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Research Paper

A General Multibody Approach for the Linear and Nonlinear Stability Analysis of Bicycle Systems. Part I: Methods of Constrained Dynamics

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Abstract. This investigation is the first contribution of a two-part research work concerning the theoretical development of a multibody approach to analyze the constrained dynamics of articulated mechanical systems. In this paper, a method for investigating the linear and nonlinear stability of the dynamic behavior of mechanical systems modeled as multibody systems subjected to holonomic and nonholonomic constraints is presented. To this end, the nonlinear equations of motions that assume a complex index-three differential-algebraic form are systematically formulated and directly linearized by using an automatic procedure based on a hybrid symbolic-numeric approach devised in this work. The proposed stability analysis method, therefore, is based on the formulation of a generalized eigenvalue problem and represents a viable computer-aided approach suitable for analyzing multibody mechanical systems having different degrees of complexity. Furthermore, an extension of the generalized coordinate partitioning algorithm is introduced in this paper for handling nonholonomic multibody systems leading to a robust and general multibody computational procedure referred to as the Robust Generalized Coordinate Partitioning Algorithm (RGCPA). Since the methodologies employed in this paper to study the stability of multibody mechanical systems are general and versatile, they can be easily implemented in general-purpose multibody computer programs and readily used to analyze several mechanical applications having engineering interest.

Keywords: Multibody System Dynamics, Holonomic and Nonholonomic Constraints, Robust Generalized Coordinate Partitioning Algorithm, Stability Analysis.

1. Introduction

In this section, an introduction to the issues addressed in this paper is reported. For this purpose, some background material is provided first to explain the significance of the present research work. Subsequently, the specific problems addressed in this investigation are formulated. Afterward, a concise literature survey on the topics of particular interest for the paper, such as the computer algorithms for simulating the dynamic behavior of multibody systems, is reported. Finally, the organization employed to structure the entire manuscript is given.

1.1 Background and Significance of the Present Research Work

The multibody approach to carry out a systematic dynamic analysis of articulated systems constrained by kinematic joints is a powerful tool that can be successfully used to model a large variety of mechanical systems, and it is also well suited for the computer implementation of the mathematical models of these complex systems [1,2]. The dynamics of multibody systems is described by differential-algebraic equations, which are characterized by highly nonlinearities that arise to describe the large rotational motion of the rigid bodies, the presence of force elements modeling the action of mechanical components, as well as the influence of nonlinear force fields that represent the interactions of the bodies with the external environment [3,4]. Apart from vehicle dynamics and robotics [5–9], the systematic approach for the analytical generation of the differential-algebraic equations of motion and their subsequent numerical implementation, typically employed for modeling articulated mechanical systems within the multibody framework [10,11], is widely used also in several biomechanical applications [12–14]. Thus, appropriate analytical approaches and effective computational procedures become necessary for properly describing the dynamic behavior of such complex systems [15].

The design and development of complex mechanical systems in general, and in particular of two-wheeled vehicles, is a challenging engineering task [16,17]. For this purpose, the use of a mathematical model of the physical system of interest based on the multibody approach to the system dynamics can simplify the design process and significantly reduce the production cost and the entire duration and effort of the engineering endeavor. For these reasons, a good model that can describe the fundamental aspects of the mechanical system of interest can be really useful for engineering applications. Besides, an appropriate model can also facilitate the designer to control the fundamental parameters of the mechanical system under study so that the virtual prototype can be

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easily modified to analyze the desired aspects and reach the prescribed performance. Therefore, for its generality and the relative ease of numerical implementation of the resulting dynamic model, the multibody approach for the dynamic analysis of mechanical systems constrained by kinematic joints suits well these objectives in general [18–23].

1.2 Formulation of the Problem of Interest for this Investigation

From a broad perspective, the stability analysis of mechanical systems can be accomplished by using two general methods, namely, a direct approach, that is based on the dynamic analysis through analytical solutions/computer simulations of the mathematical model of the system of interest in different scenarios of engineering relevance, or, on the other hand, by employing an indirect approach, that is based on the modal analysis of a linearized version of the system equations of motion, as well as considering the general Lyapunov stability methods. In the latter case, the linearization and the consequent stability analysis can be challenging for complex multibody systems, which may present both holonomic and nonholonomic algebraic constraints acting simultaneously, and special analytical approaches and computational algorithms are required to address this important issue properly. However, in general, the analytical approaches found in the literature for the solution of this fundamental problem are based on direct methods that are computationally expensive, or on the use of indirect methods based on a minimal constraint-free analytical description of the equations of motion, which becomes too involved or inapplicable when large multibody mechanical systems, such as vehicles or complex machines and mechanisms, are analyzed. Thus, in this paper, an effective analytical method is proposed to overcome these difficulties using an indirect approach. A robust numerical procedure for performing the dynamic analysis is also introduced to verify the numerical results found from the stability analysis through a direct simulation approach.

The basic mechanical model of a bicycle is an example of a simple mechanical system that posses well-known and interesting complex dynamical characteristics. In particular, many authors studied the dynamic behavior of the Whipple-Carvallo bicycle system, which is considered as a common reference, representing one of the most fundamental bicycle models [24, 25]. For instance, Meijaard et al. proposed an interesting benchmark model for this system that became a reference model for the analysis of two-wheeled vehicles in general [16]. This benchmark bicycle model has all the features necessary to show the effectiveness of the approach presented in this paper for the dynamic analysis of multibody mechanical systems. The analytical method and the computational procedure presented in this investigation well suits general systems with both holonomic and nonholonomic constraints and, therefore, can be readily used to study different complex systems such as the Whipple-Carvallo model of the bicycle system considered in this investigation. Moreover, as shown in this paper, the mechanics of bicycle systems, and, more in general, the behavior of two-wheeled vehicles, is heavily influenced by their geometric and inertial parameters. As well-known, the resulting peculiarities in the dynamical behavior of two-wheeled systems mainly originate from gyroscopic effects. An appropriate geometric parametrization of the system of interest based on a virtual prototype, as well as the multibody formulation approach to the system dynamics, can address the increasing necessity of having accurate models and the need to simulate and define in advance the characteristics of a given mechanical system before its actual construction. To demonstrate this fact, in this two-part investigation, a parametric analysis of the Whipple-Carvallo bicycle system is made to understand if the multibody approach devised in this work can correctly capture some of the well-known dynamical behaviors of two-wheeled systems, which have already been investigated for motorcycles and bicycles in the literature [26–29].

The fundamental problem of interest for this investigation and the main contributions of the paper can be summarized as reported below.

(1) This investigation deals with the kinematic and dynamic analysis of multibody mechanical systems whose motion is limited by holonomic and/or nonholonomic constraints. To address this fundamental problem, a Lagrangian approach based on a redundant set of generalized coordinates, which is referred to as the Redundant Coordinate Formulation (RCF), is employed in this work. In the paper, considering as the starting point this sound and well-known formulation approach, the analytical derivation of the differential-algebraic set of equations of motion, together with its subsequent computer implementation, is performed for a general nonholonomic multibody system to lay the foundations for developing the case study analyzed in this two-part manuscript.

(2) More specifically, this work is concerned with the stability analysis of multibody mechanical systems. For this purpose, an effective analytical methodology is proposed in the paper to systematically obtain an appropriate generalized eigenvalue problem associated with a linearized version of the differential-algebraic equations of motion of the multibody system of interest. The linearization process is based on the index-three form of the multibody equations of motion obtained using the RCP and is carried out around a given configuration whose stability characteristics are of interest for the dynamic analysis.

(3) As far as the nonlinear dynamic analysis of multibody mechanical systems is concerned, the paper makes a contribution to this field by introducing an extension of the Robust Generalized Coordinate Partitioning Algorithm (RGCPA), an effective constraint stabilization method which is capable of handling complex sets of holonomic and nonholonomic algebraic constraints at the same time. The proposed computational procedure allows for executing accurate numerical simulations of the differential-algebraic equations of motion of multibody systems by enforcing the constraint equations at the position, velocity, and acceleration levels. It can be easily implemented through the development of a general-purpose multibody numerical algorithm. The RGCPA is massively used in the paper for the dynamical simulations of nonlinear nonholonomic multibody models.

As discussed above, the first part of this two-part research paper provides the analytical and computational background for the subsequent numerical analysis of the case study of interest.

1.3 Literature Review on Multibody Algorithms

The stability analysis of dynamical systems is an effective approach that can considerably simplify the study of a mechanical system since its dynamical behavior can be explored without the necessity of solving the differential equations of the associated mathematical model [30]. In this respect, the stability analysis of a set of Differential-Algebraic Equations (DAEs) can be particularly challenging. Besides, the models obtained with the multibody approach are typically nonlinear. For these reasons, the stability analysis of a multibody mechanical system cannot ignore the implementation of a proper linearization procedure. A typical approach is to perform the analysis on an equivalent model linearized around a predetermined configuration of equilibrium to study the stability of a nonlinear system. Several methods can be employed to accomplish this task [31]. For mechanical systems whose motion is restricted by kinematic pairs, a typical analysis method is focused on a priori mathematical manipulation devoted to the reformulation and subsequent elimination of the algebraic equations to obtain a set of Ordinary Differential Equations (ODEs), which is uniquely a function of the independent coordinates [32, 33]. However, for complex multibody mechanical systems composed of many rigid bodies and kinematic joints, a more practical choice to solve the problem can be using a properly simplified dynamical model. In this respect, the work of Ripepi and Masarati specified a method to derive an appropriate reduced-order model, which can be effectively used to perform an eigenanalysis of the dynamical system under investigation [34]. Other authors proposed several interesting reduction techniques suitable for computer-aided analysis, as in the work of Lehner and Eberhard [35], and the paper of Koutsovasilis and Beitelschmidt [36].

There are substantially two diverse methodologies that can be followed to analyze the stability of constrained multibody mechanical systems. For simplicity, these two opposite strategies are called in this paper the direct analysis and the indirect analysis. The first approach (direct method) is based on the numerical simulation of the system dynamics, which is perturbed around the configuration of interest for the stability analysis, and on the consequent study of its resulting dynamical behavior [37]. Conversely, the second strategy (indirect method) is based on the numerical resolution of a particular eigenproblem related to the original system of differential equations or, in the alternative, leverages the use of one of the more general Lyapunov stability methods [31]. There is no need to perform any dynamical simulation in this second case, leading to a more compact and efficient solution algorithm. Examples that belong to the first type of approach can be found in a wide range of studies. For example, Nikravesh and Gim studied in [38] the dynamical performance of a race car by using appropriate numerical simulations carried out employing the multibody model developed in their work. In [39], Kim et al. explored the stability characteristics of a flying insect using the direct approach and compared the numerical results obtained from their dynamical model with those found by Sun et al. in [40], where an indirect technique was applied. Furthermore, other interesting demonstrations of the direct analysis using multibody models can be found in [41,42]. Regarding the indirect analysis, several authors have developed different interesting ideas and viable methods. In [43], Escalona and Chamorro devised a formulation to study the stability of a vehicle along a given trajectory as an alternative approach to the established Floquet theory. The work of Masarati et al. focuses on alternative methods to identify the eigenvalues which characterize the dynamical behavior of a mechanical system [44,45]. In [46,47], Negrut, and Ortiz developed a computational technique that is analytically equivalent to a state-space formulation and is founded on the index-three form of the DAEs representing the multibody system to be analyzed. Another indirect method elaborated for the investigation of the coupled dynamics of a vehicle with the sloshing inertial incidence can be found in the paper of Nichkawde et al. [48], while in [49] Bencsik et al. present a procedure to define the parameters of a controller for a mechanical system starting from its stability analysis.

The achievement of an accurate numerical solution for the dynamics of complex multibody systems is still a challenging task in the field of computer-aided analysis and engineering. Several researchers have explored different approaches for solving this issue during the years, starting from the pioneering work of Haug and his coworkers [4, 50]. In particular, a revisited version of the generalized coordinate partitioning method, proposed by the authors in previous investigations and extended in this paper for the first time to the case of nonholonomic multibody mechanical systems [51], is developed in this work starting from the original generalized partitioning method devised by Wehage and Haug in [52], and subsequently studied by Nikravesh and Haug in [53, 54], who also considered nonholonomic constraints and recognized the difficulty encountered in addressing this challenging problem. To address this issue, Wehage et al. proposed in [55] a strategy based on kinematic substructuring for a complex mechanism. Nada and Bashiri discussed in [56] a method based on the selection of generalized coordinates. Marques et al. analyzed in [57] different strategies to face the problem of constraints violation, including the partitioning method. In this respect, Masarati discussed in [58] a method to add kinematic constraints to a defined mechanical system, assuring that they are also satisfied up to the second-order. Terze and Naudet proposed in [59] an optimization procedure that can be applied to both holonomic and nonholonomic systems. Several other interesting research work on nonholonomic mechanical systems can also be found in the literature [60–62].

1.4 Organization of the Manuscript

The subsequent sections of this manuscript are organized as follows. In Section 2., the equations of motion of multibody mechanical systems are briefly recalled and the analytical techniques employed to systematically linearize a nonlinear set of differential-algebraic equations of motion expressed in the index-three form are described. In Section 3., the computational strategy adopted in this work for obtaining an accurate numerical solution for the differential-algebraic motion equations that consider both holonomic and nonholonomic constraints is presented. In Section 4., a summary of the work done, the conclusions of this paper, and some possible directions for future research are provided.

2. Analytical Methodology

In this section, the analytical methods and the multibody algorithms that allowed for developing the linear and nonlinear differential-algebraic equations of motion considered in this paper are reported.

2.1 Equations of Motion of Multibody Mechanical Systems

In this subsection, the equations of motion governing the dynamic behavior of multibody mechanical systems subjected to special motion restrictions described by holonomic and nonholonomic algebraic constraints are formulated. Multibody mechanical systems are formed by rigid bodies, kinematic joints, actuation elements, and force fields. To this end, consider a general multibody system in a three-dimensional space composed of N_b rigid bodies whose mechanical configuration is described by $n_q = N_b n_r$ geometric parameters, where n_r represents the number of reference generalized coordinates associated with each rigid body. Thus, the configuration of the entire multibody system under consideration is identified by a set of geometric parameters that forms a vector of generalized coordinates denoted with $\boldsymbol{q} \equiv \boldsymbol{q}(t) \in \Re^{n_q}$, where t indicates the independent time variable. Two general approaches can be followed in the formulation of the equations of motion of a given multibody system. Namely, one can employ a Minimal Coordinate Formulation (MCF) or a Redundant Coordinate Formulation (RCF). As the semantic definition suggests, in the former case, the number of generalized coordinates n_q used in the kinematic description is equal to the number of the system degrees of freedom denoted with n_f. In contrast, in the latter case, the formulation of the problem is based on a set of geometric parameters whose dimensions exceed the number of parameters strictly necessary for the description of the motion. While, in the case of the MCF, the resulting set of equations of motion is a compact set of nonlinear Ordinary Differential Equations (ODEs), the RCF leads to a larger set of nonlinear Differential-Algebraic Equations (DAEs) for the description of the dynamical model of the same multibody system. However, the MCF is challenging to be applied when dealing with large systems composed of several closed chains. Conversely, the RCF can be systematically formulated also in the case of complex multibody mechanical systems featuring open-loop and/or closed-loop topologies.

Adopting the MCF and considering the basic principle of classical mechanics, one obtains the following minimal set of dynamic equations:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_v + \boldsymbol{Q}_e \tag{1}$$

where $\boldsymbol{M} \equiv \boldsymbol{M}(\boldsymbol{q}, t) \in \Re^{n_q \times n_q}$ represents the mass matrix of the multibody system, $\boldsymbol{Q}_v \equiv \boldsymbol{Q}_v(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_q}$ is the system inertia quadratic velocity vector that absorbs the generalized inertial forces that are quadratic in the generalized velocities, such as the centrifugal and Coriolis inertial terms, and $\boldsymbol{Q}_e \equiv \boldsymbol{Q}_e(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_q}$ denotes the generalized external force vector that includes the action of the external forces and torques applied on the multibody system, as well as the dynamical effects induced by the force fields acting on the multibody system.

On the other hand, the formulation of the equations of motion of a multibody system based on the RCF exploits the systematic definition of the algebraic equations modeling the motion restrictions imposed on the system. For this purpose, two families of algebraic constraints can be distinguished, namely the holonomic algebraic constraints and the nonholonomic algebraic constraints. The holonomic constraints form a nonlinear set of $n_{c,h}$ algebraic equations defined at the position level involving the system generalized coordinate vector, which can be grouped in an holonomic constraint vector denoted with $\boldsymbol{C} \equiv \boldsymbol{C}(\boldsymbol{q}, t) \in \Re^{n_{c,h}}$. Conversely, the nonholonomic constraints form a nonlinear set of $n_{c,nh}$ algebraic equations defined at the velocity level involving the system generalized coordinate and velocity vectors, which can be grouped in a nonholonomic constraint vector denoted with $\boldsymbol{D} \equiv \boldsymbol{D}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_{c,nh}}$. Therefore, for a general multibody mechanical system described by using the RCF, the two sets of holonomic and nonholonomic constraint equations lead to the formulation of the following separate sets of algebraic equations:

$$\boldsymbol{C} = \boldsymbol{0}, \quad \boldsymbol{D} = \boldsymbol{0} \tag{2}$$

In order to adjoin the algebraic constraints to the equations of motion, the method of Lagrange multipliers can be readily employed. To this end, the first step consists of defining an additional vector of unknown variables for each set of algebraic constraints. Let $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}(t) \in \Re^{n_{c,h}}$ be a vector of Lagrange multipliers associated with the set of holonomic constraints. Similarly, let $\boldsymbol{\mu} \equiv \boldsymbol{\mu}(t) \in \Re^{n_{c,nh}}$ be a vector of Lagrange multipliers associated with the set of nonholonomic constraints. Subsequently, define the holonomic constraint Jacobian matrix as the Jacobian matrix of the vector of holonomic constraints computed with respect to the vector of generalized coordinates indicated as $\boldsymbol{C}_{\boldsymbol{q}} = \partial \boldsymbol{C}/\partial \boldsymbol{q} \equiv \boldsymbol{C}_{\boldsymbol{q}}(\boldsymbol{q}, t) \in \Re^{n_{c,h} \times n_q}$. In analogy with the case of holonomic algebraic constraints, define the nonholonomic constraint Jacobian matrix as the Jecobian matrix of the vector of nonholonomic constraints computed with respect to the vector of generalized velocities indicated as $\boldsymbol{D}_{\dot{\boldsymbol{q}}} = \partial \boldsymbol{D}/\partial \dot{\boldsymbol{q}} \equiv \boldsymbol{D}_{\dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_{c,nh} \times n_q}$.

It can be easily demonstrated that the application of the classical principles of analytical dynamics based on the RCF, namely the D'Alembert-Lagrange principle of virtual work employed in conjunction with the use of the technique of Lagrange multipliers, yields the following redundant set of differential-algebraic equations of motion:

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_v + \boldsymbol{Q}_e - \boldsymbol{C}_{\boldsymbol{q}}^T \boldsymbol{\lambda} - \boldsymbol{D}_{\dot{\boldsymbol{q}}}^T \boldsymbol{\mu} \\ \boldsymbol{C} = \boldsymbol{0} \\ \boldsymbol{D} = \boldsymbol{0} \end{cases} \tag{3}$$

where again, considering an appropriate change in the system dimensions and in the definition of the matrix and vector terms that appear in the multibody model, $\boldsymbol{M} \equiv \boldsymbol{M}(\boldsymbol{q}, t) \in \Re^{n_q \times n_q}$ identifies the system mass matrix, $\boldsymbol{Q}_v \equiv \boldsymbol{Q}_v(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_q}$ represents the system inertia quadratic velocity vector, and $\boldsymbol{Q}_e \equiv \boldsymbol{Q}_e(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_q}$ is the system generalized external force vector. The system of dynamic equations resulting from the use of the RCF forms an index-three set of differential-algebraic equations in which the generalized constraint force vectors associated with the sets of holonomic and nonholonomic constraints are respectively denoted with $\boldsymbol{Q}_{c,h} \equiv \boldsymbol{Q}_{c,h}(\boldsymbol{q}, t) \in \Re^{n_q}$ and $\boldsymbol{Q}_{c,nh} \equiv \boldsymbol{Q}_{c,nh}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_q}$. The analytical form of these two vectors is given by:

$$\boldsymbol{Q}_{c,h} = -\boldsymbol{C}_{\boldsymbol{q}}^T \boldsymbol{\lambda}, \quad \boldsymbol{Q}_{c,nh} = -\boldsymbol{D}_{\dot{\boldsymbol{q}}}^T \boldsymbol{\mu} \tag{4}$$

Furthermore, the index-three form of the differential-algebraic set of equations of motion describing the nonlinear dynamics of a multibody mechanical system subjected to both holonomic and nonholonomic constraints can be properly transformed into its index-one counterpart by substituting the set of holonomic constraints with its second derivative with respect to time and by substituting the set of nonholonomic constraints with its first derivative with respect to time. To achieve this goal for the two sets of holonomic and nonholonomic algebraic constraints, one can respectively write:

$$\boldsymbol{C} = \boldsymbol{0} \quad \Rightarrow \quad \ddot{\boldsymbol{C}} = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{C}_{\boldsymbol{q}}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_{d,h} \tag{5}$$

and

$$\boldsymbol{D} = \boldsymbol{0} \quad \Rightarrow \quad \dot{\boldsymbol{D}} = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{D}_{\dot{\boldsymbol{q}}}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_{d,nh} \tag{6}$$

where:

$$\boldsymbol{Q}_{d,h} = -\frac{\partial^2 \boldsymbol{C}}{\partial t^2} - \left(\frac{\partial \left(\boldsymbol{C}_{\boldsymbol{q}}\dot{\boldsymbol{q}}\right)}{\partial \boldsymbol{q}}\right)\dot{\boldsymbol{q}} - 2\frac{\partial^2 \boldsymbol{C}}{\partial \boldsymbol{q}\partial t}\dot{\boldsymbol{q}} = -\boldsymbol{C}_{t,t} - \left(\boldsymbol{C}_{\boldsymbol{q}}\dot{\boldsymbol{q}}\right)_{\boldsymbol{q}}\dot{\boldsymbol{q}} - 2\boldsymbol{C}_{\boldsymbol{q},t}\dot{\boldsymbol{q}} \tag{7}$$

and

$$\boldsymbol{Q}_{d,nh} = -\frac{\partial \boldsymbol{D}}{\partial t} - \frac{\partial \boldsymbol{D}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = -\boldsymbol{D}_t - \boldsymbol{D}_{\boldsymbol{q}}\dot{\boldsymbol{q}} \tag{8}$$

where $\boldsymbol{Q}_{d,h} \equiv \boldsymbol{Q}_{d,h}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_{c,h}}$ is the holonomic constraint quadratic velocity vector and $\boldsymbol{Q}_{d,nh} \equiv \boldsymbol{Q}_{d,nh}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_{c,nh}}$ is the nonholonomic constraint quadratic velocity vector. By following this approach, the index-one set of differential-algebraic equations that governs the nonlinear motion of a general multibody mechanical system can be written as:

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_v + \boldsymbol{Q}_e - \boldsymbol{C}_{\boldsymbol{q}}^T \boldsymbol{\lambda} - \boldsymbol{D}_{\dot{\boldsymbol{q}}}^T \boldsymbol{\mu} \\ \boldsymbol{C}_{\boldsymbol{q}}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_{d,h} \\ \boldsymbol{D}_{\dot{\boldsymbol{q}}}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_{d,nh} \end{cases} \tag{9}$$

By properly manipulating the index-one form of the multibody equations of motion, one obtains the following compact set of dynamic equations written in matrix form as follows:

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_b - \boldsymbol{J}^T \boldsymbol{v} \\ \boldsymbol{J}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_d \end{cases} \tag{10}$$

where:

$$\boldsymbol{Q}_b = \boldsymbol{Q}_v + \boldsymbol{Q}_e, \quad \boldsymbol{v} = \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{q}} \\ \boldsymbol{D}_{\dot{\boldsymbol{q}}} \end{bmatrix}, \quad \boldsymbol{Q}_d = \begin{bmatrix} \boldsymbol{Q}_{d,h} \\ \boldsymbol{Q}_{d,nh} \end{bmatrix} \tag{11}$$

where $n_c = n_{c,h} + n_{c,nh}$ indicates the total number of algebraic constraints, $\boldsymbol{Q}_b \equiv \boldsymbol{Q}_b(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_q}$ denotes the total vector of inertial and external generalized forces acting on the multibody system, $\boldsymbol{v} \equiv \boldsymbol{v}(t) \in \Re^{n_c}$ is the total vector of Lagrange multipliers, $\boldsymbol{J} \equiv \boldsymbol{J}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_c \times n_q}$ represents the total Jacobian matrix of the algebraic constraints, and $\boldsymbol{Q}_d \equiv \boldsymbol{Q}_d(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_c}$ identifies the total constraint quadratic velocity vector associated with the entire set of holonomic and nonholonomic algebraic constraints.

As discussed below in the manuscript, the index-three form of the equations of motion derived adopting a systematic approach based on the RCF, as well as being much easier to obtain in a compact matrix form even for complex closed-chain mechanical systems, can be directly linearized employing an effective analytical method considered in this investigation. By doing so, one can readily perform the modal analysis of a multibody system considering its linearized dynamical model and obtain the system natural frequencies and modes of vibration. Besides, the index-one set of differential-algebraic equations of motion, analytically derived herein considering the RCF, can be readily implemented in a general-purpose multibody computational algorithm for the dynamic analysis of nonlinear multibody mechanical systems. Thus, the marching on the time grid of the numerical solution of the motion equations can be effectively carried out using an appropriate numerical integration scheme.

2.2 Linearization of the Equations of Motion

In this subsection, an effective linearization method is introduced and used to obtain a linear version of the index-three equations of motion of a general multibody system considering small perturbations around a given configuration of interest. By employing the proposed approach, one can readily define and solve the stability problem of a general multibody mechanical system by analyzing its dynamic behavior considering the linearized equations of motion. To this end, the nonlinear differential equations and the nonlinear algebraic equations that form the dynamical model of a general multibody mechanical system are considered at the same time in the linearization process.

Define the integer number $n_x = n_q + n_c$ and let $\boldsymbol{x} \equiv \boldsymbol{x}(t) \in \Re^{n_x}$ be a composite vector containing the generalized coordinates and the Lagrange multipliers associated with the multibody system. Thus, for definition, the composite coordinate vector is formed by the vector of generalized coordinates together with the complete vector of Lagrange multipliers and is simply defined as:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{v} \end{bmatrix} \tag{12}$$

To apply the linearization method described herein, the equations of motion of a generic multibody system must be properly reformulated using a similar partition process. Following this strategy, the proposed approach is based on the index-three form of the differential-algebraic dynamic equations, which can be rewritten as follows:

$$\begin{cases} \boldsymbol{F} = \boldsymbol{0} \\ \boldsymbol{E} = \boldsymbol{0} \end{cases} \tag{13}$$

where:

$$\boldsymbol{F} = \boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{Q}_b + \boldsymbol{J}^T\boldsymbol{v}, \quad \boldsymbol{E} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{D} \end{bmatrix} \tag{14}$$

where $\boldsymbol{F} \equiv \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{v}, t) \in \Re^{n_q}$ denotes a nonlinear vector function containing the differential part of the equations of motion, while $\boldsymbol{E} \equiv \boldsymbol{E}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_c}$ represents a nonlinear vector function that contains the algebraic part of the equations of motion. By doing so, the equations of motion can be rewritten in an implicit matrix form and can be further rearranged in a more compact formulation given by:

$$\boldsymbol{f} = \boldsymbol{0}, \quad \boldsymbol{f} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{E} \end{bmatrix} \tag{15}$$

where $\boldsymbol{f} \equiv \boldsymbol{f}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}, t) \in \Re^{n_x}$ identifies a nonlinear vector function in which the complete set of differential-algebraic equations of motion is implicitly embedded.

The introduction of the nonlinear vector function $\boldsymbol{f}$ allows for performing a direct formulation of the stability problem, which is, in turn, expressed employing a generalized eigenvalue problem associated with the linearized equations of motion. For this purpose, consider the composite vector $\boldsymbol{x}_0$ representing the reference configuration to be used in the stability analysis. The reference composite vector $\boldsymbol{x}_0$ is made of the generalized coordinate vector $\boldsymbol{q}_0$, that identifies the reference configuration around which the stability analysis is performed, and contains the Lagrange multiplier vector $\boldsymbol{v}_0$, which embeds the reference Lagrangian multipliers that can be readily determined in correspondence to the preassigned reference configuration. In order to derive the linearized version of the equations of motion, consider a perturbation of the composite coordinate vector denoted with $\bar{\boldsymbol{x}} \equiv \bar{\boldsymbol{x}}(t) \in \Re^{n_x}$ that is defined as follows:

$$\bar{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0 \tag{16}$$

The Taylor expansion truncated at the first order of the implicit form of the equations of motion associated with the perturbation of the composite coordinate vector around an assigned reference configuration results in the following vector equation:

$$\boldsymbol{f} \simeq \boldsymbol{f}|_0 + \boldsymbol{f}_{\boldsymbol{x}}|_0\bar{\boldsymbol{x}} + \boldsymbol{f}_{\dot{\boldsymbol{x}}}|_0\dot{\bar{\boldsymbol{x}}} + \boldsymbol{f}_{\ddot{\boldsymbol{x}}}|_0\ddot{\bar{\boldsymbol{x}}} = \boldsymbol{0} \tag{17}$$

where:

$$\boldsymbol{f}|_0 = \boldsymbol{f}(\boldsymbol{x}_0, \dot{\boldsymbol{x}}_0, \ddot{\boldsymbol{x}}_0, t_0), \quad \begin{cases} \boldsymbol{f}_{\boldsymbol{x}}|_0 = \left.\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_0, \dot{\boldsymbol{x}}_0, \ddot{\boldsymbol{x}}_0, t_0} \\ \boldsymbol{f}_{\dot{\boldsymbol{x}}}|_0 = \left.\dfrac{\partial \boldsymbol{f}}{\partial \dot{\boldsymbol{x}}}\right|_{\boldsymbol{x}_0, \dot{\boldsymbol{x}}_0, \ddot{\boldsymbol{x}}_0, t_0} \\ \boldsymbol{f}_{\ddot{\boldsymbol{x}}}|_0 = \left.\dfrac{\partial \boldsymbol{f}}{\partial \ddot{\boldsymbol{x}}}\right|_{\boldsymbol{x}_0, \dot{\boldsymbol{x}}_0, \ddot{\boldsymbol{x}}_0, t_0} \end{cases} \tag{18}$$

and

$$\bar{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0, \quad \dot{\bar{\boldsymbol{x}}} = \dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_0, \quad \ddot{\bar{\boldsymbol{x}}} = \ddot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}_0 \tag{19}$$

In particular, it is assumed that the reference configuration can equally represent a preassigned configuration point or a given trajectory. In either case, since the composite reference vector, which represents the configuration around which the linearization of the equations of motion is performed, is indeed a solution of the nonlinear set of differential-algebraic motion equations, the following relation stands:

$$\boldsymbol{f}(\boldsymbol{x}_0, \dot{\boldsymbol{x}}_0, \ddot{\boldsymbol{x}}_0, t_0) = \mathbf{0} \tag{20}$$

In light of this simplification, the truncation of the Taylor series expansion can be manipulated to obtain the following mathematical expression:

$$\bar{\boldsymbol{M}}_0 \ddot{\bar{\boldsymbol{x}}} + \bar{\boldsymbol{R}}_0 \dot{\bar{\boldsymbol{x}}} + \bar{\boldsymbol{K}}_0 \bar{\boldsymbol{x}} = \mathbf{0} \tag{21}$$

where $\bar{\boldsymbol{M}}_0 \equiv \bar{\boldsymbol{M}}_0(\boldsymbol{x}_0, t_0) \in \Re^{n_x \times n_x}$, $\bar{\boldsymbol{R}}_0 \equiv \bar{\boldsymbol{R}}_0(\boldsymbol{x}_0, \dot{\boldsymbol{x}}_0, t_0) \in \Re^{n_x \times n_x}$, and $\bar{\boldsymbol{K}}_0 \equiv \bar{\boldsymbol{K}}_0(\boldsymbol{x}_0, \dot{\boldsymbol{x}}_0, \ddot{\boldsymbol{x}}_0, t_0) \in \Re^{n_x \times n_x}$ respectively represent the composite mass, damping, and stiffness matrices of the multibody system linearized around the reference configuration of interest. The generic symbolic forms of the composite mass, damping, and stiffness matrices can be obtained as follows:

$$\bar{\boldsymbol{M}}_0 = \boldsymbol{f}_{\ddot{\boldsymbol{x}}}|_0, \quad \bar{\boldsymbol{R}}_0 = \boldsymbol{f}_{\dot{\boldsymbol{x}}}|_0, \quad \bar{\boldsymbol{K}}_0 = \boldsymbol{f}_{\boldsymbol{x}}|_0 \tag{22}$$

From a mathematical perspective, the composite mass, damping, and stiffness matrices can be assembled in block matrix forms. To this end, four distinct matrix blocks can be identified and used in the definition of these composite matrices. The separation into four independent matrix blocks is also advantageous in implementing the proposed method in a general-purpose multibody code.

The composite mass matrix can be calculated using the following block formulation:

$$\bar{\boldsymbol{M}}_0 = \boldsymbol{f}_{\ddot{\boldsymbol{x}}}|_0 = \begin{bmatrix} \boldsymbol{F}_{\ddot{\boldsymbol{q}}}\big|_0 & \boldsymbol{F}_{\ddot{\boldsymbol{v}}}|_0 \\ \boldsymbol{E}_{\ddot{\boldsymbol{q}}}\big|_0 & \boldsymbol{E}_{\ddot{\boldsymbol{v}}}|_0 \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{M}}^0_{1,1} & \bar{\boldsymbol{M}}^0_{1,2} \\ \bar{\boldsymbol{M}}^0_{2,1} & \bar{\boldsymbol{M}}^0_{2,2} \end{bmatrix} \tag{23}$$

where:

$$\begin{cases} \bar{\boldsymbol{M}}^0_{1,1} = \boldsymbol{M}|_0, \quad \bar{\boldsymbol{M}}^0_{1,2} = \boldsymbol{O} \\ \bar{\boldsymbol{M}}^0_{2,1} = \boldsymbol{O}, \quad \bar{\boldsymbol{M}}^0_{2,2} = \boldsymbol{O} \end{cases} \tag{24}$$

The composite damping matrix can be calculated using the following block formulation:

$$\bar{\boldsymbol{R}}_0 = \boldsymbol{f}_{\dot{\boldsymbol{x}}}|_0 = \begin{bmatrix} \boldsymbol{F}_{\dot{\boldsymbol{q}}}\big|_0 & \boldsymbol{F}_{\dot{\boldsymbol{v}}}|_0 \\ \boldsymbol{E}_{\dot{\boldsymbol{q}}}\big|_0 & \boldsymbol{E}_{\dot{\boldsymbol{v}}}|_0 \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{R}}^0_{1,1} & \bar{\boldsymbol{R}}^0_{1,2} \\ \bar{\boldsymbol{R}}^0_{2,1} & \bar{\boldsymbol{R}}^0_{2,2} \end{bmatrix} \tag{25}$$

where:

$$\begin{cases} \bar{\boldsymbol{R}}^0_{1,1} = -\frac{\partial \boldsymbol{Q}_b}{\partial \dot{\boldsymbol{q}}}\Big|_0 + \frac{\partial (\boldsymbol{J}^T \boldsymbol{v})}{\partial \dot{\boldsymbol{q}}}\Big|_0, \quad \bar{\boldsymbol{R}}^0_{1,2} = \boldsymbol{O} \\ \bar{\boldsymbol{R}}^0_{2,1} = \frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}}\Big|_0, \quad \bar{\boldsymbol{R}}^0_{2,2} = \boldsymbol{O} \end{cases} \tag{26}$$

The composite stiffness matrix can be calculated using the following block formulation:

$$\bar{\boldsymbol{K}}_0 = \boldsymbol{f}_{\boldsymbol{x}}|_0 = \begin{bmatrix} \boldsymbol{F}_{\boldsymbol{q}}|_0 & \boldsymbol{F}_{\boldsymbol{v}}|_0 \\ \boldsymbol{E}_{\boldsymbol{q}}|_0 & \boldsymbol{E}_{\boldsymbol{v}}|_0 \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{K}}^0_{1,1} & \bar{\boldsymbol{K}}^0_{1,2} \\ \bar{\boldsymbol{K}}^0_{2,1} & \bar{\boldsymbol{K}}^0_{2,2} \end{bmatrix} \tag{27}$$

where:

$$\begin{cases} \bar{\boldsymbol{K}}^0_{1,1} = \frac{\partial (\boldsymbol{M}\ddot{\boldsymbol{q}})}{\partial \boldsymbol{q}}\Big|_0 - \frac{\partial \boldsymbol{Q}_b}{\partial \boldsymbol{q}}\Big|_0 + \frac{\partial (\boldsymbol{J}^T \boldsymbol{v})}{\partial \boldsymbol{q}}\Big|_0, \quad \bar{\boldsymbol{K}}^0_{1,2} = \boldsymbol{J}^T\big|_0 \\ \bar{\boldsymbol{K}}^0_{2,1} = \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{q}}\Big|_0, \quad \bar{\boldsymbol{K}}^0_{2,2} = \boldsymbol{O} \end{cases} \tag{28}$$

Using the previous definitions of the composite mass, damping, and stiffness matrices based on block matrix forms, the generalized eigenvalues problem associated with the study of the stability of the linearized system of equations of motion can be easily implemented. To achieve this goal, the stability analysis based on the proposed linearization method can be carried out immediately after a state-space reformulation of the linear form of the system equations of motion is performed. To this end, consider a state vector denoted with $\bar{\boldsymbol{z}} \equiv \bar{\boldsymbol{z}}(t) \in \Re^{n_z}$, where $n_z = 2n_x$ represents the dimension of the state-space, that is formulated as follows:

$$\bar{\boldsymbol{z}} = \begin{bmatrix} \bar{\boldsymbol{x}} \\ \dot{\bar{\boldsymbol{x}}} \end{bmatrix} \tag{29}$$

By introducing the previous definition of the state vector, the generalized state-space formulation of the linearized equations of motion of a general multibody system is given by:

$$\bar{\boldsymbol{U}}_0 \dot{\bar{\boldsymbol{z}}} = \bar{\boldsymbol{V}}_0 \bar{\boldsymbol{z}} \tag{30}$$

where $\bar{\boldsymbol{z}}_0$ identifies the reference point of the state-space, while $\bar{\boldsymbol{U}}_0 \equiv \bar{\boldsymbol{U}}_0(\bar{\boldsymbol{z}}_0, t_0) \in \Re^{n_z \times n_z}$ and $\bar{\boldsymbol{V}}_0 \equiv \bar{\boldsymbol{V}}_0(\bar{\boldsymbol{z}}_0, t_0) \in \Re^{n_z \times n_z}$ represent the two transition matrices of the generalized state-space dynamic model that are respectively based on the following block matrix forms:

$$\bar{\boldsymbol{U}}_0 = \begin{bmatrix} \bar{\boldsymbol{R}}_0 & \bar{\boldsymbol{M}}_0 \\ \bar{\boldsymbol{M}}_0 & \boldsymbol{O} \end{bmatrix}, \quad \bar{\boldsymbol{V}}_0 = \begin{bmatrix} -\bar{\boldsymbol{K}}_0 & \boldsymbol{O} \\ \boldsymbol{O} & \bar{\boldsymbol{M}}_0 \end{bmatrix} \tag{31}$$

For the same multibody mechanical system, the eigenvalues calculated with the proposed approach based on the definition of an appropriate generalized state-space dynamical model exactly coincide with those of the equivalent standard state-space dynamical model. The only important difference in the two eigenvalue problems, namely the standard eigenvalue problem associated with the MCF and the generalized eigenvalue problem associated with the RCF, is the presence in the generalized eigenvalue problem of additional eigenvalues equal to the infinite correlated to the Lagrange multipliers embedded in the composite coordinate vector. Conversely, the eigenvectors cannot be the same for the two formulations of the eigenproblems since they are associated with two different state vectors.

To solve the generalized eigenvalue problem derived above, consider the following trial solution labeled with the integer number k:

$$\bar{\boldsymbol{z}} = \boldsymbol{\psi}_k e^{s_k}, \quad k = 1, 2, \ldots, n_z \tag{32}$$

where s_k is a complex scalar denoting the generic eigenvalue labeled with the integer number k and $\boldsymbol{\psi}_k$ represents the corresponding eigenvector. The substitution of the trial analytical solution into the generalized state-space model leads to the mathematical expressions reported below:

$$\bar{\boldsymbol{V}}_0\boldsymbol{\psi}_k = s_k\bar{\boldsymbol{U}}_0\boldsymbol{\psi}_k \quad \Leftrightarrow \quad \left(\bar{\boldsymbol{V}}_0 - s_k\bar{\boldsymbol{U}}_0\right)\boldsymbol{\psi}_k = \boldsymbol{0} \tag{33}$$

and

$$\det(\bar{\boldsymbol{V}}_0 - s_k\bar{\boldsymbol{U}}_0) = 0 \tag{34}$$

This set of algebraic equations identifies the basic equations representing the generalized eigenvalue problem associated with the stability of a general multibody mechanical system modeled by using the RCF. As for the standard eigenvalue problem, the last condition in the previous set of equations was imposed to obtain nontrivial solutions. As already said, the eigenvalues associated with the Lagrange multipliers found with the use of this method give no significant information about the stability of the system. Thus, only the remaining eigenvalues must be considered to study and investigate the stability of the multibody system under consideration.

As mentioned in the paper, taking the opportune cautions, the stability of a nonlinear system can be analyzed through the stability of its linearized counterpart obtained by performing a Taylor series expansion around the state or the trajectory of interest. In particular, if the equilibrium state of the linearized system is of the type referred to as center, namely, the linearized system oscillates with a constant amplitude, the behavior of the trajectory of the original nonlinear system is determined by the remaining terms of the Taylor series that were neglected during the linearization process. Only in this case, the linearized system analysis gives no definitive information about the dynamical behavior of the nonlinear system, and, therefore, a more sophisticated nonlinear stability analysis becomes necessary. From a more general perspective, consider a dynamical system described by the following nonlinear state-space equations:

$$\boldsymbol{H}\dot{\boldsymbol{z}} = \boldsymbol{h} \tag{35}$$

where $\boldsymbol{z} \equiv \boldsymbol{z}(t) \in \Re^{n_z}$ denotes the time-dependent state vector, $\boldsymbol{H} \equiv \boldsymbol{H}(\boldsymbol{z}, t) \in \Re^{n_z \times n_z}$ is the so-called state mass matrix, and $\boldsymbol{h} \equiv \boldsymbol{h}(\boldsymbol{z}, t) \in \Re^{n_z}$ represents the general nonlinear state function. The state $\boldsymbol{z}_0$ of a nonlinear dynamical system is a stable equilibrium point at the time instant t_0 if, for any given strictly positive scalar $\varepsilon > 0$, there exists another strictly positive scalar $\delta \equiv \delta(\varepsilon, t_0) > 0$ such that if $\|\boldsymbol{z}(t_0) - \boldsymbol{z}_0\| < \delta$ then $\|\boldsymbol{z}(t) - \boldsymbol{z}_0\| < \varepsilon$ for all $t \geq t_0$. A nonlinear dynamical system that is stable according to this definition is also called stable in the sense of Lyapunov. In particular, if δ is independent of t_0, the system is said to be uniformly stable in the sense of Lyapunov. Moreover, the state $\boldsymbol{z}_0$ of a nonlinear dynamical system is an asymptotically stable equilibrium point at the time instant t_0 if it is stable in the sense of Lyapunov and if there exists a strictly positive scalar $\chi \equiv \chi(t_0) > 0$ such that if $\|\boldsymbol{z}(t_0) - \boldsymbol{z}_0\| < \chi$ then $\lim\limits_{t\to\infty} \|\boldsymbol{z}(t) - \boldsymbol{z}_0\| = 0$. If χ is independent of the initial time instant t_0 as well, the system is said to be uniformly asymptotically stable in the sense of Lyapunov.

In any other case, however, the stability analysis of the linearized system provides insights into the stability features of the original nonlinear dynamical system. Therefore, the linearization approach described in this section that leads to the formulation of a generalized eigenvalue problem plays a central role in the stability analysis of mechanical systems constrained by kinematic pairs. Finally, since it is based on relatively simple symbolic manipulations and on standard numeric calculations that can be efficiently handled with the computational tools readily available today, the proposed linearization approach based on the generalized eigenvalue problem can be easily implemented in general-purpose multibody computer codes and applied to the analysis of complex mechanical systems.

3. Computational Algorithm

In this section, the fundamental computational tools necessary to efficiently and effectively obtain an accurate numerical solution of the differential-algebraic equations of motion of multibody mechanical systems subjected to holonomic and/or nonholonomic constraints are presented.

3.1 Multibody Solution Procedure

In this subsection, the complete computational algorithm used for the numerical solutions of the equations of motion is presented. In general, since the complex dynamic behavior of a given multibody mechanical system is governed by a nonlinear set of differential-algebraic equations of motion, a robust and reliable computational algorithm is required for achieving an effective and efficient numerical solution. In Figure 1, the flowchart of the multibody algorithm used in this paper for formulating and solving the differential-algebraic equations of motion is illustrated.

The general flowchart of the multibody solution procedure employed in this investigation, shown in Figure 1, can be summarized in the following five essential steps. The first step is the analytical formulation of the differential-algebraic equations of motion of the multibody system of interest, which is performed in advance at the preprocessing stage before entering the numerical solution loop. In this paper, an analytical approach based on the Redundant Coordinate Formulation (RCF) is employed to derive the differential-algebraic motion equations. The second step aims to define a set of initial conditions for the multibody system that must be consistent with the algebraic constraints modeling the kinematic joints associated with position-level algebraic equations and complying with the particular limitations of the dynamical behavior imposed at the velocity level. Like the first phase of the computational procedure, the second step is carried out only once at a preprocessing stage, before the actual beginning of the numerical solution process necessary for performing the dynamical simulation. The third step of the computational algorithm focuses on stabilizing the drift of the holonomic and/or nonholonomic constraint equations. This crucial step represents the real starting point of the iterative procedure devoted to forwarding the numerical solution of the equations of motion on the time grid. As discussed below in this section, the computational technique considered in this research work for stabilizing the constraint equations is the Robust Generalized Coordinate Partitioning Algorithm (RGCPA) and its variant specifically devised in this paper for handling both holonomic and nonholonomic algebraic constraints at the same time. The fourth step of the computational procedure is devoted to calculating the vector of generalized accelerations of the multibody mechanical system necessary to define the system state function. To this end, a simple but effective matrix method based on the Augmented Formulation (AF) is employed considering the index-one form of the differential-algebraic equations of motion. Finally, in the fifth and last step of the computational procedure, the actual numerical integration of the motion equations is carried out to compute the system state vector corresponding to the next point of the discretized time axis. In particular, a numerical integration scheme based on the Adams-Bashforth Method (ABM) is used to achieve this goal, considering a standard state-space representation of the dynamical equations. By doing so, one obtains an array of real numbers containing an approximation of the numerical solution of the equations of motion corresponding to a given point on the time grid. The last three steps of the solution procedure are consecutively repeated until the entire time span of the dynamical simulation is covered. Since they represent the core part of the entire computational procedure employed to derive the

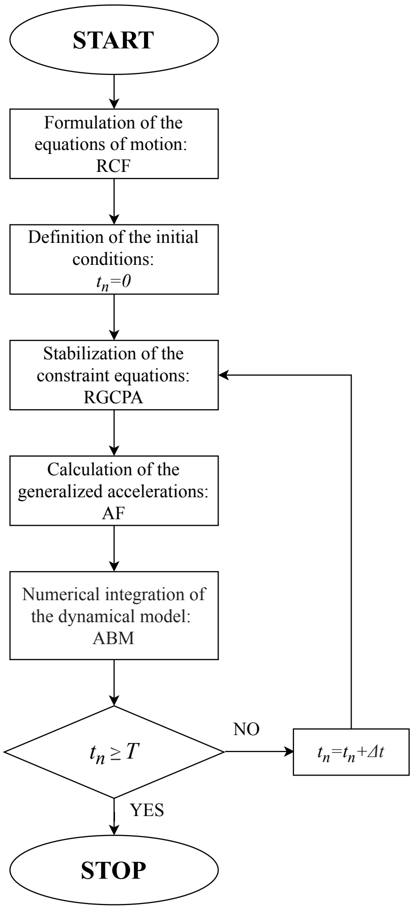


Figure 1. *Multibody solution procedure.*

numerical solution of the nonlinear set of differential-algebraic equations of motion, the last three steps of the multibody flowchart shown in Figure 1 are discussed in detail below.

3.2 Numerical Integration of the Equations of Motion

In this subsection, the numerical integration scheme employed in this paper for solving the equations of motion of index-one multibody systems is concisely described. To this end, the index-one nonlinear set of DAEs representing the multibody equations of motion can be readily rewritten in the following matrix form that is referred to as the Augmented Formulation (AF):

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{J}^T\boldsymbol{v} = \boldsymbol{Q}_b \\ \boldsymbol{J}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_d \end{cases} \Leftrightarrow \begin{bmatrix} \boldsymbol{M} & \boldsymbol{J}^T \\ \boldsymbol{J} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}_b \\ \boldsymbol{Q}_d \end{bmatrix} \tag{36}$$

or

$$\boldsymbol{M}_a\boldsymbol{q}_a = \boldsymbol{Q}_a \tag{37}$$

where the integer $n_a = n_q + n_c$ represents the characteristic dimension of the AF, $\boldsymbol{M}_a \equiv \boldsymbol{M}_a(\boldsymbol{q}, t) \in \Re^{n_a \times n_a}$ is the augmented mass matrix, $\boldsymbol{q}_a \equiv \boldsymbol{q}_a(t) \in \Re^{n_a}$ denotes the augmented generalized coordinate vector, and $\boldsymbol{Q}_a \equiv \boldsymbol{Q}_a(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_a}$ identifies an augmented vector of generalized forces. The vector and matrix quantities employed in the AF are respectively given by:

$$\boldsymbol{M}_a = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{J}^T \\ \boldsymbol{J} & \boldsymbol{O} \end{bmatrix}, \quad \boldsymbol{q}_a = \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{v} \end{bmatrix}, \quad \boldsymbol{Q}_a = \begin{bmatrix} \boldsymbol{Q}_b \\ \boldsymbol{Q}_d \end{bmatrix} \tag{38}$$

By solving the equations of motion rewritten according to the AF, which represents a system of linear equations defined at each time step of the dynamical simulation, one obtains the augmented generalized coordinate vector $\boldsymbol{q}_a$ associated with the multibody system equations of motion containing the vector of generalized accelerations $\ddot{\boldsymbol{q}}$ and the vector of Lagrange multipliers $\boldsymbol{v}$. While the physical information contained in the Lagrange multiplier vector can be used for calculating the generalized constraint forces relative to the kinematic joints acting on the multibody system, one can use the computation of the generalized acceleration vector of the mechanical system for the definition of the state function necessary for the progressive marching of the numerical simulation on the time grid considering a standard numerical integration algorithm. For this purpose, let $\boldsymbol{y} \equiv \boldsymbol{y}(t) \in \Re^{n_y}$ be the state vector associated with the state-space form of the index-one set of DAEs representing the multibody equations of motion given by:

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} \tag{39}$$

where $n_y = 2n_q$ is the dimension of the state vector. The previous straightforward introduction of the state vector denoted with $\boldsymbol{y}$

allows for obtaining the following mathematical formulation of the dynamical problem associated with the equations of motion:

$$\begin{cases} \dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}, t) \\ \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases} \tag{40}$$

where $\boldsymbol{y}_0$ represents the vector of initial conditions that identify the initial configuration of the multibody system and its initial set of generalized velocities, while the nonlinear vector function denoted with $\boldsymbol{g} \equiv \boldsymbol{g}(\boldsymbol{y}, t) \in \Re^{n_y}$ is the state function of the multibody system. These vectors are respectively defined as follows:

$$\boldsymbol{y}_0 = \begin{bmatrix} \boldsymbol{q}_0 \\ \dot{\boldsymbol{q}}_0 \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} \end{bmatrix} \tag{41}$$

where $\boldsymbol{q}_0$ and $\dot{\boldsymbol{q}}_0$ respectively represent the vectors containing the initial generalized coordinates and velocities of the multibody system, while the system generalized acceleration vector can be readily determined by using the computational method based on the AF mentioned before.

Considering the state-space representation of the equations of motion of a general multibody mechanical system, provided that an appropriate constraint stabilization algorithm is implemented in the iterative loop representing the multibody solution procedure, to obtain an approximate solution of the index-one set of DAEs that governs the dynamics of the multibody system under study, one can directly use a proper integration method that is selected between the robust numerical integration schemes developed for solving a system of ODEs. To this end, a fourth-order explicit multistep scheme based on the Adams-Bashforth Method (ABM) is employed in this paper, which can be summarized by considering the following integration formula:

$$\begin{aligned} \boldsymbol{Y}_{n+4} = \boldsymbol{Y}_{n+3} + \tfrac{\Delta t}{24}\left(55\boldsymbol{g}(\boldsymbol{Y}_{n+3}, t_{n+3}) - 59\boldsymbol{g}(\boldsymbol{Y}_{n+2}, t_{n+2})\right) \\ + \tfrac{\Delta t}{24}\left(37\boldsymbol{g}(\boldsymbol{Y}_{n+1}, t_{n+1}) - 9\boldsymbol{g}(\boldsymbol{Y}_{n}, t_{n})\right) \end{aligned} \tag{42}$$

where n is an integer number associated with the time discretization, the scalar quantity denoted with Δt represents the discretized time interval employed in the numerical integration scheme based on the ABM, while the discrete vector denoted with $\boldsymbol{Y}_{n+i}$ identifies the numerical approximation of the exact solution for the exact state vector $\boldsymbol{y}_n = \boldsymbol{y}(t_n)$ defined at the discretized time instant $t_{n+i} = (n+i)\,\Delta t$, and i is a positive integer number. As discussed below, since the AF is a computational method for the determination of the system generalized acceleration vector based on an index-one form of the differential-algebraic equations of motion necessary for the computer implementation of the ABM, it is important to emphasize that an effective constraint stabilization algorithm is also required to complete the multibody solution procedure correctly. While the enforcement of the constraint equations at the acceleration level is ensured through the use of the AF, the chosen constraint stabilization algorithm must be implemented in conjunction with the numerical integration scheme employed for the step-by-step computation of the numerical solution on the discretized time axis to eliminate the detrimental drift of the holonomic and nonholonomic algebraic constraints at both the position and velocity levels.

3.3 Robust Generalized Coordinate Partitioning Algorithm for Holonomic Multibody Systems

In this subsection, the fundamental aspects of the Robust Generalized Coordinate Partitioning Algorithm (RGCPA) [51, 63], developed as a constraint stabilization method for holonomic multibody systems starting from the original version of the generalized coordinate partitioning method [52], are recalled to clarify the subsequent derivation of the same computational technique in the case of nonholonomic multibody systems. To this end, consider a multibody mechanical system described by n_q generalized coordinates embedded in the generalized coordinate vector $\boldsymbol{q}$ and subjected only to a nonlinear set of $n_{c,h}$ holonomic constraints grouped in the holonomic constraint vector $\boldsymbol{C}$ associated with a vector of Lagrange multipliers denoted as $\boldsymbol{\lambda}$. In this case, the set of differential-algebraic equations of motion can be formulated in the index-three form and can be subsequently rewritten in the index-one form as follows:

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_b - \boldsymbol{C}_{\boldsymbol{q}}^T\boldsymbol{\lambda} \\ \boldsymbol{C} = \boldsymbol{0} \end{cases} \quad \Rightarrow \quad \begin{cases} \boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_b - \boldsymbol{C}_{\boldsymbol{q}}^T\boldsymbol{\lambda} \\ \boldsymbol{C}_{\boldsymbol{q}}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_{d,h} \end{cases} \tag{43}$$

As mentioned before, while the index-three form of the equations of motion is well suited for the linearization procedure previously described in the paper, the corresponding index-one form of the dynamic equations is more advantageous for the computer implementation of the AF necessary for performing dynamical simulations. For this purpose, Equation (43) can be rewritten according to the matrix structure of Equation (37). In particular, in the case of holonomic multibody mechanical systems, one obtains:

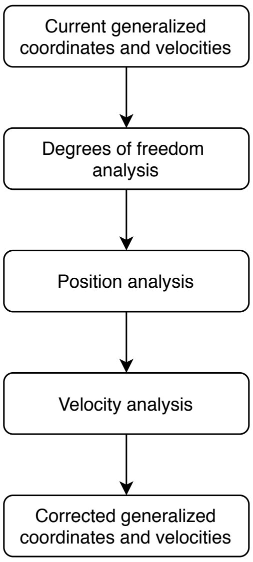
$$\boldsymbol{M}_a = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{C}_{\boldsymbol{q}}^T \\ \boldsymbol{C}_{\boldsymbol{q}} & \boldsymbol{O} \end{bmatrix}, \quad \boldsymbol{q}_a = \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix}, \quad \boldsymbol{Q}_a = \begin{bmatrix} \boldsymbol{Q}_b \\ \boldsymbol{Q}_{d,h} \end{bmatrix} \tag{44}$$

where, in this case of holonomic systems, the characteristic dimension of the AF is defined by the integer number $n_a = n_q + n_{c,h}$. The RGCPA applied to holonomic multibody systems is structured following four consecutive computational steps: the degrees of freedom analysis, the position analysis, the velocity analysis, and the acceleration analysis. The flowchart of the RGCPA for holonomic multibody systems is shown in Figure 2a.

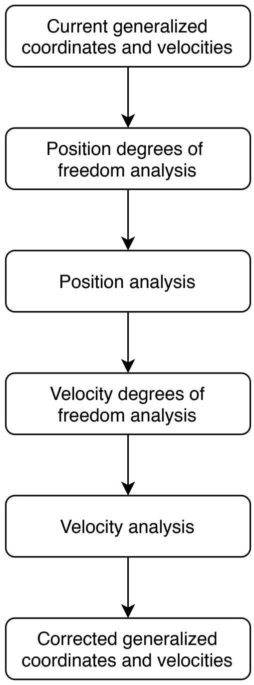
As shown in Figure 2a, the computational step devoted to the determination of the generalized acceleration vector is not included in the algorithm flowchart since, in principle, one can also employ an analytical method that differs from the AF for carrying out the acceleration analysis. To this end, several well-known methods can be found in the literature, and the general computational algorithm developed in this work can equally include each one of these techniques [64]. As discussed below, the AF is based on the index-one form of the motion equations and is employed only in the acceleration analysis.

The first step of the RGCPA is the degrees of freedom analysis. In this phase, the generalized coordinate vector of the holonomic multibody system $\boldsymbol{q}$ is partitioned into independent and dependent coordinates. Thus, the generalized coordinate vector $\boldsymbol{q}$ can be rearranged as follows:

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_i \\ \boldsymbol{q}_d \end{bmatrix} \tag{45}$$



(a) *Holonomic systems.*



(b) *Nonholonomic systems.*

Figure 2. *Flowcharts of the RGCPA for holonomic and nonholonomic multibody systems.*

where $n_{q,i}$ is the number of independent coordinates, $n_{q,d}$ is the number of dependent coordinates, $\boldsymbol{q}_i \equiv \boldsymbol{q}_i(t) \in \Re^{n_{q,i}}$ represents the independent generalized coordinates vector, and $\boldsymbol{q}_d \equiv \boldsymbol{q}_d(t) \in \Re^{n_{q,d}}$ represents the dependent generalized coordinates vector. The partitioning into dependent and independent generalized coordinates can be readily accomplished through the numerical analysis of the structure of the constraint Jacobian matrix $\boldsymbol{C}_{\boldsymbol{q}}$. Therefore, different computational methods are available to identify the vectors of independent and dependent coordinates $\boldsymbol{q}_i$ and $\boldsymbol{q}_d$. For example, a Gaussian elimination or a LU factorization with partial of full pivoting of the constraint Jacobian matrix $\boldsymbol{C}_{\boldsymbol{q}}$ can be applied at each time step of the numerical simulation to accomplish this task [4, 50]. By doing so, the set of independent coordinates $\boldsymbol{q}_i$ is identified by the column of the upper triangular matrix in which the pivots are not found. Conversely, the set of dependent coordinates $\boldsymbol{q}_d$ is associated with the column of the upper triangular matrix in which the pivots are found.

It is important to note that there are several cases quite common in engineering applications in which the set of constraint equations is redundant, resulting in a scenario in which the constraint Jacobian matrix $\boldsymbol{C}_{\boldsymbol{q}}$ does not have a full row rank. To handle this situation, a preliminary Gaussian elimination procedure can be used to find and remove the redundant constraint equations in the vector of holonomic constraints denoted with $\boldsymbol{C}$. In this case, one can define a set of independent algebraic constraints of dimension denoted with $\bar{n}_{c,h}$ that are embedded in the holonomic constraint vector indicated as $\bar{\boldsymbol{C}} \equiv \bar{\boldsymbol{C}}(\boldsymbol{q}, t) \in \Re^{\bar{n}_{c,h}}$, as well as an independent Jacobian matrix of the holonomic constraint equations indicated as $\bar{\boldsymbol{C}}_{\boldsymbol{q}} = \partial\bar{\boldsymbol{C}}/\partial\boldsymbol{q} \equiv \bar{\boldsymbol{C}}_{\boldsymbol{q}}(\boldsymbol{q}, t) \in \Re^{\bar{n}_{c,h} \times n_q}$. Identifying the system degrees of freedom and the redundant constraint equations at each time step of the dynamical simulation is essential for correctly solving the differential-algebraic set of dynamic equations describing the motion of the multibody system under investigation. Repeating this procedure at each time step during the simulation significantly increases the numerical robustness of the algorithm, assuring that, for every discrete time step, the independent generalized coordinates are successfully found. However, although it is not recommended, for simplicity, one can identify the set of redundant algebraic constraints only at the beginning of the numerical simulation, while the identification of the sets of independent and dependent generalized coordinates must be, in any case, repeated at each time step of the dynamic analysis to obtain accurate numerical results for complex multibody mechanical systems [3].

The second step of the RGCPA is the position analysis. To carry out this task, a Newton-Raphson method can be implemented to calculate the vector of dependent coordinates of the multibody system denoted with $\boldsymbol{q}_d$, holding fixed the set of independent generalized coordinates denoted with $\boldsymbol{q}_i$ and, at the same time, satisfying the holonomic constraints at the position level. For a small perturbation around an actual configuration of the multibody system consistent with the kinematic constraints, the holonomic constraint equations imposed on the multibody system, which are described by the first set of algebraic equations that appear in Equation (2), can be approximated by computing a Taylor series expansion truncated at the first order. For this purpose, one can write:

$$\boldsymbol{C} = \boldsymbol{0} \quad \Rightarrow \quad \Delta\boldsymbol{C} = \boldsymbol{C} + \boldsymbol{C}_{\boldsymbol{q}_i}\Delta\boldsymbol{q}_i + \boldsymbol{C}_{\boldsymbol{q}_d}\Delta\boldsymbol{q}_d = \boldsymbol{C} + \boldsymbol{C}_{\boldsymbol{q}}\Delta\boldsymbol{q} = \boldsymbol{0} \tag{46}$$

or equivalently:

$$\boldsymbol{C}_{\boldsymbol{q}}\Delta\boldsymbol{q} = -\boldsymbol{C} \tag{47}$$

where $\Delta\boldsymbol{C}$ represents the first-order Taylor expansion of the nonlinear constraint vector function $\boldsymbol{C}$, $\Delta\boldsymbol{q}$ identifies the vector of Newton differences of the generalized coordinates of the multibody system, $\boldsymbol{C}_{\boldsymbol{q}_i} = \partial\boldsymbol{C}/\partial\boldsymbol{q}_i \equiv \boldsymbol{C}_{\boldsymbol{q}_i}(\boldsymbol{q}, t) \in \Re^{n_{c,h} \times n_{q,i}}$ denotes the Jacobian matrix associated to the holonomic constraints that can be obtained by differentiating the vector of holonomic constraints $\boldsymbol{C}$ with respect to the generalized independent coordinates $\boldsymbol{q}_i$, $\boldsymbol{C}_{\boldsymbol{q}_d} = \partial\boldsymbol{C}/\partial\boldsymbol{q}_d \equiv \boldsymbol{C}_{\boldsymbol{q}_d}(\boldsymbol{q}, t) \in \Re^{n_{c,h} \times n_{q,d}}$ denotes the Jacobian matrix associated to the holonomic constraints that can be obtained by differentiating the vector of holonomic constraints $\boldsymbol{C}$ with respect to the generalized dependent coordinates $\boldsymbol{q}_d$, while the vectors $\Delta\boldsymbol{q}_i$ and $\Delta\boldsymbol{q}_d$ represent, respectively, the vectors of the Newton differences of the independent and dependent generalized coordinates. Since the independent coordinates vector $\boldsymbol{q}_i$ is kept fixed in the position analysis, the equivalent vector of Newton differences must be set equal to zero:

$$\Delta\boldsymbol{q}_i = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{B}_i\Delta\boldsymbol{q} = \boldsymbol{0} \tag{48}$$

where $\boldsymbol{B}_i$ is an appropriate Boolean matrix used to select the independent generalized coordinates $\boldsymbol{q}_i$ from the total generalized coordinate vector of the multibody system denoted with $\boldsymbol{q}$, as described by the following simple matrix equation:

$$\boldsymbol{q}_i = \boldsymbol{B}_i \boldsymbol{q} \tag{49}$$

At this stage, Equations (47) and (48) can be combined in the following compact matrix form:

$$\begin{cases} \boldsymbol{C}_{\boldsymbol{q}} \Delta \boldsymbol{q} = -\boldsymbol{C} \\ \boldsymbol{B}_i \Delta \boldsymbol{q} = \boldsymbol{0} \end{cases} \quad \Leftrightarrow \quad \boldsymbol{J}_c \Delta \boldsymbol{q} = \boldsymbol{b}_c \tag{50}$$

where the matrix $\boldsymbol{J}_c \equiv \boldsymbol{J}_c(\boldsymbol{q}, t) \in \Re^{n_q \times n_q}$ can be defined as the augmented holonomic constraint matrix and the vector $\boldsymbol{b}_c \equiv \boldsymbol{b}_c(\boldsymbol{q}, t) \in \Re^{n_q}$ is the right-hand side augmented vector of holonomic constraints at the position level. These vector and matrix quantities are respectively given by:

$$\boldsymbol{J}_c = \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{q}} \\ \boldsymbol{B}_i \end{bmatrix}, \quad \boldsymbol{b}_c = \begin{bmatrix} -\boldsymbol{C} \\ \boldsymbol{0} \end{bmatrix} \tag{51}$$

The system of algebraic equations represented by the Equation (50) must be solved iteratively in a while-type loop. To this end, the vector of Newton differences $\Delta \boldsymbol{q}$ is updated until the convergence condition is reached for a predetermined tolerance value for the norms of both the holonomic constraint vector $\boldsymbol{C}$ and the Newton differences of the independent coordinates $\Delta \boldsymbol{q}_i$. By doing so, one can iteratively improve the correction of the generalized coordinate vector $\boldsymbol{q}$ starting from a realistic initial guess of the system position configuration indicated as $\boldsymbol{q}_0$ coming from the previous discrete time step:

$$\boldsymbol{q} = \boldsymbol{q}_0 + \Delta \boldsymbol{q} \tag{52}$$

where the vector of Newton differences $\Delta \boldsymbol{q}$ and the starting point for the generalized coordinate vector $\boldsymbol{q}_0$ are constantly updated in the while-type loop.

In the case of a multibody system subjected to a redundant set of holonomic constraints, the partitioning procedure at the position level is identical, except for the necessity of respectively using, instead of the complete constraint vector $\boldsymbol{C}$ and the complete Jacobian matrix $\boldsymbol{C}_{\boldsymbol{q}}$, the vector of independent algebraic constraints $\bar{\boldsymbol{C}}$ and the relative Jacobian matrix $\bar{\boldsymbol{C}}_{\boldsymbol{q}}$ already defined before. Consequently, in this case, Equation (50) associated with the iterative procedure of the position analysis must be modified as follows:

$$\bar{\boldsymbol{J}}_c \Delta \boldsymbol{q} = \bar{\boldsymbol{b}}_c \tag{53}$$

where the independent augmented holonomic constraint matrix indicated as $\bar{\boldsymbol{J}}_c \equiv \bar{\boldsymbol{J}}_c(\boldsymbol{q}, t) \in \Re^{n_q \times n_q}$ and the independent right-hand side augmented vector of the holonomic constraints at the position level indicated as $\bar{\boldsymbol{b}}_c \equiv \bar{\boldsymbol{b}}_c(\boldsymbol{q}, t) \in \Re^{n_q}$ are respectively defined as:

$$\bar{\boldsymbol{J}}_c = \begin{bmatrix} \bar{\boldsymbol{C}}_{\boldsymbol{q}} \\ \boldsymbol{B}_i \end{bmatrix}, \quad \boldsymbol{b}_c = \begin{bmatrix} -\bar{\boldsymbol{C}} \\ \boldsymbol{0} \end{bmatrix} \tag{54}$$

As in the case of a multibody system with no redundant coordinates equations already described above, the system of algebraic equations defined in Equation (53) must be iteratively solved until a numerical convergence is reached to determine the dependent set of generalized coordinates from the independent ones.

The third step of the RGCPA is the velocity analysis. This stage aims to force the generalized velocities of the multibody system to satisfy the constraint equations at the velocity level. Thus, the dependent velocities grouped in the vector $\dot{\boldsymbol{q}}_d$ must be calculated holding fixed the vector of independent velocities denoted with $\dot{\boldsymbol{q}}_i$ and the entire generalized coordinate vector $\boldsymbol{q}$ of the multibody system already corrected in the phase of the position analysis. To achieve this goal, the constraint equations at the velocity level mathematically derived below must be numerically solved:

$$\dot{\boldsymbol{C}} = \boldsymbol{0} \quad \Rightarrow \quad \dot{\boldsymbol{C}} = \boldsymbol{C}_t + \boldsymbol{C}_{\boldsymbol{q}_i} \dot{\boldsymbol{q}}_i + \boldsymbol{C}_{\boldsymbol{q}_d} \dot{\boldsymbol{q}}_d = \boldsymbol{C}_t + \boldsymbol{C}_{\boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{0} \tag{55}$$

or equivalently:

$$\boldsymbol{C}_{\boldsymbol{q}} \dot{\boldsymbol{q}} = -\boldsymbol{C}_t \tag{56}$$

where $\boldsymbol{C}_t$ denotes the partial derivative of the vector $\boldsymbol{C}$ of holonomic constraint equations computed with respect to the time variable t. Since both the independent velocities $\dot{\boldsymbol{q}}_i$ and the corrected generalized coordinates $\boldsymbol{q}$ of the multibody system are kept fixed, the following identity must be taken into account:

$$\dot{\boldsymbol{q}}_i = \dot{\boldsymbol{q}}_i \quad \Leftrightarrow \quad \boldsymbol{B}_i \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_i \tag{57}$$

Equations (56) and (57) can be arranged in the following compact matrix form:

$$\begin{cases} \boldsymbol{C}_{\boldsymbol{q}} \dot{\boldsymbol{q}} = -\boldsymbol{C}_t \\ \boldsymbol{B}_i \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_i \end{cases} \quad \Leftrightarrow \quad \boldsymbol{J}_c \dot{\boldsymbol{q}} = \boldsymbol{d}_c \tag{58}$$

where $\boldsymbol{J}_c$ is the same augmented holonomic constraint matrix encountered in the position analysis described by Equation (50), while $\boldsymbol{d}_c \equiv \boldsymbol{d}_c(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_q}$ is the right-hand side holonomic constraint vector at the velocity level having the following mathematical form:

$$\boldsymbol{d}_c = \begin{bmatrix} -\boldsymbol{C}_t \\ \dot{\boldsymbol{q}}_i \end{bmatrix} \tag{59}$$

To consistently perform the velocity analysis, since the problem of the stabilization of the constraint equations at the velocity level is exactly formulated in terms of a linear system of algebraic equations, Equation (58) can be solved numerically in only one step to obtain the corrected generalized velocity vector $\dot{\boldsymbol{q}}$.

Even in this case, it is important to point out that the approach used for handling the velocity analysis can also be applied when dealing with redundant constraints to obtain the correct value of the generalized velocity vector $\dot{\boldsymbol{q}}$ of the multibody system. To this end, the independent algebraic equations embedded in the vector $\bar{\boldsymbol{C}}$ found in the first step of the algorithm must be used and, consequently, Equation (58) must be reformulated as follows:

$$\bar{\boldsymbol{J}}_c\dot{\boldsymbol{q}} = \bar{\boldsymbol{d}}_c \tag{60}$$

where $\bar{\boldsymbol{J}}_c$ is the independent augmented holonomic constraint matrix employed in Equation (53) and $\bar{\boldsymbol{d}}_c \equiv \bar{\boldsymbol{d}}_c(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_q}$ is the independent right-hand side augmented vector of holonomic constraints at the velocity level associated with the redundant set of algebraic constraints given by:

$$\bar{\boldsymbol{d}}_c = \begin{bmatrix} -\bar{\boldsymbol{C}}_t \\ \dot{\boldsymbol{q}}_i \end{bmatrix} \tag{61}$$

where $\bar{\boldsymbol{C}}_t$ represents the partial derivative of the independent constraint vector $\bar{\boldsymbol{C}}$ computed with respect to the time variable t.

The fourth and final step of the RGCPA is the acceleration analysis. In this last step, the objective is to force the numerical solution of the equations of motion to satisfy the constraint equations also at the acceleration level. This goal can be reached considering the AF based on the index-one form of the equation of motion described by Equation (43). By doing so, since the correct values of the generalized coordinate and velocity vectors previously found, respectively denoted with $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, are used to define the numerical values of the augmented mass matrix $\boldsymbol{M}_a$ and the augmented total generalized force vector $\boldsymbol{Q}_a$, the set of differential-algebraic equations of motion can be consistently solved to obtain the augmented generalized acceleration vector $\boldsymbol{q}_a$. As already discussed above, since they correspond to the first n_q entries of the augmented generalized acceleration vector defined in Equation (44), the generalized accelerations can be extracted from $\boldsymbol{q}_a$, and the numerical integration of the dynamic equations can be readily performed using a standard numerical scheme.

3.4 Robust Generalized Coordinate Partitioning Algorithm for Nonholonomic Multibody Systems

In this subsection, the RGCPA is extended to the case of multibody systems subjected to both holonomic and nonholonomic constraints. In the case of a multibody mechanical system subjected to a nonlinear set of $n_{c,h}$ holonomic constraints, grouped in the holonomic constraint vector indicated as $\boldsymbol{C}$, and conditioned by a nonlinear set of $n_{c,nh}$ nonholonomic constraints, grouped in the nonholonomic constraint vector indicated as $\boldsymbol{D}$, the differential-algebraic equations of motion can be reformulated considering their index-three structure leading to the following set of equations:

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{Q}_b - \boldsymbol{J}^T\boldsymbol{v} \\ \boldsymbol{E} = \boldsymbol{0} \end{cases} \tag{62}$$

To determine the generalized acceleration vector, a computational method based on the index-one form of the equations of motion can be effectively used, leading to the construction of the matrix formulation that characterizes the AF described by Equation (37). Consequently, in the comprehensive computational procedure employed in this paper, the AF represents the key method used to carry out the acceleration analysis for both holonomic and nonholonomic multibody systems. Besides, as in the case of holonomic constraints, the RGCPA implemented for nonholonomic multibody systems can be divided into four consecutive computational steps that are conceptually identical to those employed in the case of holonomic multibody systems, namely the degrees of freedom analysis, the position analysis, the velocity analysis, and the acceleration analysis. More precisely, as discussed in detail below, for the proper numerical solution of the equations of motion of nonholonomic multibody systems, an additional step is required in the computational algorithm, resulting in an actual number of five computational steps for the computational procedure. The flowchart of the RGCPA for nonholonomic multibody systems is shown in Figure 2b. Similarly to the RGCPA for holonomic systems represented in Figure 2a, the flowchart of the dual algorithm for nonholonomic systems shown in Figure 2b does not include the acceleration analysis because this particular computational step can also be performed by using another approach different from the AF, which represents another method that takes part in the computational procedure [64].

The first and second steps of the RGPCA for multibody systems whose motion is limited by both holonomic and nonholonomic constraints, namely the degrees of freedom analysis and the position analysis, are the same as in the case in which there are only holonomic constraints. All the hypotheses assumed for the analysis of holonomic multibody systems are indeed still valid for nonholonomic systems. In particular, since the set of nonholonomic constraints encapsulated in the nonlinear vector $\boldsymbol{D}$ does not affect the configuration space of a general multibody system, which is only influenced by the holonomic constraint vector $\boldsymbol{C}$, the first two computational stages of the RGCPA can be applied without any change to this second case of nonholonomic systems. On the other hand, the presence of the nonholonomic constraints directly affects the velocity space, which is restricted by both the time derivative of the holonomic constraint vector $\dot{\boldsymbol{C}}$ and by the nonholonomic constraint vector $\boldsymbol{D}$ itself.

When both holonomic and nonholonomic constraints act at the same time on a given multibody mechanical system, the partition into dependent and independent generalized coordinates, respectively denoted with $\boldsymbol{q}_d$ and $\boldsymbol{q}_i$, is fundamentally different from the partition into dependent and independent generalized velocities, respectively indicated as $\dot{\boldsymbol{q}}_d$ and $\dot{\boldsymbol{q}}_i$. Therefore, a second analysis of the independent generalized velocities taking into account also the limitations imposed by the set of nonholonomic constraints must be performed and included as an additional step of the RGCPA for nonholonomic multibody systems. For simplicity, this fundamental additional step of the algorithm is referred to as the "degrees of freedom analysis at the velocity level" or it is simply called the "velocity degrees of freedom analysis".

The velocity degrees of freedom analysis represents a fundamental computational step of the RGCPA applied to the case of nonholonomic systems. The lack of identification of the difference between the subsets of independent and dependent generalized coordinates, respectively indicated as $\boldsymbol{q}_i$ and $\boldsymbol{q}_d$, and the subsets of independent and dependent generalized velocities, respectively indicated as $\dot{\boldsymbol{q}}_i$ and $\dot{\boldsymbol{q}}_d$, may lead to inaccurate or incorrect numerical results in the dynamic analysis of relatively complex multibody systems subjected to holonomic and nonholonomic constraints, as in the case study considered in this investigation. To better clarify this point, Figure 2 shows a comparison between the computational steps of the RGCPA for holonomic and nonholonomic multibody mechanical systems.

As discussed above, the third necessary step of the RGCPA for multibody systems subjected to both holonomic and nonholonomic constraints is the velocity degrees of freedom analysis. Similarly to the position degrees of freedom analysis, to define the independent generalized velocities denoted with $\dot{\boldsymbol{q}}_i$ and the dependent generalized velocities denoted with $\dot{\boldsymbol{q}}_d$, a QR factorization or a LU factorization with partial of full pivoting of the total constraint Jacobian matrix $\boldsymbol{J}$ can be applied at each time step of the computational procedure. The set of independent generalized velocities $\dot{\boldsymbol{q}}_i$ are identified by the column of the upper triangular matrix in which the pivots are not found, while the set of dependent generalized velocities $\dot{\boldsymbol{q}}_d$ are identified by the column of the upper triangular matrix in which the pivots are found. Consequently, the generalized velocity vector $\dot{\boldsymbol{q}}$ can be partitioned as follows:

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{\boldsymbol{q}}_i \\ \dot{\boldsymbol{q}}_d \end{bmatrix} \tag{63}$$

The introduction of the velocity degrees of freedom analysis represents a crucial step for the correct fulfillment of the subsequent velocity analysis.

The fourth step of the RGCPA for nonholonomic multibody mechanical systems is the velocity analysis. This computational stage is focused on forcing the numerical solution to satisfy the constraint equations at the velocity level. For this purpose, the vector of the dependent generalized velocities of the multibody system denoted with $\dot{\boldsymbol{q}}_d$ must be calculated holding fixed the set of independent velocities indicated as $\dot{\boldsymbol{q}}_i$ and, at the same time, the entire generalized coordinate vector denoted with $\boldsymbol{q}$ that was already corrected during the position analysis. To achieve this goal, the constraint equations at the velocity level that must be numerically solved belong to two families. The first set of velocity-level algebraic constraints results from the time derivative of the holonomic constraint vector denoted with $\dot{\boldsymbol{C}}$ provided in Equation (56), while the second set of velocity-level constraint equations is represented by the nonholonomic algebraic constraints embedded in the constraint vector $\boldsymbol{D}$ given in the complete set of kinematic constraints provided in Equation (2). Without loss of generality, one can prove that the nonlinear vector containing the set of nonholonomic constraints can be typically expressed according to the following analytical structure called Pfaffian form:

$$\boldsymbol{D} = \boldsymbol{w}_d + \boldsymbol{D}_{\dot{\boldsymbol{q}}_i}\dot{\boldsymbol{q}}_i + \boldsymbol{D}_{\dot{\boldsymbol{q}}_d}\dot{\boldsymbol{q}}_d = \boldsymbol{w}_d + \boldsymbol{D}_{\dot{\boldsymbol{q}}}\dot{\boldsymbol{q}} = \boldsymbol{0} \tag{64}$$

or equivalently:

$$\boldsymbol{D}_{\dot{\boldsymbol{q}}}\dot{\boldsymbol{q}} = -\boldsymbol{w}_d \tag{65}$$

where $\boldsymbol{w}_d \equiv \boldsymbol{w}_d(\boldsymbol{q},t) \in \Re^{n_{c,nh}}$ denotes a nonlinear nonholonomic right-hand side vector, $\boldsymbol{D}_{\dot{\boldsymbol{q}}_i} = \partial\boldsymbol{D}/\partial\dot{\boldsymbol{q}}_i \equiv \boldsymbol{D}_{\dot{\boldsymbol{q}}_i}(\boldsymbol{q},t) \in \Re^{n_{c,nh}\times n_{q,i}}$ represents the Jacobian matrix associated with the nonholonomic constraints that can be calculated as the partial derivative of the nonholonomic constraint vector $\boldsymbol{D}$ with respect to the independent generalized velocities $\dot{\boldsymbol{q}}_i$, and $\boldsymbol{D}_{\dot{\boldsymbol{q}}_d} = \partial\boldsymbol{D}/\partial\dot{\boldsymbol{q}}_d \equiv \boldsymbol{D}_{\dot{\boldsymbol{q}}_d}(\boldsymbol{q},t) \in \Re^{n_{c,nh}\times n_{q,d}}$ identifies the Jacobian matrix associated with the nonholonomic constraints that can be calculated as the partial derivative of the nonholonomic constraint vector $\boldsymbol{D}$ with respect to the dependent generalized velocities $\dot{\boldsymbol{q}}_d$. Moreover, the identity expressed by Equation (57) still remains valid in the case of a multibody system constrained by both holonomic and nonholonomic constraints. Thus, one can combine Equations (56), (64), and (57) to obtain the following matrix expression:

$$\begin{cases} \boldsymbol{C}_{\boldsymbol{q}}\dot{\boldsymbol{q}} = -\boldsymbol{C}_t \\ \boldsymbol{D}_{\dot{\boldsymbol{q}}}\dot{\boldsymbol{q}} = -\boldsymbol{w}_d \\ \boldsymbol{B}_i\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_i \end{cases} \quad \Leftrightarrow \quad \boldsymbol{J}_{c,v}\dot{\boldsymbol{q}} = \boldsymbol{d}_{c,v} \tag{66}$$

where $\boldsymbol{J}_{c,v} \equiv \boldsymbol{J}_{c,v}(\boldsymbol{q},t) \in \Re^{n_q\times n_q}$ represents the augmented nonholonomic constraint matrix and $\boldsymbol{d}_{c,v} \equiv \boldsymbol{d}_{c,v}(\boldsymbol{q},t) \in \Re^{n_q}$ is the right-hand side augmented vector of nonholonomic constraints at the velocity level which can be respectively assembled as reported below:

$$\boldsymbol{J}_{c,v} = \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{q}} \\ \boldsymbol{D}_{\dot{\boldsymbol{q}}} \\ \boldsymbol{B}_i \end{bmatrix}, \quad \boldsymbol{d}_{c,v} = \begin{bmatrix} -\boldsymbol{C}_t \\ -\boldsymbol{w}_d \\ \dot{\boldsymbol{q}}_i \end{bmatrix} \tag{67}$$

The linear system of algebraic equations reported in Equation (66) can be simultaneously solved at each time step of the numerical simulation to obtain the correct generalized velocity vector $\dot{\boldsymbol{q}}$ that satisfies the holonomic and nonholonomic constraint equations at the velocity level respectively embedded in the nonlinear vectors $\dot{\boldsymbol{C}}$ and $\boldsymbol{D}$.

If the set of nonholonomic constraint equations does not feature a Pfaffian form, one can still perform the velocity analysis by following the approach used for the RGCPA proposed in this paper. For this purpose, the computational step associated with the velocity analysis became characterized by an iterative calculation whose structure is identical to that of the position analysis. To clarify this fact, assume that the nonlinear set of nonholonomic algebraic constraints has a general structure different from the Pfaffian form and consider the following Taylor series expansion truncated at the first order that is performed in terms of the generalized velocities holding fixed the generalized coordinates:

$$\boldsymbol{D} = \boldsymbol{0} \quad \Rightarrow \quad \Delta\boldsymbol{D} = \boldsymbol{D} + \boldsymbol{D}_{\dot{\boldsymbol{q}}_i}\Delta\dot{\boldsymbol{q}}_i + \boldsymbol{D}_{\dot{\boldsymbol{q}}_d}\Delta\dot{\boldsymbol{q}}_d = \boldsymbol{D} + \boldsymbol{D}_{\dot{\boldsymbol{q}}}\Delta\dot{\boldsymbol{q}} = \boldsymbol{0} \tag{68}$$

or equivalently:

$$\boldsymbol{D}_{\dot{\boldsymbol{q}}}\Delta\dot{\boldsymbol{q}} = -\boldsymbol{D} \tag{69}$$

where $\Delta\boldsymbol{D}$ identifies the first-order Taylor expansion of the nonlinear constraint vector function $\boldsymbol{D}$, $\Delta\dot{\boldsymbol{q}}$ represents the vector of Newton differences of the generalized velocities of the multibody system, whereas $\Delta\dot{\boldsymbol{q}}_i$ and $\Delta\dot{\boldsymbol{q}}_d$ are, respectively, the vectors of the Newton differences of the independent and dependent generalized velocities. In analogy with the position analysis, since the independent velocities vector $\dot{\boldsymbol{q}}_i$ is considered constant in the velocity analysis, the corresponding vector of Newton differences must be set equal to zero:

$$\Delta\dot{\boldsymbol{q}}_i = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{B}_i\Delta\dot{\boldsymbol{q}} = \boldsymbol{0} \tag{70}$$

Furthermore, one must also include in the velocity analysis the velocity-level version of the holonomic constraint equations expressed in terms of the vector of Newton differences for the generalized velocities given by:

$$\boldsymbol{C}_{\boldsymbol{q}}\Delta\dot{\boldsymbol{q}} = -\boldsymbol{C}_t \tag{71}$$

By combining Equations (69), (70), and (71), one obtains the key equations associated with the velocity analysis of multibody systems having a general non-Pfaffian structure of the nonholonomic constraints:

$$\begin{cases} \boldsymbol{C}_{\boldsymbol{q}}\Delta\dot{\boldsymbol{q}} = -\boldsymbol{C}_t \\ \boldsymbol{D}_{\dot{\boldsymbol{q}}}\Delta\dot{\boldsymbol{q}} = -\boldsymbol{D} \\ \boldsymbol{B}_i\Delta\dot{\boldsymbol{q}} = \boldsymbol{0} \end{cases} \quad \Leftrightarrow \quad \boldsymbol{J}_{c,v}\Delta\dot{\boldsymbol{q}} = \boldsymbol{d}^*_{c,v} \tag{72}$$

where the augmented nonholonomic constraint matrix $\boldsymbol{J}_{c,v}$ is the same matrix arising in the velocity analysis of Pfaffian nonholonomic constraints provided in Equation (67), while $\boldsymbol{d}^*_{c,v} \equiv \boldsymbol{d}^*_{c,v}(\boldsymbol{q},\dot{\boldsymbol{q}},t) \in \Re^{n_q}$ denotes the modified right-hand side augmented vector of nonholonomic constraints at the velocity level defined as follows:

$$\boldsymbol{d}^*_{c,v} = \begin{bmatrix} -\boldsymbol{C}_t \\ -\boldsymbol{D} \\ \boldsymbol{0} \end{bmatrix} \tag{73}$$

Similarly to the position analysis, the system of algebraic equations provided in Equation (72) must be solved iteratively in a while-type loop. To this end, the vector of Newton differences $\Delta\dot{\boldsymbol{q}}$ is updated until the convergence condition is reached for a predetermined tolerance value for the norms of both the nonholonomic constraint vector $\boldsymbol{D}$ and the Newton differences of the independent velocities $\Delta\dot{\boldsymbol{q}}_i$. By doing so, one can iteratively improve the correction of the generalized velocity vector $\dot{\boldsymbol{q}}$ starting from a realistic initial guess of the system velocity configuration indicated as $\dot{\boldsymbol{q}}_0$ coming from the previous discrete time step:

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_0 + \Delta\dot{\boldsymbol{q}} \tag{74}$$

where the vector of Newton differences $\Delta\dot{\boldsymbol{q}}$ and the starting point for the generalized velocity vector $\dot{\boldsymbol{q}}_0$ are constantly updated in the while-type loop.

Even in this case of nonholonomic systems, it can be observed that, if the multibody system analyzed presents some redundant holonomic and/or nonholonomic constraint equations, the structure of the algorithm is kept the same, except for the necessity of using the independent holonomic and nonholonomic constraint vectors respectively denoted with $\bar{\boldsymbol{C}}$ and $\bar{\boldsymbol{D}}$. Consequently, in the most general scenario, including redundant sets of holonomic and nonholonomic constraints, Equation (66) must be reformulated as follows:

$$\bar{\boldsymbol{J}}_{c,v}\dot{\boldsymbol{q}} = \bar{\boldsymbol{d}}_{c,v} \tag{75}$$

where the independent augmented nonholonomic constraint matrix denoted with $\bar{\boldsymbol{J}}_{c,v} \equiv \bar{\boldsymbol{J}}_{c,v}(\boldsymbol{q}, t) \in \Re^{n_q \times n_q}$ and the independent right-hand side augmented vector of the nonholonomic constraints at the velocity level denoted with $\bar{\boldsymbol{d}}_{c,v} \equiv \bar{\boldsymbol{d}}_{c,v}(\boldsymbol{q}, t) \in \Re^{n_q}$ are respectively defined as:

$$\bar{\boldsymbol{J}}_{c,v} = \begin{bmatrix} \bar{\boldsymbol{C}}_{\boldsymbol{q}} \\ \bar{\boldsymbol{D}}_{\dot{\boldsymbol{q}}} \\ \boldsymbol{B}_i \end{bmatrix}, \quad \bar{\boldsymbol{d}}_{c,v} = \begin{bmatrix} -\bar{\boldsymbol{C}}_t \\ -\bar{\boldsymbol{w}}_d \\ \dot{\boldsymbol{q}}_i \end{bmatrix} \tag{76}$$

where the integer number $\bar{n}_{c,nh}$ indicates the number of independent nonholonomic constraint equations, the vector $\bar{\boldsymbol{D}} \equiv \bar{\boldsymbol{D}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{\bar{n}_{c,nh}}$ is the vector of independent nonholonomic algebraic constraints, $\bar{\boldsymbol{D}}_{\dot{\boldsymbol{q}}} = \partial\bar{\boldsymbol{D}}/\partial\dot{\boldsymbol{q}} \equiv \bar{\boldsymbol{D}}_{\dot{\boldsymbol{q}}}(\boldsymbol{q}, t) \in \Re^{\bar{n}_{c,nh} \times n_q}$ represents the Jacobian matrix of the independent nonholonomic constraint vector $\bar{\boldsymbol{D}}$ that can be obtained as the partial derivative of $\bar{\boldsymbol{D}}$ with respect to the generalized velocity vector $\dot{\boldsymbol{q}}$, whereas the the vector $\bar{\boldsymbol{w}}_d \equiv \bar{\boldsymbol{w}}_d(\boldsymbol{q}, t) \in \Re^{\bar{n}_{c,nh}}$ identifies the independent nonlinear right-hand side vector associated with the set of nonholonomic constraints. Moreover, if the multibody system of interest features redundant holonomic and nonholonomic constraints, as well as a non-Pfaffian structure of the nonholonomic constraint equations, the velocity analysis can be still carried out employing the approach proposed in this paper leading to the following set of linear equations written in terms of the Newton differences of the generalized velocities:

$$\bar{\boldsymbol{J}}_{c,v}\Delta\dot{\boldsymbol{q}} = \bar{\boldsymbol{d}}^*_{c,v} \tag{77}$$

where the augmented nonholonomic constraint matrix associated to the non-redundant set of algebraic constraints $\bar{\boldsymbol{J}}_{c,v}$ is provided in Equation (76), while $\bar{\boldsymbol{d}}^*_{c,v} \equiv \bar{\boldsymbol{d}}^*_{c,v}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \Re^{n_q}$ denotes the non-redundant right-hand side augmented vector of nonholonomic constraints at the velocity level defined as follows:

$$\bar{\boldsymbol{d}}^*_{c,v} = \begin{bmatrix} -\bar{\boldsymbol{C}}_t \\ -\bar{\boldsymbol{D}} \\ \boldsymbol{0} \end{bmatrix} \tag{78}$$

By iteratively solving Equation (77), one can readily obtain the correct set of generalized velocities consistent with the non-redundant holonomic algebraic constraints as well as with the nonholonomic equations belonging to the general category of non-Pfaffian vector functions.

The fifth and final step of the RGCPA for nonholonomic systems is represented by the acceleration analysis, whose objective is to satisfy the constraint equations also at the acceleration level. Since the correct values of the generalized coordinate vector $\boldsymbol{q}$ and of the generalized velocity vector $\dot{\boldsymbol{q}}$ were previously found, the set of differential-algebraic equations describing the dynamics of the multibody system can be readily solved by adopting the AF. As mentioned before, this last computational step can also be collocated outside the flowchart describing the numerical procedure of the RGCPA for both holonomic and nonholonomic multibody mechanical systems shown in Figure 2.

4. Summary, Conclusions, and Future Work

This work can be seen as part of a more comprehensive research plan which can be divided into three areas of research, namely the kinematics and dynamics of multibody mechanical systems [65, 66], the analytical methods to develop control strategies suitable for guiding the dynamical behavior of nonlinear systems [67, 68], and the identification or estimation of mathematical models of dynamical systems based on the analysis of input-output experimental data [69, 70]. In this paper, the linear and nonlinear dynamics of articulated mechanical systems are studied by adopting a versatile multibody approach. For this purpose, a general analytical method is proposed for handling nonholonomic systems in conjunction with a robust computational procedure for effectively performing computer simulations of such complex systems. In particular, this manuscript is the first contribution of a two-part research paper.

This investigation is devoted to analyzing the stability of constrained mechanical systems in general, whose mathematical model is developed within the multibody framework, employing an alternative linearization technique. Thus, the main focus of the paper is on performing a systematic parametric analysis useful to understand the influence of the fundamental model parameters on the system dynamics. In particular, after a systematic linearization around a given set point of the original nonlinear differential-algebraic equations describing the dynamics of the multibody system of interest, the stability analysis is carried out by solving an appropriate generalized eigenvalue problem. The general nonlinear multibody framework presented in this manuscript is obtained and analyzed through a symbolic/numeric approach. For this purpose, a useful analytical method is introduced and used to perform the stability analysis of multibody mechanical systems subjected to a general set of holonomic and nonholonomic algebraic constraints through the systematic formulation of an appropriate generalized eigenvalue problem. Furthermore, a new computational procedure referred to as the Robust Generalized Coordinate Partitioning Algorithm (RGCPA) is proposed in this work to enforce the holonomic and nonholonomic sets of constraint equations at the position, velocity, and acceleration levels during the entire nonlinear dynamic simulation of complex multibody systems.

As the next step, the paper is also aimed at showing a practical example of the use of the analytical and numerical methodology developed in this study considering the dynamic analysis of a fully nonlinear multibody model of the Whipple-Carvallo bicycle system. The thorough analysis of the bicycle system that will be carried out in the second part of this paper is based on a nonlinear model derived from the use of the computer implementation of the systematic multibody approach proposed in the first part of this investigation.

Author Contributions

This research paper was principally developed by the first author (Carmine Maria Pappalardo). Great support was provided by the second author (Antonio Lettieri). The detailed review carried out by the third author (Domenico Guida) considerably improved the quality of the work. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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