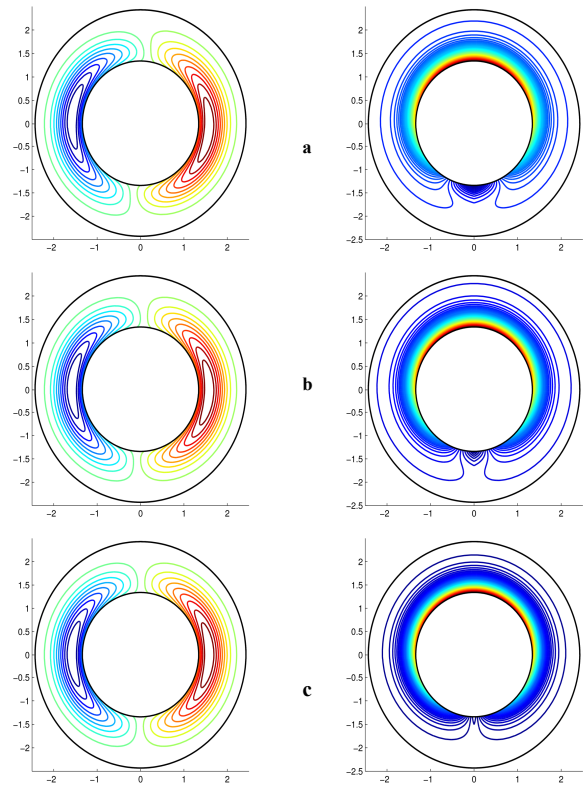


Table 1. Absolute error comparison between YTHPM and HPM of stream function at $R = 2, Gr = 1000$ and $Pr = 0.7$.

θ	r	$t = 0.01$		$t = 0.1$	
		YTHPM	HPM	YTHPM	HPM
0	1	2.82×10^{-1}	56.983	8.921	196.254
	1.5	5.44×10^{-4}	1.10×10^{-1}	4.775	105.047
	2	3.91×10^{-12}	7.91×10^{-10}	7.32×10^{-1}	16.109
30	1	5.85×10^{-2}	11.811	1.463	32.164
	1.5	2.09×10^{-4}	4.23×10^{-2}	7.54×10^{-1}	16.574
	2	1.02×10^{-6}	2.08×10^{-4}	1.03×10^{-1}	2.254
60	1	2.64×10^{-1}	53.355	8.472	186.363
	1.5	6.10×10^{-4}	1.23×10^{-1}	4.544	99.954
	2	3.69×10^{-7}	6.98×10^{-5}	7.01×10^{-1}	15.419
90	1	1.39×10^{-1}	28.240	4.071	89.597
	1.5	2.30×10^{-5}	4.55×10^{-3}	2.152	47.371
	2	1.01×10^{-6}	1.97×10^{-4}	1.18×10^{-1}	7.001
180	1	1.56×10^{-1}	31.681	5.274	115.973
	1.5	6.65×10^{-4}	1.14×10^{-1}	2.847	62.603
	2	9.24×10^{-7}	1.79×10^{-4}	4.48×10^{-1}	9.837

Table 2. Absolute error comparison between YTHPM and HPM of isotherm function at $R = 2, Gr = 1000$ and $Pr = 0.7$.

θ	r	$t = 0.01$		$t = 0.1$	
		YTHPM	HPM	YTHPM	HPM
0	1	3.52×10^{-3}	17.571	3.23×10^{-2}	3.635
	1.5	2.593×10^{-5}	3.38×10^{-3}	2.76×10^{-2}	5.73×10^{-1}
	2	3.731×10^{-13}	5.595×10^{-11}	7.01×10^{-3}	8.58×10^{-2}
30	1	1.81×10^{-2}	318.479	1.586	380.428
	1.5	2.58×10^{-5}	2.95×10^{-3}	2.81×10^{-1}	72.411
	2	3.7310×10^{-13}	5.596×10^{-11}	1.55×10^{-3}	1.183
60	1	3.48×10^{-3}	115.391	4.85×10^{-1}	121.08
	1.5	2.596×10^{-5}	3.99×10^{-3}	6.78×10^{-2}	22.358
	2	3.7312×10^{-13}	5.597×10^{-11}	5.51×10^{-3}	2.73×10^{-1}
90	1	1.22×10^{-2}	282.258	1.405	342.265
	1.5	2.597×10^{-5}	4.74×10^{-3}	2.91×10^{-1}	65.903
	2	3.7311×10^{-13}	5.597×10^{-11}	1.17×10^{-2}	1.271
180	1	1.40×10^{-2}	270.701	1.289	311.133
	1.5	2.591×10^{-5}	3.44×10^{-3}	2.22×10^{-1}	58.827
	2	3.731×10^{-13}	5.597×10^{-11}	2.69×10^{-3}	9.25×10^{-1}

**Fig. 4.** Analytical solution of stream (left) isotherm (right) by YTHPM at $R = 1.2, Pr = 0.7$ and (a) $Gr = 500$, (b) $Gr = 1000$, (c) $Gr = 5000$.**Fig. 5.** Analytical solution of stream (left) isotherm (right) by YTHPM at $R = 1.5, Pr = 0.7$ and (a) $Gr = 500$, (b) $Gr = 1000$, (c) $Gr = 5000$.

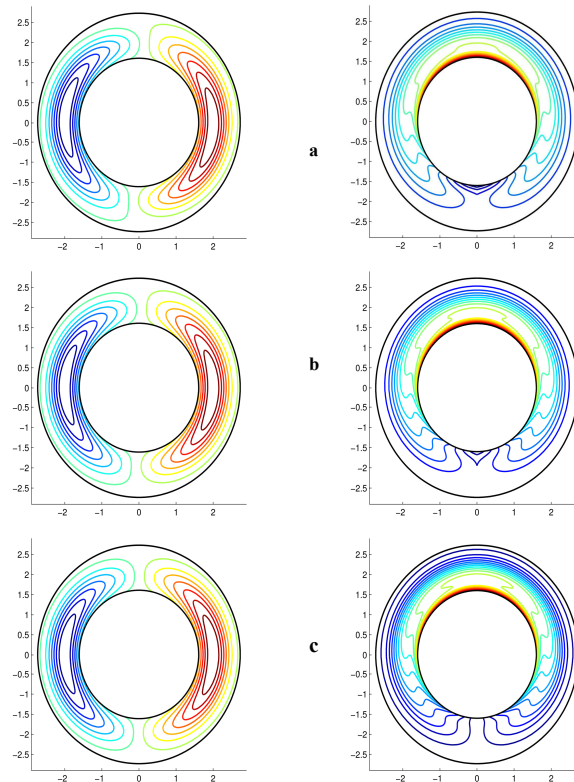


Fig. 6. Analytical solution to the stream (left) isotherm (right) by YTHPM at $R = 2, Pr = 0.7$ and (a) $Gr = 500$, (b) $Gr = 1000$, (c) $Gr = 5000$.

In Tables (1) and (2), the absolute errors of YTHPM and HPM for stream and isotherm functions are compared at $R = 2, Gr = 1000, Pr = 0.7$ and $t = 0.01, 0.1$, respectively. As clear in the tables below, YTHPM has less errors, more efficiency and higher accuracy than HPM to solve this problem. Then we can say that the new method represents a development for HPM and YT.

Figures (4), (5) and (6) show the analytical solution offered by YTHPM for the stream (left) and isotherm (right) at $Pr = 0.7, Gr = 500, 1000, 5000$ and $R = 1.2, 1.5, 2$, respectively. From the Figures (4-6), we can note the effect of Grashof with radius proportion on the flow of fluid and temperature. Also, we can see that when the Grashof increases, the pattern of the flow appears as kidney-shaped with an increase in the radius proportion. Moreover, the temperature pattern at all Grashof number it is similar to the circles when the radius proportion is small (i.e. $R = 1.2, 1.5$) due to the weak influence of convection currents. And when the radius proportion increases with the increase in the Grashof number, the distribution of temperature becomes misshaped, a matter that indicates an increase in the heat convection.

Now, the tangent velocity can be calculated by using the relation (25d), as in Figure (7) which demonstrates the velocity at $Pr = 0.71, t = 0.01, Gr = 1000$ at various radius proportions ($R = 1.2, R = 1.5, R = 2$) and different angle.

All states in Figure (7) manifest the effect of radius proportion and the angle on the velocity tangent. It was noted that the maximum values for the velocity at $\theta = 60, 150, 120, 90$, respectively. And the minimum value for velocity is at $\theta = 30$ and when $R = 2$ the velocity value converges to the unity.

Now, we illustrate the importance of the Nusselt number of the outer and inner radius and how it is affected by Gr, Pr and the radius proportion R . The Nusselt number of the outer, inner radius and the mean of Nusselt number (outer and inner) are defined respectively as follows:

$$Nu_o = -\ln(R) \left(r \frac{\partial T}{\partial r} \right)_{r=R_o} \quad (30a)$$

$$Nu_i = -\ln(R) \left(r \frac{\partial T}{\partial r} \right)_{r=R_i} \quad (30b)$$

$$\overline{Nu}_o = -\frac{\ln(R)}{\pi} \int_0^\pi \left(r \frac{\partial T}{\partial r} \right)_{r=R_o} d\theta \quad (30c)$$

$$\overline{Nu}_i = -\frac{\ln(R)}{\pi} \int_0^\pi \left(r \frac{\partial T}{\partial r} \right)_{r=R_i} d\theta \quad (30d)$$



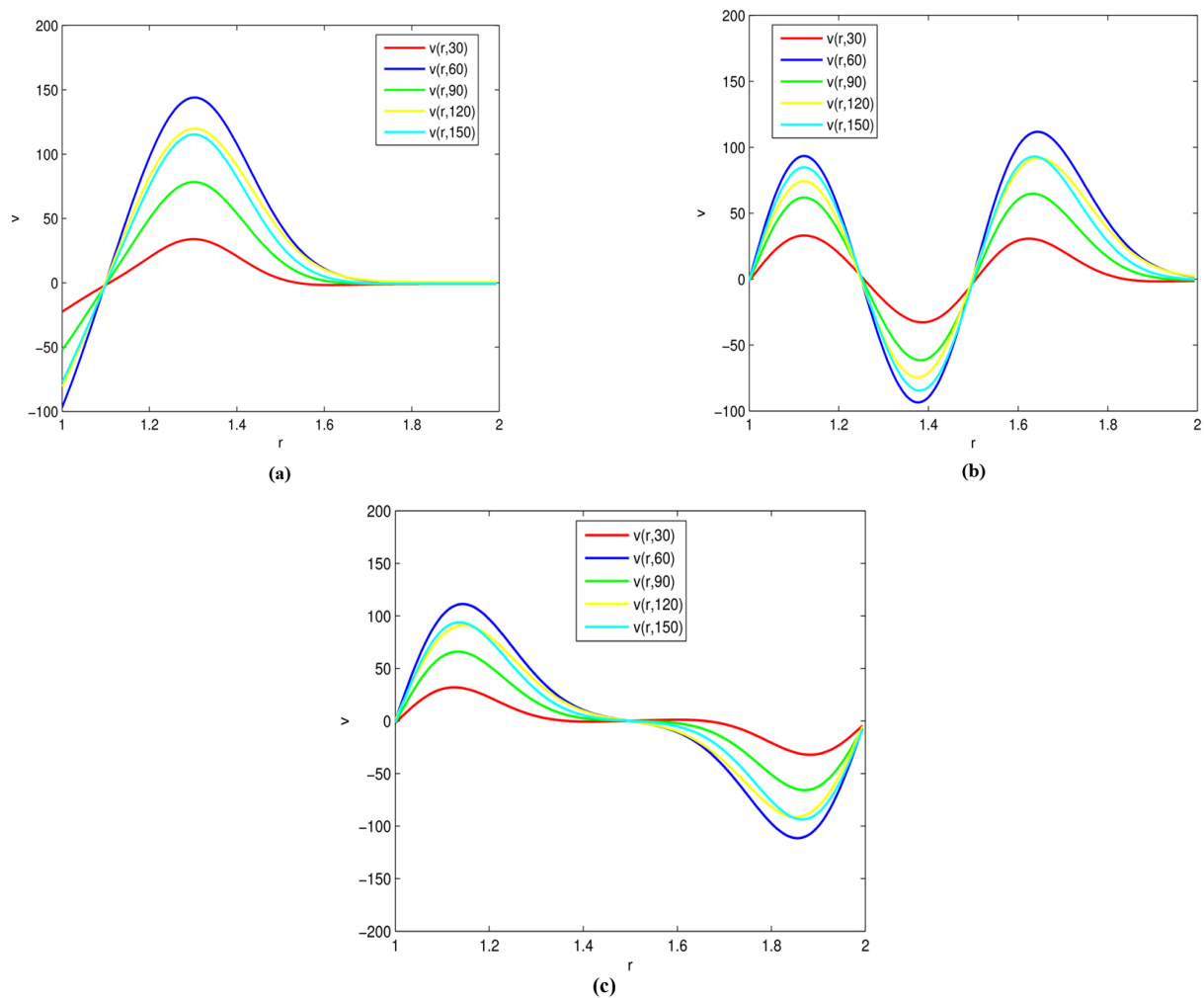


Fig. 7. Analytical solutions by YTHPM for the velocity at various values of θ and $Pr = 0.71$, $t = 0.01$, $Gr = 1000$ and (a) $R = 1.2$, (b) $R = 1.5$, (c) $R = 2$.

Figure (8) shows the mean of the Nusselt number vs. the dimensionless time for the outer and inner radius at $Gr = 4850$, $Pr = 0.7$ and $R = 1.5$. On another note, Figures (9) and (10) illustrate the mean of the Nusselt number of the outer and inner radius at $R = 1.5$, $Pr = 0.7$ and $Gr = 11500$, $Gr = 26200$, respectively. Figure (11) explains the mean of the Nusselt number for the outer and inner radius at $Gr = 10000$, $Pr = 0.7$ and $R = 2$. Furthermore, Figures (12,13) exhibit the mean of the Nusselt number for the outer and inner radius at $Pr = 0.7$, $R = 2$ and $Gr = 38800$, $Gr = 88000$ respectively. Finally, Figure (14) illustrates the mean of the Nusselt number for the outer and inner radius at $Gr = 732$, $Pr = 0.7$ and $R = 1.2$.

As clear in the figures, when t increases, the mean of $\bar{Nu}$ (outer, inner) comes close to the steady-state values. And when the Gr increases, the mean of $\bar{Nu}$ (outer and inner) increases for the same radius proportion $R = 1.5$ as shown in Figures (8-10). Moreover, Figures (11-13) illustrate that when the Gr increases, the mean of $\bar{Nu}$ (outer and inner) increases for radius proportion $R = 2$. The last Figure (14) shows that both means of the Nusselt number (outer, inner) converge to unity, this means that convection is almost non-existent at $Gr = 732$, $Pr = 0.7$ and $R = 1.2$.

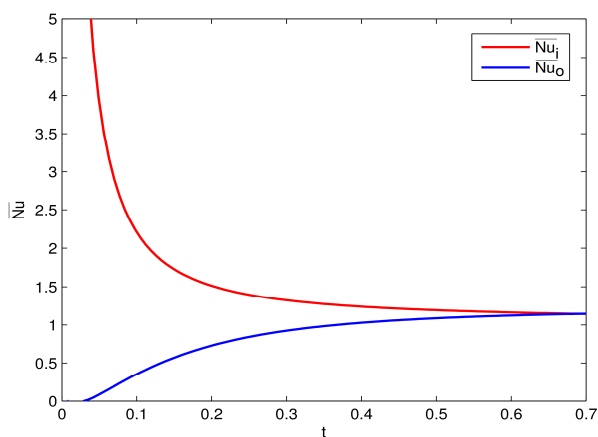


Fig. 8. The mean of Nusselt number at $Gr = 4850$, $Pr = 0.7$ and $R = 1.5$.

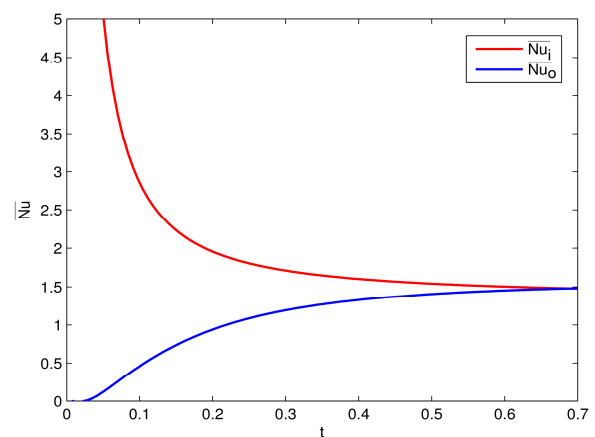


Fig. 9. The mean of Nusselt number at $Gr = 11500$, $Pr = 0.7$ and $R = 1.5$.



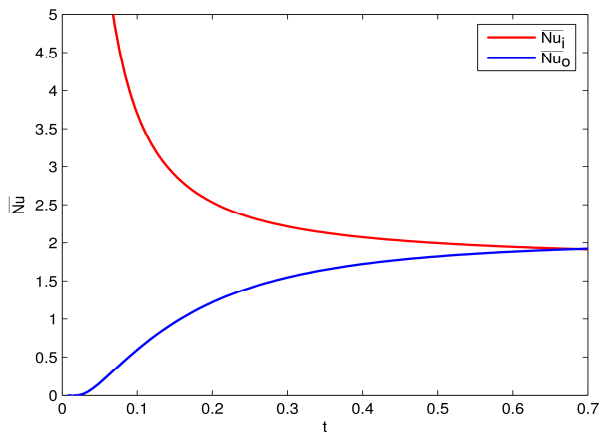


Fig. 10. The mean of Nusselt number at $Gr = 26200$, $Pr = 0.7$ and $R = 1.5$.

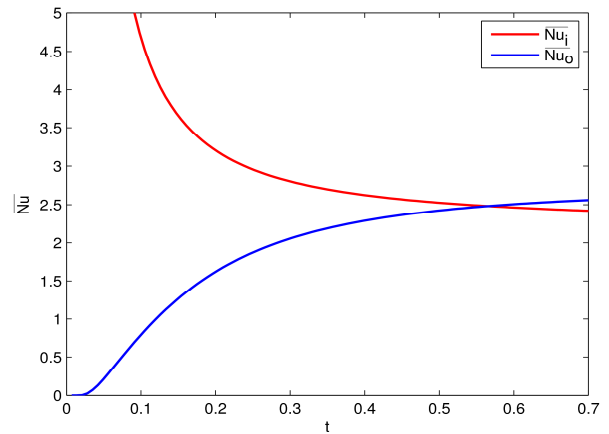


Fig. 11. The mean of Nusselt number at $Gr = 38800$, $Pr = 0.7$ and $R = 2$.

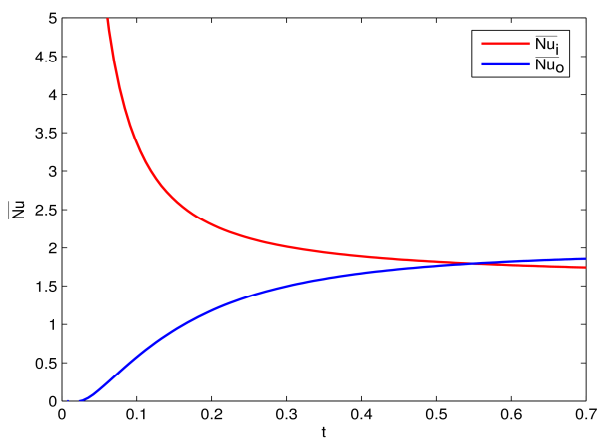


Fig. 12. The mean of Nusselt number at $Gr = 10000$, $Pr = 0.7$ and $R = 2$.

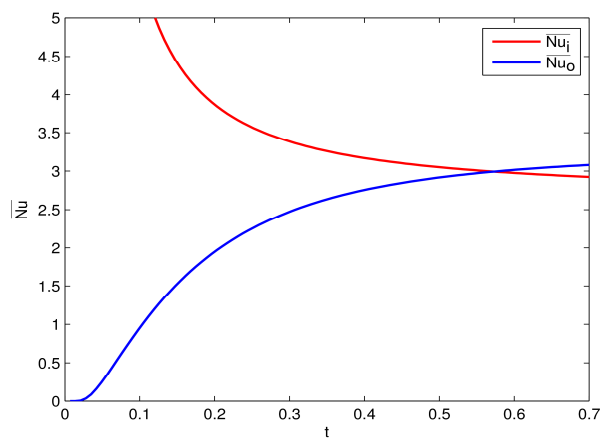


Fig. 13. The mean of Nusselt number at $Gr = 88000$, $Pr = 0.7$ and $R = 2$.

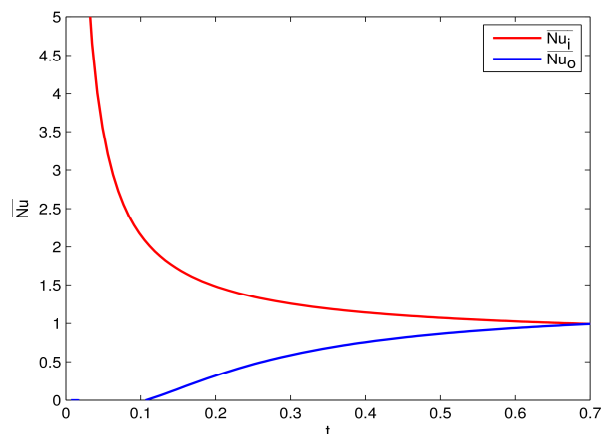


Fig. 14. The mean of Nusselt number at $Gr = 732$, $Pr = 0.7$ and $R = 1.2$.

6. Convergence Analysis of YTHPM

In this part, which is concerned with convergence analysis, we will study the convergence for the analytical approximate solution (29) obtained by using the new method (YTHPM) and as follows:

Definition: Assume that X_1 is the Banach space and $N: X_1 \rightarrow \mathbb{R}$ is a nonlinear mapping where $\mathbb{R}$ is the real numbers. Then, the sequence of the solutions can be written in the following form:



$$W_m = N(W_{m-1}), \quad W_{m-1} = \sum_{i=0}^{m-1} w_i, \quad m = 1, 2, 3, \dots \quad (31)$$

where N satisfies the Lipschitz condition such that:

$$\forall \gamma \in \mathbb{R}; \quad \|N(W_m) - N(W_{m-1})\| \leq \gamma \|W_m - W_{m-1}\|, \quad 0 < \gamma < 1. \quad (32)$$

Theorem (1): The series of analytical approximate solutions $\Psi = \sum_{k=0}^{\infty} \Psi_k$ and $T = \sum_{k=0}^{\infty} T_k$ obtained by YTHPM convergence if it satisfies the following condition:

$$\|W_m - W_n\| \rightarrow 0 \quad \text{when } n \rightarrow \infty, 0 < \gamma < 1.$$

Proof:

$$\begin{aligned} \|W_m - W_n\| &= \left\| \sum_{i=0}^m w_i - \sum_{i=0}^n w_i \right\| \\ &= \left\| w_0 + \sum_{i=1}^m N(w_i^*) - w_0 + \sum_{i=1}^n N(w_i^*) \right\|, \\ &= \left\| \sum_{i=1}^m N(w_i^*) + \sum_{i=1}^n N(w_i^*) \right\|, \quad W_m = N(W_{m-1}) \\ &= \left\| N\left(\sum_{i=0}^{m-1} w_i^*\right) + N\left(\sum_{i=0}^{n-1} w_i^*\right) \right\| \\ &\leq \gamma \|W_{m-1} - W_{n-1}\| \end{aligned}$$

Because N satisfies the Lipschitz condition and

$$\begin{aligned} w_i^* &= \sum_{j=0}^n \int_0^t \left(Gr \left(\cos(\theta) \frac{\partial T_j}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial T_j}{\partial \theta} \right) + \sum_{k=0}^j \left(\nabla^2 (-\nabla^2 \Psi_k) + \frac{1}{r} \left(\frac{\partial \Psi_k}{\partial \theta} \frac{\partial}{\partial r} (-\nabla^2 \Psi_{j-k}) + \frac{\partial \Psi_k}{\partial r} \frac{\partial}{\partial \theta} (-\nabla^2 \Psi_{j-k}) \right) \right) \right) dt \\ w_i^* &= \sum_{j=0}^n \int_0^t \left(\frac{1}{Pr} \nabla^2 T_j - \sum_{k=0}^j \left(\frac{1}{r} \left(\frac{\partial \Psi_k}{\partial \theta} \frac{\partial T_{j-k}}{\partial r} + \frac{\partial \Psi_k}{\partial r} \frac{\partial T_{j-k}}{\partial \theta} \right) \right) \right) dt \end{aligned}$$

Let, $m = n + 1$ then, we have $\|W_{n+1} - W_n\| \leq \gamma \|W_n - W_{n-1}\|$, then

$$\|W_n - W_{n-1}\| \leq \gamma \|W_{n-1} - W_{n-2}\| \leq \dots \leq \gamma^{n-1} \|W_1 - W_0\| \quad (33)$$

From (33), we have:

$$\begin{aligned} \|W_2 - W_1\| &\leq \gamma \|W_1 - W_0\| \\ \|W_3 - W_2\| &\leq \gamma^2 \|W_1 - W_0\| \\ &\vdots \\ \|W_n - W_{n-1}\| &\leq \gamma^{n-1} \|W_1 - W_0\| \end{aligned}$$

By using the triangle inequality:

$$\begin{aligned} \|W_m - W_n\| &= \|W_m - W_{m-1} - \dots - W_{n+1} - W_n\| \\ &\leq \|W_m - W_{m-1}\| + \|W_{m-1} - W_{m-2}\| + \dots + \|W_{n+1} - W_n\| \\ &\leq (\gamma^{m-1} + \gamma^{m-2} + \dots + \gamma^n) \|W_1 - W_0\| \\ &\leq \frac{\gamma^n}{1 - \gamma} \|W_1 - W_0\| \end{aligned}$$

when $n \rightarrow \infty$ we have $\|W_m - W_n\| \rightarrow 0$, then W_m is the Cauchy sequence in Banach space X_1 .

Theorem (2): The solution by the new method (YTHPM) $\Psi = \sum_{k=0}^{\infty} \Psi_k$ and $T = \sum_{k=0}^{\infty} T_k$ converges and is close to the solution problems (27) if the following property is achieved:



$$L^{-1}N(\Psi) = L^{-1}N(T) = \lim_{m \rightarrow \infty} L^{-1}N(W_m), \quad \text{where} \quad L^{-1}(\ast) = \int_0^t (\ast) dt.$$

Proof: For any $W \in X_1$ define an operator from X_1 to X_1 , $\ell(W) = W_0 + L^{-1}N(W)$.

Let, $W_1, W_2 \in X_1$, then we have:

$$\begin{aligned} \|\ell(W_1) - \ell(W_2)\| &= \|W_0 + L^{-1}N(W_1) - W_0 - L^{-1}N(W_2)\| \\ &= \|L^{-1}N(W_1) - L^{-1}N(W_2)\| \\ &\leq \|L^{-1}\| \|N(W_1) - N(W_2)\| \\ &\leq \gamma \|W_1 - W_2\| \end{aligned}$$

Therefore, the mapping ℓ is contractive, and by the Banach fixed point theorem for contractive, there is a unique solution for problems (27). Now, we prove that the series solution Ψ, T satisfies problems (27):

$$\begin{aligned} L^{-1}N(\Psi) &= L^{-1}N\left(\sum_{k=0}^{\infty} \Psi_k\right) \\ &= L^{-1}N\left(\lim_{m \rightarrow \infty} \sum_{i=0}^m w_i\right) \\ &= L^{-1}N\left(\lim_{m \rightarrow \infty} W_m\right) \\ &= \lim_{m \rightarrow \infty} L^{-1}N(W_m) \end{aligned}$$

and

$$\begin{aligned} L^{-1}N(T) &= L^{-1}N\left(\sum_{k=0}^{\infty} T_k\right) \\ &= L^{-1}N\left(\lim_{m \rightarrow \infty} \sum_{i=0}^m w_i\right) \\ &= L^{-1}N\left(\lim_{m \rightarrow \infty} W_m\right) \\ &= \lim_{m \rightarrow \infty} L^{-1}N(W_m) \end{aligned}$$

From Theorems (1) and (2), the values of the parameter γ^m must be calculated to obtain convergence by using the following relationship:

$$\gamma^m = \begin{cases} \frac{\|W_{m+1} - W_m\|}{\|W_1 - W_0\|} = \frac{\|\Psi_{m+1}\|}{\|\Psi_1\|} = \frac{\|T_{m+1}\|}{\|T_1\|}, & \|\Psi_1\| \neq 0, \|T_1\| \neq 0, \quad m = 1, 2, 3, \dots \\ 0, & \|\Psi_1\| = 0, \|T_1\| = 0 \end{cases}$$

Now, by using this definition, we find the convergence of the problem as follows:

$$\begin{aligned} \|\Psi_1 - \Psi_0\| &= \left\| \left(\frac{1}{4\sqrt{\pi} r^4 t^{\frac{3}{2}}} (T_0 [24t^3 \sin(\theta) + \dots]) + \frac{1}{4\sqrt{\pi} r^4 t^{\frac{3}{2}}} (T_1 [r^4 e^{-\zeta^2} (\sin(\theta) - 6t \sin(\theta)) - \dots]) + \dots \right) - (2T_0 \sqrt{t} \sin(\theta)(\dots) + \dots) \right\| \\ \|\Psi_2 - \Psi_1\| &\leq \|\Psi_1 - \Psi_0\| \gamma, \quad \gamma = 0.210 < 1, \\ \|\Psi_3 - \Psi_2\| &\leq \|\Psi_1 - \Psi_0\| \gamma^2, \quad \gamma^2 = 0.00608 < 1, \\ &\vdots \\ \|\Psi_m - \Psi_{m-1}\| &\leq \|\Psi_1 - \Psi_0\| \gamma^m. \end{aligned}$$

and

$$\begin{aligned} \|T_1 - T_0\| &= \left\| \left(\frac{T_0 \sqrt{t} e^{-\zeta^2} (2R - 2r)}{6\sqrt{\pi} \text{Pr}} + \frac{T_1 \sqrt{t} e^{-\zeta^2} (2r - 2)}{6\sqrt{\pi} \text{Pr}} + \dots \right) - (T_0 \operatorname{erfc}(\zeta) + T_1 \operatorname{erfc}(\xi)) \right\| \\ \|T_2 - T_1\| &\leq \|T_1 - T_0\| \gamma, \quad \gamma = 0.00578 < 1, \\ \|T_3 - T_2\| &\leq \|T_1 - T_0\| \gamma^2, \quad \gamma^2 = 0.0005008 < 1, \\ &\vdots \\ \|T_m - T_{m-1}\| &\leq \|T_1 - T_0\| \gamma^m. \end{aligned}$$

where; $\Psi_0 = \Psi_0^o + \Psi_0^i + \Psi_0^c$, $T_0 = T_0^o + T_0^i + T_0^c$ and $\Psi_1 = \Psi_1^o + \Psi_1^i + \Psi_1^c$, $T_1 = T_1^o + T_1^i + T_1^c$, and so on.



7. Conclusion

In this study, we suggested a newly developed method (YTHPM) for solving the two-dimensional transient natural convection in a horizontal cylindrical concentric annulus bounded by two isothermal surfaces analytically. The effects of parameters (Pr, Gr, R) on the flow of the fluid, isotherm, velocity and Nusselt number were illustrated. The results show that the Grashof number and radius proportion have more effect than the Prandtl number. We note that when the Grashof increases, the pattern of flow appears as kidney-shaped with an increase in the radius proportion. Moreover, the temperature pattern is similar to the circles when the radius proportion is small, and when the radius proportion increases with an increase of Grashof number, the distribution of temperature becomes misshapen. The evidence of the validity and efficiency of the new method was proved by comparing it with the previously published results and a good agreement has been obtained. Furthermore, we conclude that the YTHPM is an effective method with a high accuracy to find the analytical approximate solutions for the two-dimensional transient natural convection in a horizontal cylindrical concentric annulus from the first iteration. Finally, from the analysis of results, we can see that the new method is simple, easy-to-use and has high-precision. It can be used to handle various complicated fluid flow problems that have many applications in life, a situation that opens the door wide for future studies.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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References

- [1] Crawford, L., Lemlich, R., Natural convection in horizontal concentric cylindrical annuli, *Industrial and Engineering Chemistry Fundamentals*, 1(4), 1962, 260-264.
- [2] Yang, X., Kong, S., Numerical study of natural convection states in a horizontal concentric cylindrical annulus using smoothed particle hydrodynamics, *Engineering Analysis with Boundary Elements*, 102, 2019, 11-20.
- [3] Pop, I., Ingham, D., Cheng, P., Transient natural convection in a horizontal concentric annulus filled with a porous medium, *Journal of Heat Transfer*, 114, 1992, 990-997.
- [4] Kuehn, T., Goldstein, J., An experimental and theoretical study of natural convection in the annulus between horizontal concentric cylinders, *Journal of Fluid Mechanics*, 74, 1976, 695-719.
- [5] Tsui, Y., Trembaly, B., On transient natural convection heat transfer in the annulus between convective horizontal cylinders with isothermal surfaces, *International Journal of Heat and Mass Transfer*, 27(1), 1983, 103-111.
- [6] Sano, T., Kuribayashi, K., Transient natural convection around a horizontal circular cylinder, *Fluid Dynamics Research*, 10, 1992, 25-37.
- [7] Hassan, A., Al-Lateef, J., Numerical simulation of two dimensional transient natural convection heat transfer from isothermal horizontal cylindrical annuli, *Engineering and Technology Journal*, 25(6), 2007, 728-745.
- [8] Abdulateef, J., Effect of Prandtl number and diameter ratio on laminar natural convection from isothermal horizontal cylindrical annuli, *Diyala Journal of Engineering Sciences*, 9(1), 2016, 39-45.
- [9] Mack, L., Bishop, E., Natural convection between Horizontal concentric cylinders for low Rayleigh numbers, *The Quarterly Journal of Mechanics and Applied Mathematics*, XXI, 1968, 223-241.
- [10] Yang, M., Ding, Z., Lou, Q., Wang, Z., Lattice Boltzmann Method Simulation of Natural Convection Heat Transfer in Horizontal Annulus, *Journal of Thermophysics and Heat Transfer*, 31(3), 2017, 700-711.
- [11] Bhowmik, H., Gharibi, A., Yaarubi, A., Alawi, N., Transient natural convection heat transfer analyses from a horizontal cylinder, *Case Studies in Thermal Engineering*, 14, 2019, 1-8.
- [12] Khudhayer, S., Jasim, Y., Ibrahim, T., Natural Convection in a Porous Horizontal Cylindrical Annulus Under Sinusoidal Boundary Condition, *Journal of Mechanical Engineering Research and Developments*, 43(4), 2020, 230-244.
- [13] Mohamad, A., Taler, J., Oclon, P., Transient natural convection in a thermally insulated annular cylinder exposed to a high temperature from the inner radius, *Energies*, 13, 2020, 1-13.
- [14] Mahmood, M., Mustafa, M., Al-Azzawi, M., Abdullah, A., Natural convection heat transfer in a concentric annulus vertical cylinders embedded with porous media, *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences*, 66, 2020, 65-83.
- [15] Zubkov, P., Narygin, E., Numerical Study of Unsteady Natural Convection in a Horizontal Annular Channel, *Microgravity Science and Technology*, 32, 2020, 579-586.
- [16] Wei, Y., Wang, Z., Qian, Y., Guo, W., Study on bifurcation and dual solutions in natural convection in a horizontal annulus with rotating inner cylinder using thermal immersed boundary-lattice Boltzmann method, *Entropy*, 20, 2018, 1-15.
- [17] Medebber M., Retiel N., Said B., Aissa A. and El-Ganaoui M., Transient Numerical Analysis of Free Convection in Cylindrical Enclosure, *MATEC Web of Conferences*, 307, 2020, 1-9.
- [18] Touzani, S., Cheddadi, A., Ouazzani, M., Natural Convection in a Horizontal Cylindrical Annulus with Two Isothermal Blocks in Median Position: Numerical Study of Heat Transfer Enhancement, *Journal of Applied Fluid Mechanics*, 13(1), 2020, 327-334.
- [19] Hu, Y., Li, D., Shu, S., Niu, X., Study of multiple steady solutions for the 2D natural convection in a concentric horizontal annulus with a constant heat flux wall using immersed boundary-lattice Boltzmann method, *International Journal of Heat and Mass Transfer*, 81, 2015, 591-601.
- [20] Yuan, X., Tavakkoli, F., Vafai, K., Analysis of natural convection in horizontal concentric annuli of varying inner shape, *Numerical Heat Transfer, Part A: Applications: An International Journal of Computation and Methodology*, 68, 2015, 1155-1174.
- [21] Mehri, A., A., Farhadi, M., Shayamehr, S., Natural convection flow of Cu-Water nanofluid in horizontal cylindrical annuli with inner triangular cylinder using lattice Boltzmann method, *International Communications in Heat and Mass Transfer*, 44, 2013, 147-156.
- [22] Kumar, R., Study of natural convection in horizontal annuli, *International Journal of Heat and Mass Transfer*, 31(6), 1988, 1137-1148.
- [23] Yang, X. J., A new integral transform method for solving steady heat-transfer problem, *Thermal Science*, 20, 2016, S639-S642.
- [24] Yang, X.-J., Gao, F., A New Technology for solving diffusion and heat equations, *Thermal Science*, 21(1A), 2017, 133-140.
- [25] Aminikhah, H., Hemmatnezhad, M., An efficient method for quadratic Riccati differential equation, *Communications in Nonlinear Science and Numerical Simulation*, 15, 2010, 835-839.
- [26] Mirzazadeh, M., Ayati, Z., New homotopy perturbation method for system of Burgers equations, *Alexandria Engineering Journal*, 55, 2016, 1619-1624.



[27] Rubab, S., Ahmad, J., Muhammad, N., Exact solution of Klein Gordon equation via homotopy perturbation Sumudu transform method, *International Journal of Hybrid Information Technology*, 7(6), 2014, 445-452.

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