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A Novel Fractional-Order System: Chaos, Hyperchaos and Applications to Linear Control

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* Dedicated in memory of Prof. E. Ahmed

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Abstract. Chaos and hyperchaos are generated from a new fractional-order system. Local stability of the system's three equilibria is analyzed when the fractional parameter belongs to (0,2]. According to Hopf bifurcation theory in fractional-order systems, approximations to the periodic solutions around the system's three equilibria are explored. Lyapunov exponents, Lyapunov spectrum and bifurcation diagrams are computed and chaotic (hyperchaotic) attractors are depicted. Furthermore, a linear control technique (LFGC) based on Lyapunov stability theory is implemented to derive the hyperchaotic states of the proposed system to its three equilibrium points. Numerical results are used to validate the theoretical results.

Keywords: Fractional-order; Hopf bifurcation; Chaos; Hyperchaos; Linear control.

1. Introduction

Nowadays, fractional calculus is widely used in science and technology [1-10]. The Caputo type fractional-order derivative [11] is described as

$$D^{\alpha}\varpi(\mathbf{t}) = J^{s-\alpha}\varpi^{(s)}(\mathbf{t}),\tag{1}$$

$$J^{\mu}\rho(\mathbf{t}) = \left[\int_{0}^{t} (\mathbf{t} - \beta)^{\mu-1}\rho(\beta)\mathbf{d}\beta\right] / \Gamma(\mu),$$
⁽²⁾

such that $\alpha > 0, \mu > 0$, s is an integer given that $s - 1 < \alpha < s$.

Dynamic analysis in fractional dynamical systems (FDS) has received increasing attention [12-14]. So, this paper focuses on exploring dynamical behaviors in a novel fractional-order system with three quadratic nonlinearities which is expected to generate hyperchaos. Besides, the hyperchaotic case of FDS has important applications [15-16]. According to [17], the equilibrium points of linear FDS is locally asymptotically stable (LAS) if

$$\arg(\lambda_i) > \alpha \pi / 2, i = 1, ..., n,$$
(3)

where λ_i is an arbitrary eigenvalue of the Jacobian matrix of FDS and n represents number of equations in FDS and $\alpha \in (0,2]$. Conditions (3) are known as Matignon's inequalities. In addition, the fractional Routh-Hurwitz (FRH) scheme is efficiently used to determine the local stability of the corresponding nonlinear FDS [15,18].

Controlling chaos (hyperchaos) in FDS is also considered as focal topic for research owing to its numerous essential applications [15,19-20]. Moreover, employing fractional controllers in FDS is appropriate in such systems with long memory transients and anomalous dynamics since they provide more robustness and high performance.

In [21], a new integer-order system that generates hyperchaotic attractors was introduced and analyzed by Matouk. The existence and type of Hopf bifurcations were analyzed in this system. In addition, the hyperchaotic attractors in Matouk's system were suppressed to its equilibrium states using a simple linear control scheme which is based on the classical Lyapunov stability theory. Moreover, chaos synchronization was achieved in a fractional version of Matouk's system [22] where the fractional



parameter lies in (0,1). In [23], the influence of fractional parameter on stabilizing this system to its equilibrium points was shown within a specific parameter set generating hyperchaos. This work addresses the fractional form of Matouk's system with new parameter sets that generate rich variety of chaotic and hyperchaotic dynamics; Here, chaotic (hyperchaotic) attractors appear less than and above one. Therefore, calculations of Lyapunov exponents and bifurcation diagrams are performed in order to verify the existence of this variety of chaotic dynamics. On the other hand, the stability analysis in the fractional Matouk's system is carried out based on the four-dimensional FRH conditions given by Matouk [24-25] in which the fractional parameter lies in (0,2]. Furthermore, conditions for approximating periodic solutions are obtained in the fractional Matouk's system via Hopf bifurcation theory in fractional-order systems. Finally, a linear feedback gains control (LFGC) criterion is applied to the fractional Matouk's system so that its new hyperchaotic regions are successfully stabilized to all the system's equilibria.

2. The Proposed System

In [21], new integer-order hyperchaotic system was introduced. The system's equations are described as

$$\frac{dx}{dt} = au - ay + hx - xu,$$

$$\frac{dy}{dt} = bx + u - xz,$$

$$\frac{dz}{dt} = x^{2} - cz,$$

$$\frac{du}{dt} = du,$$
(4)

where the constants $a, b, c, d, h \in \mathbb{R}$. The fractional form of system (4) is described as

$$D^{\alpha}x = au - ay + hx - xu,$$

$$D^{\alpha}y = bx + u - xz,$$

$$D^{\alpha}z = x^{2} - cz,$$

$$D^{\alpha}u = du,$$
(5)

where $0 < \alpha \le 2$. The system (5) has three equilibria $S_i(x, y, z, u), i = 0, 1, 2$ given that $S_0 = (0, 0, 0, 0), S_1 = (\chi, \chi h / a, b, 0), S_2 = (-\chi, -\chi h / a, b, 0), \chi = \sqrt{bc}, a \neq 0, bc \ge 0$. Moreover, system (5) has the unique equilibrium $S_0(0, 0, 0, 0)$ if a = 0 or bc < 0.

Recently, some numerical methods for solving FDS have been appeared, e.g., variational iteration method (VIM) [26]; method of transfer function approximation in the frequency domain [27]; Adomian's decomposition method (ADM) [28] and the method of predictor-correctors (PECE) which shows more efficiency and is widely used in practical applications [29]. Henceforth, the fractional systems in this work are integrated using the PECE scheme for solving FDS or more precisely, Predict, Evaluate, Correct, Evaluate. Here, the PECE algorithm requires total number of points $n = 5 \times 10^4$ and discretization step 2×10^{-3} .

3. Local Stability

In [15, 24], Matouk introduced and proved the following fractional Routh-Hurwitz (FRH) criterion in the case of four-dimensional FDS:

Let H_1 , H_2 , H_3 be defined as follows

$$\mathbf{H}_{1} = \eta_{1}, \quad \mathbf{H}_{2} = \begin{vmatrix} \eta_{1} & \mathbf{1} \\ \eta_{3} & \eta_{2} \end{vmatrix}, \quad \mathbf{H}_{3} = \begin{vmatrix} \eta_{1} & \mathbf{1} & \mathbf{0} \\ \eta_{3} & \eta_{2} & \eta_{1} \\ \mathbf{0} & \eta_{2} & \eta_{2} \end{vmatrix}, \tag{6}$$

where $|\cdot|$ refers to a matrix determinant and η_i , i = 1, 2, 3, 4 are coefficient of the characteristic polynomial $P(\lambda)$ of an equilibrium $\overline{S} = (\overline{s}_v, \overline{s}_v, \overline{s}_v, \overline{s}_v, \overline{s}_v)$ given that

$$P(\lambda) = \lambda^4 + \eta_1 \lambda^3 + \eta_2 \lambda^2 + \eta_3 \lambda + \eta_4 = 0,$$

whose discriminant is determined as

$$Disc(P(\lambda)) = (\eta_1 \eta_2 \eta_3)^2 - 4(\eta_1 \eta_3)^3 + 18\eta_1 \eta_2 \eta_3^3 - 4\eta_1^2 \eta_2^3 \eta_4 - 6(\eta_1 \eta_3)^2 \eta_4 + 144 \eta_2 \eta_3^2 \eta_4 - 80\eta_1 \eta_2^2 \eta_3 \eta_4 - 192 \eta_1 \eta_3 \eta_4^2 + 18\eta_1^3 \eta_2 \eta_3 \eta_4 + 144 \eta_1^2 \eta_2 \eta_4^2 - 4\eta_2^3 \eta_3^2 - 27\eta_3^4 - 27\eta_1^4 \eta_4^2 - 128\eta_2^2 \eta_4^2 + 16\eta_2^4 \eta_4 + 256\eta_4^3.$$
(7)

Hence, we introduce the following stability theorem given by Matouk [15, 24].

Theorem 1. (Matouk's stability theory in four-dimensional FDS)

(i) If $0 < \alpha \le 2$, $H_1 > 0$, $H_2 > 0$, $H_3 = 0$ and $\eta_4 > 0$ then $\overline{S} = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_u)$ is LAS.

(ii) If $2/3 < \alpha \le 2$, $\text{Disc}(P(\lambda)) > 0$, $\eta_1 > 0$ and $\eta_2 < 0$ then $\overline{S} = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_u)$ does not achieve Matignon's inequalities.

(iii) If $0 < \alpha < 1/3$, $\text{Disc}(P(\lambda)) < 0$ and $\eta_i > 0$, i = 1, 2, 3, 4 then $\overline{S} = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_u)$ is LAS. In addition, if $0 < \alpha \le 2$, $\text{Disc}(P(\lambda)) < 0$, $\eta_1 < 0$, $\eta_3 < 0$, $\eta_2 > 0$ and $\eta_4 > 0$ then $\overline{S} = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_u)$ does not achieve Matignon's inequalities.

If $0 < \alpha < 1$, $\text{Disc}(P(\lambda)) < 0$, $\eta_i > 0$, i = 1, 2, 3, 4 and $\frac{\eta_3}{\eta_1 \eta_2} + \frac{\eta_1 \eta_4}{\eta_2 \eta_3} - 1 = 0$ then $\overline{S} = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_u)$ is LAS. $1 \le \alpha \le 2$, $\text{Disc}(P(\lambda)) < 0$ and $\frac{\eta_3}{\eta_1 \eta_2} + \frac{\eta_1 \eta_4}{\eta_2 \eta_3} - 1 = 0$ then $\overline{S} = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_u)$ does not achieve Matignon's inequalities. (iv) is LAS. For

If $0 < \alpha \leq 2$ then the condition $\eta_4 > 0$ must necessarily be satisfied to achieve local stability of $\overline{S} = (\overline{S}_v, \overline{S}_v, \overline{S}_v, \overline{S}_v, \overline{S}_v)$. (v)

The matrix represents the Jacobian of system (5), calculated at $\overline{S} = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_y)$ is

$$J = \begin{pmatrix} h - \overline{s}_{u} & -a & 0 & a - \overline{s}_{x} \\ b - \overline{s}_{z} & 0 & -\overline{s}_{x} & 1 \\ 2\overline{s}_{x} & 0 & -c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$
 (8)

Theorem 2. The equilibrium $S_0(0,0,0,0)$ of the fractional-order system (5) is:

(i) LAS inside $\alpha \in (0,2]$ when d < 0, c > 0 if ab > 0, h < 0 or $h^2 < 4ab$, $ab > 0, \alpha < \frac{2}{\pi} \arctan(\frac{\sqrt{4ab - h^2}}{h})$; (ii) Saddle point inside $\alpha \in (0,1]$ if $(ab < 0 \text{ and } h \neq 0)$ or dc > 0.

Proof. The Jacobian (8) calculated at $S_0(0,0,0,0)$ is

$$J(S_{0}(0,0,0,0)) = \begin{pmatrix} h & -a & 0 & a \\ b & 0 & 0 & 1 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$
(9)

Also, the Jacobian $J(S_0(0,0,0,0))$ has four eigenvalues $\lambda_1 = d, \lambda_2 = -c, \lambda_{3,4} = \frac{h \pm \sqrt{h^2 - 4ab}}{2}$. Therefore, if d < 0, c > 0, then $\lambda_1 < 0, \lambda_2 < 0$ and we have the following cases:

- As ab > 0, h < 0, then $\text{Re}(\lambda_{3,4}) \in \mathbb{R}^-$ which imply that $S_0(0,0,0,0)$ is LAS $\forall \alpha \in (0,2]$.
- Furthermore, if $h^2 < 4ab$, ab > 0, $\alpha < \frac{2}{\pi} \arctan(\frac{\sqrt{4ab h^2}}{h})$, then all the eigenvalues satisfy the stability conditions (3) which also imply that $S_0(0,0,0,0)$ is LAS $\forall \alpha \in (0,2]$. These two items completes that proof of part (i).

To prove part (ii), we recall that $S_0(0,0,0,0)$ is saddle if it has two different eigenvalues λ_i, λ_j such that $|\arg(\lambda_j)| < \alpha \pi / 2$ and $|\arg(\lambda_i)| > \alpha \pi/2$. So, the condition for saddle point is fulfilled if $\alpha \in (0,1]$, ab < 0, $h \neq 0$ since $|\arg(\lambda_i)| = 0$ and $|\arg(\lambda_i)| = \pi$. Similarly, the condition for saddle point holds if $\alpha \in (0,1]$, dc > 0 which completes the proof of part (ii).

Both the equilibrium points $S_{1,2}(\pm\chi,\pm\frac{\chi h}{a},b,0)$ have the same characteristic polynomial. So, the Jacobian (8) computed at $S_{1,2}(\pm\chi,\pm\frac{\chi h}{a},b,0)$ yields the same characteristic equation, that is

> $\lambda^4 + (\mathbf{c} - \mathbf{d} - \mathbf{h})\lambda^3 + (\mathbf{h}\mathbf{d} - \mathbf{h}\mathbf{c} - \mathbf{c}\mathbf{d})\lambda^2 + (\mathbf{h}\mathbf{c}\mathbf{d} - 2a\chi^2)\lambda + 2ad\chi^2 = 0.$ (10)

Clearly, the following inequalities are imperative conditions for $\eta_i > 0$, i = 1, 2, 3, 4:

$$n < c - d, h(d - c) > cd, hcd > 2abc, abcd > 0.$$
 (11)

Hence, based on the above-mentioned FRH stability scheme, the following results are straightforwardly obtained.

Theorem 3. The points $S_{1,2}(\pm\chi,\pm\frac{\chi h}{a},b,0)$ are LAS if they fulfill any of the following statements:

- a) $0 < \alpha \leq 2$, $H_1 > 0$, $H_2 > 0$, $H_3 = 0$ and abcd > 0;
- b) $0 < \alpha < 1/3$, Disc(P(λ)) < 0 and conditions (11) hold;
- c) $0 < \alpha < 1$, Disc(P(λ)) < 0, conditions (11) hold and h equals one of the following quantities:

$$\frac{c\pm\sqrt{c^2-8ab}}{2} \quad \text{or} \quad \frac{2abc+d^2(d-c)}{d(c-d)}, \, d\neq c, \, c^2>8ab$$

However, $S_{1,2}(\pm\chi,\pm\frac{\chi h}{a},b,0)$ is not LAS if they satisfy one of the following:

- 2 / 3 < $\alpha \le$ 2, Disc(P(λ)) > 0, h < c d and h(d c) < cd;
- $0 < \alpha < 2$, Disc(P(λ)) < 0, h > c d, h(d c) > cd, hcd < 2abc and abcd > 0;



•
$$1 \le \alpha \le 2$$
, $\text{Disc}(P(\lambda)) < 0$ and h equals one of the following quantities:
 $\frac{c \pm \sqrt{c^2 - 8ab}}{c^2 - 8ab}$ or $\frac{2abc + d^2(d - c)}{c^2 - 8ab}, d \ne c, c^2 > 8ab.$

$$\frac{1}{2} \quad \text{or} \quad \frac{1}{d(c-d)}, d \neq c, c$$

4. Hopf Bifurcation (HB)

In autonomous fractional-order system (AFOS) the HB is expected to exist near its exact solution as the system's dynamical parameter ς crosses the critical value ς_{crh} such that the following conditions hold:

- 1) The AFOS, with order less than one, must have a pair of complex conjugate eigenvalues $\lambda_{1,2} = P(\varsigma_{crh}) \pm iQ(\varsigma_{crh})$,
- $i = \sqrt{-1}$, P > 0, Q \neq 0, and other eigenvalues are negative.
- 2) There exists a function $\Omega(\varsigma)$ such that $\Omega(\varsigma_{crh}) = 0$ and $d\Omega(\varsigma) / d\varsigma|_{\varsigma=\varsigma_{crh}} \neq 0$.

This gives rise to some asymptotically periodic signals that tend to limit cycles owing to the fact that exact periodic solutions do not exist in AFOS [30].

4.1 Approximation to the periodic solution around $S_0 = (0,0,0,0)$

According to conditions (3), it is clear that α has essential role in changing the stability of the fractional system. Therefore, α can be selected as a parameter of Hopf bifurcation (HB). Then, we define the function $\Omega(\alpha) = \alpha \pi / 2 - \arctan(\sqrt{4ab - h^2} / h)$. Thus, S_0 changes its stability around $\alpha_{crh} = 2\arctan(\sqrt{4ab - h^2} / h) / \pi$. It is also evident that $d\Omega(\alpha) / d\alpha |_{\alpha = \alpha_{orh}} = \pi / 2 \neq 0$. For a = 3, b = 15, c = 0.6, d = -0.0001, h = 1.5, the critical value for the fractional parameter is $\alpha_{crh} = 9286746211 \times 10^{-10}$. So, a periodic solution is expected near S_0 for these parameter values and $\alpha = 0.9289$. The results are depicted in Fig. 1.



(a) (b) **Fig. 1.** The trajectory of system (5) with a = 3, b = 15, c = 0.6, d = -0.0001, h = 1.5, $\alpha = 0.9289$, converges to a limit cycle around S_0 .



Fig. 2. The creation of an approximating periodic solution around S_0 of system (5) when a = 3, b = 15, c = 0.6, d = -0.0001, h = 1 and $\alpha = 0.95$.



Fig. 3. The creation of approximating periodic solutions around the two points $S_{1,2} = (\pm \chi, \pm \chi h / a, b, 0)$ when a = -3, b = 15, c = 0.6, d = -0.0001, h = 1.5 and $\alpha = 0.60935$.



Fig. 4. Chaotic attractor of system (5) appears when a = -3, b = 15, c = 0.6, d = -0.0001, h = -1 and $\alpha = 0.95$.



On the other hand, the parameter h can be considered as HB parameter by defining the function $\Omega(h) = \alpha \pi / 2 - \arctan(\sqrt{4ab - h^2} / h)$. Also at the critical value $h_{ch} = \pm 2\sqrt{ab(1 + \tan^2(\alpha \pi / 2))} / [1 + \tan^2(\alpha \pi / 2)] \neq 0$, $\alpha \in (0,1)$, the transversality condition holds since $d\Omega(h) / dh |_{h=h_{ch}} = 1 / (h_{ch} \tan(\alpha \pi / 2)) \neq 0$. For a = 3, b = 15, c = 0.6, d = -0.0001, the critical value for the dynamical parameter h is $h_{ch} = 1052639222 \times 10^{-9}$. So, a periodic solution is expected near S_0 using the above-mentioned selection of parameters, $\alpha = 0.95$ and h = 1. The results are depicted in Fig. 2. Here, the related maximal Lyapunov exponent (MLE) is vanishing.

4.2 Approximation to the periodic solution around $S_{1,2} = (\pm \chi, \pm \chi h / a, b, 0)$

If $\operatorname{Disc}(P(\lambda)) < 0$, where $P(\lambda)$ is defined in Eq. (10), then $S_{1,2}$ has a pair of complex conjugate eigenvalues $\lambda_{1,2} = P \pm iQ$ and two real eigenvalues $\lambda_3 = d$, $\lambda_4 = \lambda_0$. We define the function $\Omega(\alpha) = \alpha \pi / 2 - \operatorname{arctan}(Q / P)$, where $P^2 + Q^2 = \gamma$, $\gamma^3 + ch\gamma^2 - 2abc(c-h)\gamma - 4a^2b^2c^2 = 0$. The equilibrium points $S_{1,2} = (\pm \chi, \pm \chi h / a, b, 0)$ change their stability around $\alpha_{crh} = 2\operatorname{arccos}(\sqrt{P^2} / \gamma) / \pi$. It is also evident that $d\Omega(\alpha) / d\alpha |_{\alpha=\alpha_{crh}} = \pi / 2 \neq 0$. For a = -3, b = 15, c = 0.6, d = -0.0001, h = 1.5, the critical value for the fractional parameter is $\alpha_{crh} = 6093611980 \times 10^{-10}$. So, a periodic solution is expected near $S_{1,2} = (\pm \chi, \pm \chi h / a, b, 0)$ using these parameter values and $\alpha = 0.60935$. The results are depicted in Fig. 3. Also, the MLE here is very close to zero.



Fig. 5. Hyperchaotic attractor of system (5) appears when a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5 and $\alpha = 0.95$.



Fig. 6. Lyapunov spectrum of system (5) using: (a) b = 15, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$ and varying *a*, (b) a = -3, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$ and varying *b*, (c) a = -3, b = 15, d = -0.0001, h = -1.5, $\alpha = 0.95$ and varying *c*, (d) a = -3, b = 15, c = 0.6, d = -1.5, $\alpha = 0.95$ and varying *d*, and (e) a = -3, b = 15, c = 0.6, d = -0.0001, $\alpha = 0.95$ and varying *h*.





Fig. 7. Bifurcations diagrams of system (5) using: (a) b = 15, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$ and varying *a*, (b) a = -3, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$ and varying *b*, (c) a = -3, b = 15, d = -0.0001, h = -1.5, $\alpha = 0.95$ and varying *c*, (d) a = -3, b = 15, c = 0.6, d = -0.0001, $\alpha = 0.95$ and varying *h*. Here x_m refers to x_{max} .

5. Chaotic and Hyperchaotic Attractors

А chaotic attractor in the 4D fractional-order system parameters (5) appears using the as a = -3, b = 15, c = 0.6, d = -0.0001, h = -1 and $\alpha = 0.95$. The results are depicted in Fig. 4. For these values of parameters, the Lyapunov exponents (LEs) $\Lambda_{i,s}$ are computed based on Wolf's algorithm [31] as follows; $\Lambda_1 = 0.0172$, $\Lambda_2 = -0.0001$, $\Lambda_3 = -0.9154$, $\Lambda_4 = -1.1197$. If the maximal Λ_i (MLE) is greater than zero then the system is chaotic. However, the system is hyperchaotic if it has two $\Lambda_{i,s} > 0$. Other values of the parameters a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$, are used to generate hyperchaos in system (5). Figure 5 shows the hyperchaotic attractor of system (5) as using this set of parameters. Also, the corresponding $\Lambda_{i,s}$ are computed as follows; $\Lambda_1 = 0.7861$, $\Lambda_2 = 0.0204$, $\Lambda_3 = -0.0001$, $\Lambda_4 = -3.4472$.

As a result, computations of Lyapunov spectrum and bifurcations diagrams are performed and illustrated in Figs. 6 and 7, respectively. In Fig. 6(a-e), it is clear that the largest LE (see red curve) is positive which ensures the existence of chaotic case and another LE (see blue curve) is very close to zero and sometimes positive (but near to zero) which ensures the existence of hyperchaotic case. Also, figures 7(a-d), respectively, show that the dynamics in this system exhibit sensitive dependence on initial conditions when $a \in [-4, -0.3]$, b = 15, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$; a = -3, $b \in [2,16]$, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$; a = -3, b = 15, $c \in [0,2.3]$, d = -0.0001, h = -1.5, $\alpha = 0.95$ and a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$.

On the other hand, hyperchaos can be generated from system (5) when the fractional parameter $\alpha > 1$. This interesting foundation is depicted in Fig. 8 when using the parameters a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5 and fractional-order $\alpha = 1.02$.

Also, more numerical simulations are performed by fixing a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5 and varying the fractional parameter α . Therefore, more hyperchaotic attractors exist as $\alpha = 0.98$, $\alpha = 0.96$, $\alpha = 0.94$, $\alpha = 0.92$ and $\alpha = 0.90$ which are illustrated in Fig. 9. The corresponding Lyapunov spectrum verifies the existence of hyperchaos as depicted in Fig. 10 which draw similar conclusion as pointed out in Fig. 6.





Fig. 8. Hyperchaotic attractor of system (5) appears when a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5 and $\alpha = 1.02$.



Fig. 9. Hyperchaotic attractors of system (5) appear when a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5 and (a) $\alpha = 0.98$; (b) $\alpha = 0.96$; (c) $\alpha = 0.94$; (d) $\alpha = 0.92$; (e) $\alpha = 0.90$.









(e)

Fig. 9. Continued.



Fig. 10. Hyperchaos is shown via the Lyapunov spectrum of system (5) as α is varied and with a = -3, b = 15, c = 0.6, d = -0.0001 and h = -1.5.



6. Chaos Control via a Linear Feedback Gains Control (LFGC) Scheme

We take into our consideration the fractional-order controlled system

$$\mathsf{D}^{\alpha}\xi(\mathsf{t}) = \mathsf{N}(\xi(\mathsf{t})) + \mathsf{U}(\xi(\mathsf{t})), \ \alpha \in (0, 1],$$
(12)

where $\xi(t) \in \mathbb{R}^n$, N is nonlinear vector function and $U(\xi(t))$ is a linear control vector function. Suppose that \overline{S} be an origin equilibrium point for the uncontrolled form of system (12). Hence, we present the following lemma [22]:

Lemma 1. If a Lyapunov function $V(\xi(t))$ exists for the controlled system (12) as $\alpha = 1$, then the origin equilibrium \overline{S} is at least LAS as $0 < \alpha < 1$.

6.1 Stabilizing $S_0 = (0,0,0,0)$ via LFGC

For $S_0 = (0,0,0,0)$ and the feedback control gains (FCGs) $k_i \in \mathbb{R}^+$, i = 1,2,3,4, a controlled version of eqns. (5) is

$$D^{\alpha}x = au - ay + (h - k_{1})x - xu,$$

$$D^{\alpha}y = bx + u - xz - k_{2}y,$$

$$D^{\alpha}z = x^{2} - (k_{3} + c)z,$$

$$D^{\alpha}u = (d - k_{4})u.$$
(13)

So, we have

Theorem 4. The hyperchaotic attractors in eqns. (13) are suppressed to $S_0 = (0,0,0,0)$ provided that

$$\begin{aligned} k_{1} &> \frac{1}{4k_{2}k_{4} - (4d+1)} \left\{ \left(|a+b|^{2} \left(k_{4} - d \right) + \frac{|a||a+b|}{2} \right) + \left(\frac{|a||a+b|}{2} + |a|^{2} k_{2} \right) + \frac{\varepsilon_{y}^{2}}{(c+k_{3})} \left(k_{2}(k_{4} - d) - \frac{1}{4} \right) \right\} \\ &+ \varepsilon_{z} + \varepsilon_{u} + h, \\ k_{2} &> \frac{1}{4k_{4} - 4d}, \\ k_{3} &> -c, \quad k_{4} > d, \end{aligned}$$

$$(14)$$

where $|\mathbf{y}| < \varepsilon_{\mathbf{y}}, |\mathbf{z}| < \varepsilon_{\mathbf{z}}, |\mathbf{u}| < \varepsilon_{\mathbf{u}}.$

Proof. The following function is candidate for the controlled hyperchaotic system (13) to be its Lyapunov function as $\alpha = 1$

$$V(\xi_1,\xi_2,\xi_3,\xi_4) = \sum_{i=1}^{4} \xi_i^2 / 2,$$
(15)

where ξ_i refers to a state variable of eqns. (5), i.e. $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) = (x, y, z, u)$. Hence, we get

$$D^{1}V = \sum_{i=1}^{1} \xi_{i}D^{1}\xi_{i}$$

$$= (\xi_{3} - \xi_{4} + h - k_{1})\xi_{1}^{2} - k_{2}\xi_{2}^{2} - (k_{3} + c)\xi_{3}^{2} + (d - k_{4})\xi_{4}^{2} + (b - a)\xi_{1}\xi_{2} - \xi_{1}\xi_{2}\xi_{3} + a\xi_{1}\xi_{4} + \xi_{2}\xi_{4}$$

$$\leq (\varepsilon_{z} + \varepsilon_{u} - k_{1} + h)|\xi_{1}|^{2} - k_{2}|\xi_{2}|^{2} - (k_{3} + c)|\xi_{3}|^{2} + (d - k_{4})|\xi_{4}|^{2} + |b + a||\xi_{1}||\xi_{2}| + \varepsilon_{y}|\xi_{1}||\xi_{3}| + |a||\xi_{1}||\xi_{4}| + |\xi_{2}||\xi_{4}| = -\zeta^{T}M\zeta,$$
(16)

where

$$\zeta = \begin{bmatrix} |\mathbf{x}| & |\mathbf{y}| & |\mathbf{z}| & |\mathbf{u}| \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{M} = \begin{pmatrix} (-\mathbf{h} - \varepsilon_{z} - \varepsilon_{w} + \mathbf{k}_{1}) & -|\mathbf{b} + \mathbf{a}|/2 & -\varepsilon_{y}/2 & -|\mathbf{a}|/2 \\ -|\mathbf{b} + \mathbf{a}|/2 & \mathbf{k}_{2} & \mathbf{0} & -\mathbf{1}/2 \\ -\varepsilon_{y}/2 & \mathbf{0} & \mathbf{k}_{3} + \mathbf{c} & \mathbf{0} \\ -|\mathbf{a}|/2 & -\mathbf{1}/2 & \mathbf{0} & \mathbf{k}_{4} - \mathbf{d} \end{pmatrix}.$$

The Hermitian matrix M is strictly positive if all the inequalities (14) are satisfied. Therefore, it follows that $D^1V(\xi_1,\xi_2,\xi_3,\xi_4) < 0$ for all $(\xi_1,\xi_2,\xi_3,\xi_4) \neq (0,0,0,0)$ belongs to a domain $\Psi \subset C^3$ that contains a neighborhood of $(\xi_1,\xi_2,\xi_3,\xi_4) = (0,0,0,0)$. Therefore, it is shown that the function V represents a Lyapunov function for the system (13) with $\alpha = 1$. Thus, based on Lemma 1, we conclude that the origin equilibrium point of eqns. (13) is at least LAS when $0 < \alpha < 1$. This implies that the hyperchaotic states of eqns. (5) are controlled to the origin $S_0 = (0,0,0,0)$. \Box

System (13) is integrated with a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$ and the FCGs $k_1 = 165$, $k_2 = k_3 = k_4 = 1$ which satisfy Theorem 4. Also, according to Fig. 5, the positive bounds $\varepsilon_y, \varepsilon_z, \varepsilon_u$ are specified as $\varepsilon_y = 20$, $\varepsilon_z = 40$, $\varepsilon_u = 0.02$. So, Fig. 11 depicts the successful stabilization results.



Fig. 11. The trajectories of eqns. (13) tend to S_0 using a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$ and FCGs $k_1 = 165$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$.



Fig. 12. The trajectories of eqns. (17) tend to $S_1 = (\chi, \chi h / a, b, 0)$ using a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$, FCGs $k_1 = 165$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$ and controllers (18).

6.2 Stabilizing $S_{1,2} = (\pm \chi, \pm \chi h / a, b, 0)$ via LFGC

Now, suppose that $\overline{S}' = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_u)$ represents the non-origin equilibrium points S_1 or S_2 . So, we use the transformation $\xi' = \xi - \overline{S}', \quad \xi' = (y_1, y_2, y_3, y_4)^T$ to translate the point $\overline{S}' = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_u)$ to the origin of coordinates.

Then a controlled version of eqns. (5) to the equilibrium point $\overline{S}' = (\overline{s}_x, \overline{s}_y, \overline{s}_z, \overline{s}_u)$, is introduced by

$$D^{\alpha}y_{1} = ay_{4} - ay_{2} + hy_{1} - y_{1}y_{4} + u'_{1},$$

$$D^{\alpha}y_{2} = by_{1} + y_{4} - y_{1}y_{3} + u'_{2},$$

$$D^{\alpha}y_{3} = y_{1}^{2} - cy_{3} + u'_{3},$$

$$D^{\alpha}y_{4} = dy_{4} + u'_{4},$$
(17)

where u'_1, u'_2, u'_3, u'_4 are linear control functions given as

$$u_{1}' = -a\overline{s}_{u} + a\overline{s}_{y} + \overline{s}_{u}y_{1} + \overline{s}_{x}y_{4} + \overline{s}_{x}\overline{s}_{u} - h\overline{s}_{x} - k_{1}y_{1},$$

$$u_{2}' = -b\overline{s}_{x} + \overline{s}_{z}y_{1} + \overline{s}_{x}y_{3} + \overline{s}_{x}\overline{s}_{z} - k_{2}y_{2},$$

$$u_{3}' = c\overline{s}_{z} - 2\overline{s}_{x}y_{1} - \overline{s}_{x}^{2} - k_{3}y_{3},$$

$$u_{4}' = -d\overline{s}_{u} - k_{4}y_{4}.$$
(18)

Hence, we get

Journal of Applied and Computational Mechanics, Vol. 7, No. 2, (2021), 701-714





Fig. 13. The trajectories of eqns. (17) tend to $S_2 = (-\chi, -\chi h / a, b, 0)$ using a = -3, b = 15, c = 0.6, d = -0.0001, h = -1.5, $\alpha = 0.95$, FCGs $k_1 = 165$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$ and controllers (18).

Lemma 2. The hyperchaotic states in system (17) are suppressed to $S_{1,2} = (\pm \chi, \pm \chi h / a, b, 0)$ if the linear controllers (18) are implemented provided that the conditions (14) hold.

For the previously mentioned parameter set, FCGs and fractional order $\alpha = 0.95$, the controlled system (17) is numerically integrated with the linear controllers (18). So, based on Lemma 2, all trajectories of eqns. (17) approach to $S_{1,2} = (\pm \chi, \pm \chi h / a, b, 0)$. Therefore, Fig. 12 and Fig. 13 depict the successful stabilization results to S_1 and S_2 , respectively.

7. Conclusion

The fractional-order system given by Matouk is discussed. Local stability of the system's three equilibria has been examined when the fractional $\alpha \in (0,2]$. Approximations to the periodic solutions around the system's three equilibria have been explored based on Hopf bifurcation's conditions in fractional-order systems. Lyapunov exponents, Lyapunov spectrum and bifurcation diagrams have been computed for the fractional Matouk's system and rich variety of new chaotic and hyperchaotic attractors have been reported within $\alpha \in (0,2]$. Moreover, based on Lyapunov stability theory, the LFGC technique has been implemented to derive the hyperchaotic states of Matouk's system with fractional orders to its three equilibrium points. Discussions on circuit realization of integer and fractional-order Matouk's systems with applications to text encryption, secure communications and obtaining conditions for its chaotifiction are suggested points for future studies.

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