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Research Paper

Nonlinear Control for Attitude Stabilization of a Rigid Body Forced by Nonstationary Disturbances with Zero Mean Values

A. Yu. Aleksandrov [1], A. A. Tikhonov [2,3]

[1] Faculty of Applied Mathematics and Control Processes, Saint Petersburg State University, 7-9 Universitetskaya nab., Saint Petersburg, 199034, Russia, Email: a.u.aleksandrov@spbu.ru

[2] Department of Theoretical and Applied Mechanics, Saint Petersburg State University, 7-9 Universitetskaya nab., Saint Petersburg, 199034, Russia, Email: a.tikhonov@spbu.ru

[3] Department of Mechanics, Saint Petersburg Mining University, 2, 21st Line, St. Petersburg, 199106, Russia

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Corresponding author: A.Yu. Aleksandrov (a.u.aleksandrov@spbu.ru)

Abstract. A rigid body forced by a nonstationary perturbing torque with zero mean value is under consideration. The control strategy for attitude stabilization of the rigid body is based on the usage of dissipative and restoring torques. It is assumed that the dissipative torque is linear, while restoring and perturbing torques are purely nonlinear. A theorem on sufficient conditions for asymptotic stability of the body angular position is proved on the basis of the decomposition method, the Lyapunov direct method and the averaging technique. Computer simulation results illustrating the theorem are presented.

Keywords: Rigid body, Triaxial stabilization, Lyapunov function, Decomposition, Nonstationary disturbance, Averaging method.

1. Introduction

The success of the analytical and numerical study of the dynamics of a mechanical system largely depends on the quality of its mechanical model and on the correct account of the acting forces and torques. As a rule, the assessment of the significance of a particular force effect is based primarily on the consideration of its magnitude [1]. In the case of the action of time-varying force effects, it is also necessary to take into account their average values, amplitudes, frequencies, etc. [2]. Based on the magnitudes of the average values, some preliminary estimates of the influence of certain forcing on the dynamics of a mechanical system can be made. However, the proximity of the average value to zero does not allow, generally, to conclude that the corresponding force effect on the mechanical system is insignificant. In this connection it should be noted that non-linearities in a multi degree-of-freedom system may cause many "unexpected"' effects in the response [2-4]. At the same time, taking into account such force factors that cause the appearance of periodic and almost periodic functions in a mathematical model results in its fundamental complication. Analytical investigation of the behavior of a mechanical system becomes especially difficult in the presence of several almost periodic perturbations that are close to each other in magnitude. This situation is well known from the numerous examples arising from the study of the attitude dynamics of a spacecraft interacting with geophysical fields and with a rarefied Earth's atmosphere [5-8]. At the same time, the importance of solving problems of spacecraft attitude dynamics subject to perturbations with zero mean values stimulates the emergence of new researches in this difficult area of nonlinear mechanics [9-12]. In particular, a considerable stream of publications is devoted to topical problems of orientation and stabilization of a spacecraft under nonstationary disturbances with zero mean values [5, 12, 13]. This article also addresses this issue.

2. Statement of the Problem

Consider the problem of triaxial stabilization of a rigid body [14]. Assume that the body rotates around its mass center O under the action of a control torque $\vec{L}$ with an angular velocity $\vec{\omega}$.

Let $Ox_1x_2x_3$ be the principal central axes of inertia of the body. Choose two right triples of mutually orthogonal unit vectors $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ and $\vec{\eta}_1$, $\vec{\eta}_2$, $\vec{\eta}_3$, where vectors $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ are constant in the inertial frame, whereas vectors $\vec{\eta}_1$, $\vec{\eta}_2$, $\vec{\eta}_3$ are constant in the body-fixed frame. Hence, vectors $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ rotate with respect to the system $Ox_1x_2x_3$ with the angular velocity $-\vec{\omega}$.

As a result, we obtain the Euler dynamical equations

$$\mathbf{J}\dot{\vec{\omega}} + \vec{\omega} \times \mathbf{J}\vec{\omega} = \vec{L} \tag{1}$$

and the Poisson kinematic equations

$$\dot{\vec{\xi}}_i = -\vec{\omega} \times \vec{\xi}_i, \quad i = 1,2,3, \tag{2}$$

where $\mathbf{J} = diag\{A_1, A_2, A_3\}$ is inertia tensor of the body in the axes $Ox_1x_2x_3$ [1].

The problem of triaxial stabilization consists of the designing a control torque $\vec{L}$ under which the system (1), (2) admits asymptotically stable equilibrium position

$$\vec{\omega} = \vec{0}, \qquad \vec{\xi}_i = \vec{\eta}_i, \quad i = 1,2,3. \tag{3}$$

From the results of [14], it follows that the torque $\vec{L}$ can be chosen as a sum of a dissipative component $\vec{L}_d$ and a restoring one $\vec{L}_r$: $\vec{L} = \vec{L}_d + \vec{L}_r$, where

$$\vec{L}_d = \mathbf{B}\vec{\omega},$$

$$\vec{L}_r = -ch^{\nu}(\vec{\xi}_1, \vec{\xi}_2)\left(a_1\vec{\xi}_1 \times \vec{\eta}_1 + a_2\vec{\xi}_2 \times \vec{\eta}_2\right).$$

Here $\mathbf{B}$ is a constant symmetric and negative definite matrix, c, a_1, a_2 are positive constants,

$$h(\vec{\xi}_1, \vec{\xi}_2) = \frac{1}{2}\left(a_1 \| \vec{\xi}_1 - \vec{\eta}_1 \|^2 + a_2 \| \vec{\xi}_2 - \vec{\eta}_2 \|^2\right),$$

$\nu \geq 0$, and $\| \cdot \|$ denotes the Euclidean norm of a vector.

In the case where $\nu > 0$ the restoring torque is purely (strong) nonlinear. It is worth noting that purely nonlinear restoring forces are used for modeling of wide classes of mechanical systems [15-18]. Furthermore, it is known (see [18-23]), that systems with such forces are more robust with respect to time-varying disturbances and delays than linear ones.

In the present contribution, we will study the impact of time-varying disturbances on the stability of the equilibrium position (3). Let, along with the control torque $\vec{L}$, a perturbing torque $\vec{L}_p$ acts on the body. We assume that

$$\vec{L}_p = \mathbf{D}(t)\vec{G}(\vec{\xi}_1 - \vec{\eta}_1, \vec{\xi}_2 - \vec{\eta}_2), \tag{4}$$

where components of the vector $\vec{G}(\vec{u}, \vec{v})$ are continuously differentiable for $\vec{u}, \vec{v} \in \mathbb{R}^3$ homogeneous functions (with respect to the standard dilation) of the order $2\nu + 1$, the matrix $\mathbf{D}(t)$ is continuous and bounded for $t \geq 0$. In addition, let

$$\frac{1}{T}\int_0^T \mathbf{D}(\tau)d\tau \to \mathbf{0} \qquad \text{as} \qquad T \to +\infty. \tag{5}$$

Hence, the homogeneity order of the perturbing torque is equal to that of the restoring torque, whereas the matrix $\mathbf{D}(t)$ has the zero mean value. It is worth noting that disturbances with zero mean values often appear in problems of attitude control of satellites (see [1, 4, 5, 12]).

It is well known [24, 25] that, if restoring and perturbing torques are linear ($\nu = 0$), then disturbances of a such type might destroy the stability of the equilibrium position. On the over hand, in [19, 20, 22, 23, 26], some approaches to the stability analysis of nonlinear nonstationary homogeneous systems were developed. On the basis of these approaches, it was proved that if the homogeneity order of a considered system is greater than one, then time-varying disturbances with zero mean values do not destroy the asymptotic stability.

However, compared with (5), in [19, 20, 22, 23, 26], a more conservative constraint on the matrix $\mathbf{D}(t)$ was imposed. Hence, the results of [19, 20, 22, 23, 26] are nonapplicable to our problem.

Therefore, to derive conditions under which the asymptotic stability of the equilibrium position (3) can be ensured for the disturbed system, we will propose another approach. This approach is based on the decomposition method [27], the comparison principle [25] and the averaging technique [24, 25].

3. Stability Analysis

Consider the disturbed system composed of the Poisson kinematic equations (2) and the Euler dynamic equations

$$\mathbf{J}\dot{\vec{\omega}} + \vec{\omega} \times \mathbf{J}\vec{\omega} = \mathbf{B}\vec{\omega} - ch^{\nu}(\vec{\xi}_1, \vec{\xi}_2)\left(a_1\vec{\xi}_1 \times \vec{\eta}_1 + a_2\vec{\xi}_2 \times \vec{\eta}_2\right) + \vec{L}_p. \tag{6}$$

Theorem. *Let the perturbed torque $\vec{L}_p$ be of the form (4). If $\nu > 0$ and the matrix $\mathbf{D}(t)$ satisfies the condition (5), then the equilibrium position (3) of the system (2), (6) is asymptotically stable.*

Proof. First, with the aid of the decomposition method in the form developed in [21], transform the system (2), (6) to a complex system describing the interaction of two subsystems.

Instead of $\vec{\xi}_1$, $\vec{\xi}_2$, we introduce new variables by the formulae

$$\vec{\zeta}_i = \vec{\xi}_i - \vec{\eta}_i + \left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right) \times \vec{\eta}_i, \qquad i = 1,2.$$

We arrive at the system

$$\begin{aligned}
&\mathbf{J}\dot{\vec{\omega}} + \vec{\omega}\times\mathbf{J}\vec{\omega} = \mathbf{B}\vec{\omega} + \mathbf{D}(t)\vec{Q}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\\
&-c\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\sum_{j=1}^{2}a_j\left(\vec{\zeta}_j-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_j\right)\times\vec{\eta}_j,\\
&\dot{\vec{\zeta}}_i = -\vec{\omega}\times\vec{\zeta}_i + \vec{\omega}\times\left(\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right)\\
&+c\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\vec{\eta}_i\times\left(\mathbf{B}^{-1}\sum_{j=1}^{2}a_j\left(\vec{\zeta}_j-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_j\right)\times\vec{\eta}_j\right)\\
&+\vec{\eta}_i\times\left(\mathbf{B}^{-1}\left(\vec{\omega}\times\mathbf{J}\vec{\omega}\right)\right)-\vec{\eta}_i\times\left(\mathbf{B}^{-1}\mathbf{D}(t)\vec{Q}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\right),\quad i=1,2.
\end{aligned}\tag{7}$$

Here

$$\varphi(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega}) = \frac{1}{2}\left(a_1\left\|\vec{\zeta}_1-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_1\right\|^2 + a_2\left\|\vec{\zeta}_2-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_2\right\|^2\right),$$

$$\vec{Q}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega}) = \vec{G}\left(\vec{\zeta}_1-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_1,\vec{\zeta}_2-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_2\right).$$

The system (7) can be interpreted as a complex system describing interaction of the subsystems

$$\mathbf{J}\dot{\vec{\omega}} + \vec{\omega}\times\mathbf{J}\vec{\omega} = \mathbf{B}\vec{\omega}\tag{8}$$

and

$$\dot{\vec{\zeta}}_i = c\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{0})\vec{\eta}_i\times\sum_{j=1}^{2}a_j\left(\vec{\zeta}_j\times\vec{\eta}_j\right),\quad i=1,2.\tag{9}$$

Choose Lyapunov function candidates for the isolated subsystems (8) and (9), respectively, as follows

$$V_1(\vec{\omega}) = \frac{1}{2}\vec{\omega}^{\mathrm{T}}\mathbf{J}\vec{\omega},\quad V_2(\vec{\zeta}_1,\vec{\zeta}_2) = \frac{1}{2}\left(a_1\|\vec{\zeta}_1\|^2 + a_2\|\vec{\zeta}_2\|^2\right).$$

Next, aggregate these functions into a Lyapunov function for the complex system (7). Let

$$V(\omega,\vec{\zeta}_1,\vec{\zeta}_2) = V_1^{\gamma_1}(\vec{\omega}) + V_2^{\gamma_2}(\vec{\zeta}_1,\vec{\zeta}_2),\tag{10}$$

where $\gamma_i \ge 1$, $i = 1,2$.

Consider the derivative of the function (10) along the solutions of (7). We obtain

$$\begin{aligned}
&\dot{V} = \gamma_1 V_1^{\gamma_1-1}(\vec{\omega})(\vec{\omega}^{\mathrm{T}}\mathbf{B}\vec{\omega}+\vec{\omega}^{\mathrm{T}}\mathbf{D}(t)\vec{Q}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\\
&-c\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\vec{\omega}^{\mathrm{T}}\sum_{j=1}^{2}a_j\left(\vec{\zeta}_j-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_j\right)\times\vec{\eta}_j\Big)\\
&+\gamma_2 V_2^{\gamma_2-1}(\vec{\zeta}_1,\vec{\zeta}_2)\sum_{i=1}^{2}a_i\vec{\zeta}_i^{\mathrm{T}}(\vec{\omega}\times\left(\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right)\\
&+c\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\vec{\eta}_i\times(\mathbf{B}^{-1}\sum_{j=1}^{2}a_j\left(\vec{\zeta}_j-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_j\right)\times\vec{\eta}_j)\\
&+\vec{\eta}_i\times\left(\mathbf{B}^{-1}\left(\vec{\omega}\times\mathbf{J}\vec{\omega}\right)\right)-\vec{\eta}_i\times\left(\mathbf{B}^{-1}\mathbf{D}(t)\vec{Q}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\right))\\
&\le -b_1\|\vec{\omega}\|^{2\gamma_1} + b_2\|\vec{\omega}\|^{2\gamma_1-1}\left(\left\|\vec{\zeta}_1-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_1\right\|^{2\nu+1}\right.\\
&\left.+\left\|\vec{\zeta}_2-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_2\right\|^{2\nu+1}\right) + b_3\|\vec{\omega}\|^2\left(\|\vec{\zeta}_1\|^{2\gamma_2-1}+\|\vec{\zeta}_2\|^{2\gamma_2-1}\right)\\
&+c\gamma_2\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})V_2^{\gamma_2-1}(\vec{\zeta}_1,\vec{\zeta}_2)\sum_{i=1}^{2}a_i\vec{\zeta}_i^{\mathrm{T}}\left(\vec{\eta}_i\times\left(\mathbf{B}^{-1}\sum_{j=1}^{2}a_j\left(\vec{\zeta}_j-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_j\right)\times\vec{\eta}_j\right)\right)\\
&-\gamma_2 V_2^{\gamma_2-1}(\vec{\zeta}_1,\vec{\zeta}_2)\sum_{i=1}^{2}a_i\vec{\zeta}_i^{\mathrm{T}}\left(\vec{\eta}_i\times\left(\mathbf{B}^{-1}\mathbf{D}(t)\vec{Q}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\right)\right),
\end{aligned}$$

where b_1, b_2, b_3 are positive constants.

Using negative definiteness of the matrix $\mathbf{B}$, properties of homogeneous functions (see [18, 28]) and Lemma 1 from [29], it can be verified that there exist positive numbers $\delta, b_4, b_5, b_6, b_7$ such that

$$\begin{aligned}
&\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\sum_{i=1}^{2}a_i\vec{\zeta}_i^{\mathrm{T}}\left(\vec{\eta}_i\times\left(\mathbf{B}^{-1}\sum_{j=1}^{2}a_j\left(\vec{\zeta}_j-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_j\right)\times\vec{\eta}_j\right)\right)\\
&=\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\left(\sum_{i=1}^{2}a_i\vec{\zeta}_i\times\vec{\eta}_i\right)^{\mathrm{T}}\mathbf{B}^{-1}\sum_{i=1}^{2}a_i\left(\vec{\zeta}_i\times\vec{\eta}_i-\left(\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right)\times\vec{\eta}_i\right)\\
&\le\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\left(\sum_{i=1}^{2}a_i\left(\vec{\zeta}_i\times\vec{\eta}_i-\left(\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right)\times\vec{\eta}_i\right)\right)^{\mathrm{T}}\mathbf{B}^{-1}\sum_{i=1}^{2}a_i(\vec{\zeta}_i\times\vec{\eta}_i
\end{aligned}$$

$$-\left(\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right)\times\vec{\eta}_i\big)+b_4\|\vec{\omega}\|\sum_{i=1}^{2}\left\|\vec{\zeta}_i-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right\|^{2\nu+1}$$

$$\le -b_5\varphi^{\nu}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\left\|\sum_{i=1}^{2}a_i\left(\vec{\zeta}_i-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right)\times\vec{\eta}_i\right\|^2$$

$$+b_4\|\vec{\omega}\|\sum_{i=1}^{2}\left\|\vec{\zeta}_i-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right\|^{2\nu+1}$$

$$\le -b_6\sum_{i=1}^{2}\left\|\vec{\zeta}_i-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right\|^{2\nu+2}+b_4\|\vec{\omega}\|\sum_{i=1}^{2}\left\|\vec{\zeta}_i-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right\|^{2\nu+1}$$

$$\le -b_7\left(\|\vec{\zeta}_1\|^{2\nu+2}+\|\vec{\zeta}_2\|^{2\nu+2}\right)+b_8\|\vec{\omega}\|^{2\nu+2}+b_4\|\vec{\omega}\|\sum_{i=1}^{2}\left\|\vec{\zeta}_i-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right\|^{2\nu+1}$$

for

$$\sum_{i=1}^{2}\left\|\vec{\zeta}_i-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right\|^2<\delta^2.$$

Hence,

$$\dot{V}\le -b_1\|\vec{\omega}\|^{2\gamma_1}+b_8\|\vec{\omega}\|^{2\gamma_1-1}\left(\left\|\vec{\zeta}_1\right\|^{2\nu+1}+\left\|\vec{\zeta}_2\right\|^{2\nu+1}\right)$$

$$+b_9\|\vec{\omega}\|^{2\gamma_1+2\nu}+b_3\|\vec{\omega}\|^2\left(\|\vec{\zeta}_1\|^{2\gamma_2-1}+\|\vec{\zeta}_2\|^{2\gamma_2-1}\right)$$

$$-b_{10}\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu}\right)+b_{11}\|\vec{\omega}\|^{2\nu+2}\left(\|\vec{\zeta}_1\|^{2\gamma_2-2}+\|\vec{\zeta}_2\|^{2\gamma_2-2}\right)$$

$$+b_{12}\|\vec{\omega}\|\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu-1}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu-1}\right)$$

$$-\gamma_2 V_2^{\gamma_2-1}(\vec{\zeta}_1,\vec{\zeta}_2)\sum_{i=1}^{2}a_i\vec{\zeta}_i^{\mathrm{T}}\left(\vec{\eta}_i\times\left(\mathbf{B}^{-1}\mathbf{D}(t)\vec{Q}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\right)\right)$$

For,

$$\sum_{i=1}^{2}\left\|\vec{\zeta}_i-\left(\mathbf{B}^{-1}\mathbf{J}\vec{\omega}\right)\times\vec{\eta}_i\right\|^2<\delta^2,$$

where $b_8,b_9,b_{10},b_{11},b_{12}$ are positive constants.

If,

$$\max\left\{1;\frac{2\nu+1}{2}\right\}<\frac{\gamma_2+\nu}{\gamma_1}<2\nu+1,$$

then one can choose a number $\bar{\delta}>0$ such that

$$\dot{V}\le -\frac{1}{2}\left(b_1\|\vec{\omega}\|^{2\gamma_1}+b_{10}\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu}\right)\right)$$

$$-\gamma_2 V_2^{\gamma_2-1}(\vec{\zeta}_1,\vec{\zeta}_2)\sum_{i=1}^{2}a_i\vec{\zeta}_i^{\mathrm{T}}\left(\vec{\eta}_i\times\left(\mathbf{B}^{-1}\mathbf{D}(t)\vec{Q}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\right)\right)$$

for $\|\vec{\omega}\|^2+\|\vec{\zeta}_1\|^2+\|\vec{\zeta}_2\|^2<\bar{\delta}^2$.

Next, according to the approach to the application of the averaging technique proposed in [30], modify the Lyapunov function (10) as follows

$$\widetilde{V}(t,\omega,\vec{\zeta}_1,\vec{\zeta}_2)=V(\omega,\vec{\zeta}_1,\vec{\zeta}_2)$$

$$+\gamma_2 V_2^{\gamma_2-1}(\vec{\zeta}_1,\vec{\zeta}_2)\sum_{i=1}^{2}a_i\vec{\zeta}_i^{\mathrm{T}}\left(\vec{\eta}_i\times\left(\mathbf{B}^{-1}\int_0^t \mathbf{D}(s)ds\vec{G}(\vec{\zeta}_1,\vec{\zeta}_2)\right)\right).$$

The function $\widetilde{V}(t,\omega,\vec{\zeta}_1,\vec{\zeta}_2)$ and its derivative with respect to the system (7) satisfy the estimates

$$\begin{aligned} d_1\left(\|\vec{\omega}\|^{2\gamma_1}+\|\vec{\zeta}_1\|^{2\gamma_2}+\|\vec{\zeta}_2\|^{2\gamma_2}\right)-d_3(t+1)\psi(t)\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu}\right)&\\ \le\widetilde{V}(t,\omega,\vec{\zeta}_1,\vec{\zeta}_2)\le d_2\left(\|\vec{\omega}\|^{2\gamma_1}+\|\vec{\zeta}_1\|^{2\gamma_2}+\|\vec{\zeta}_2\|^{2\gamma_2}\right)&\\ +d_3(t+1)\psi(t)\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu}\right),& \end{aligned} \tag{11}$$

$$\dot{\widetilde{V}}\le -\frac{1}{2}\left(b_1\|\vec{\omega}\|^{2\gamma_1}+b_{10}\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu}\right)\right)$$

$$+d_4(t+1)\psi(t)\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu-1}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu-1}\right)\left(\|\vec{\omega}\|^{2\nu+1}\right.$$

$$\left.+\|\vec{\omega}\|^2+\|\vec{\omega}\|\left(\|\vec{\zeta}_1\|+\|\vec{\zeta}_2\|\right)+\|\vec{\zeta}_1\|^{2\nu+1}+\|\vec{\zeta}_2\|^{2\nu+1}\right)$$

$$
\begin{aligned}
&+\gamma_2 V_2^{\gamma_2-1}(\vec{\zeta}_1,\vec{\zeta}_2)\sum_{i=1}^{2} a_i\vec{\zeta}_i^{\mathrm{T}}\left(\vec{\eta}_i\times\left(\mathbf{B}^{-1}\mathbf{D}(t)\left(\vec{G}(\vec{\zeta}_1,\vec{\zeta}_2)-\vec{Q}(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\omega})\right)\right)\right)\\
&\le -\frac{1}{2}\left(b_1\|\vec{\omega}\|^{2\gamma_1}+b_{10}\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu}\right)\right)\\
&+d_4(t+1)\psi(t)\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu-1}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu-1}\right)\left(\|\vec{\omega}\|^{2\nu+1}\right.\\
&\left.+\|\vec{\omega}\|^2+\|\vec{\omega}\|\left(\|\vec{\zeta}_1\|+\|\vec{\zeta}_2\|\right)+\|\vec{\zeta}_1\|^{2\nu+1}+\|\vec{\zeta}_2\|^{2\nu+1}\right)\\
&+d_5\|\vec{\omega}\|^{2\nu+1}\left(\|\vec{\zeta}_1\|^{2\gamma_2-1}+\|\vec{\zeta}_2\|^{2\gamma_2-1}\right)+d_6\|\vec{\omega}\|\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu-1}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu-1}\right)
\end{aligned}
$$

for $t\ge 0$, $\|\vec{\omega}\|^2+\|\vec{\zeta}_1\|^2+\|\vec{\zeta}_2\|^2<\bar{\delta}^2$. Here $d_1,\ldots,d_6$ are positive constants and

$$\psi(t)=\frac{1}{t+1}\left\|\int_0^t D(s)ds\right\|\to 0 \ \text{ as } \ t\to+\infty.$$

If $\bar{\delta}$ is sufficiently small, then

$$
\begin{aligned}
\tilde{V}&\le -\frac{1}{3}\left(b_1\|\vec{\omega}\|^{2\gamma_1}+b_{10}\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu-1}\right)\right)\\
&+d_4(t+1)\psi(t)\left(\|\vec{\zeta}_1\|^{2\gamma_2+2\nu-1}+\|\vec{\zeta}_2\|^{2\gamma_2+2\nu-1}\right)\left(\|\vec{\omega}\|^{2\nu+1}\right.\\
&\left.+\|\vec{\omega}\|^2+\|\vec{\omega}\|\left(\|\vec{\zeta}_1\|+\|\vec{\zeta}_2\|\right)+\|\vec{\zeta}_1\|^{2\nu+1}+\|\vec{\zeta}_2\|^{2\nu+1}\right)
\end{aligned}
\tag{12}
$$

for $t\ge 0$, $\|\vec{\omega}\|^2+\|\vec{\zeta}_1\|^2+\|\vec{\zeta}_2\|^2<\bar{\delta}^2$.

Let $(\vec{\omega}^{\mathrm{T}}(t,\vec{\omega}_0,\vec{\zeta}_{10},\vec{\zeta}_{20},t_0),\vec{\zeta}_1^{\mathrm{T}}(t,\vec{\omega}_0,\vec{\zeta}_{10},\vec{\zeta}_{20},t_0),\vec{\zeta}_2^{\mathrm{T}}(t,\vec{\omega}_0,\vec{\zeta}_{10},\ \vec{\zeta}_{20},t_0))^{\mathrm{T}}$ denote the solution of (7) passing through the point $(\vec{\omega}_0^{\mathrm{T}},\vec{\zeta}_{10}^{\mathrm{T}},\vec{\zeta}_{20}^{\mathrm{T}})^{\mathrm{T}}$ at $t=t_0$.

With the aid of the estimates (11), (12) and the comparison principle, it can be proved the existence of positive numbers $\tilde{t},\tilde{\delta}_1,\tilde{\delta}_2$ such that if

$$\max\left\{1;2\nu;\frac{2\nu+1}{2}\right\}<\frac{\gamma_2+\nu}{\gamma_1}<2\nu+1,$$

$$t_0\ge\tilde{t},\quad \|\vec{\omega}_0\|^{2\gamma_1}+\|\vec{\zeta}_{10}\|^{2\gamma_2}+\|\vec{\zeta}_{20}\|^{2\gamma_2}<\tilde{\delta}_1 t_0^{-\frac{\gamma_2}{\nu}},$$

Then

$$\|\vec{\omega}(t,\vec{\omega}_0,\vec{\zeta}_{10},\vec{\zeta}_{20},t_0)\|^{2\gamma_1}+\|\vec{\zeta}_1(t,\vec{\omega}_0,\vec{\zeta}_{10},\vec{\zeta}_{20},t_0)\|^{2\gamma_2}+\|\vec{\zeta}_2(t,\vec{\omega}_0,\vec{\zeta}_{10},\vec{\zeta}_{20},t_0)\|^{2\gamma_2}<\tilde{\delta}_2 t^{-\frac{\gamma_2}{\nu}}$$

for $t\ge t_0$. Hence, the zero solution of (7) is asymptotically stable, and this implies the asymptotic stability of the equilibrium position (3) of the system (2), (6). This completes the proof.

4. A Numerical Simulation

To illustrate the analytical approach for construction of control strategy for rigid body attitude stabilization, consider the following numerical example, where all values are taken in International System of Units. Let $A_1=5$, $A_2=6$, $A_3=4$ are the components of the inertial tensor of a given rigid body. The equilibrium position (3) of the body is such that "aircraft"' angles φ, ϑ, ψ (Fig. 1) are all equal to zero in the inertial coordinate system.

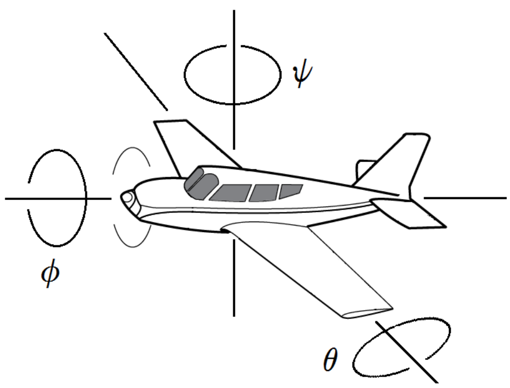

Fig. 1. Aircraft attitude motion and angles φ, θ, ψ.

The disturbing torque is taken in the form (4), where

$$\mathbf{D}(t) = diag\{\sin\sqrt{t}, 1.5\sin\sqrt{2t}, 2\sin\sqrt{t}\},$$

$$\vec{G}(\vec{\xi}_1 - \vec{\eta}_1, \vec{\xi}_2 - \vec{\eta}_2) = \left(\| \vec{\xi}_1 - \vec{\eta}_1 \|^2 + \| \vec{\xi}_2 - \vec{\eta}_2 \|^2\right)^{\nu} (\vec{\xi}_1 \times \vec{\eta}_1 + \vec{\xi}_2 \times \vec{\eta}_2).$$

Choose the matrix $\mathbf{B}$ of dissipative control torque in the form $\mathbf{B} = -\mathrm{diag}\{2,2,2\}$. Let coefficients in the restoring control torque be $a_1 = a_2 = 1$. Consider the control process governed by the system (2), (6) for different values of ν and the same initial conditions $\varphi(0) = 0.5$, $\vartheta(0) = 0.5$, $\psi(0) = -0.5$, $\omega_1(0) = \omega_2(0) = \omega_3(0) = 1.5$.

First, we take $\nu = 0$. In this case disturbing and control torques are linear and the process doesn't converge to the equilibrium position as can be seen from Fig. 2.

The theorem proved in the paper gives us the possibility to reach the goal of stabilization process. This possibility is based on applying nonlinear restoring control torque ($\nu > 0$). As is shown in Fig. 3, asymptotic stability of the rigid body equilibrium position is achievable for $\nu = 1/3$ and the same dissipative torque.

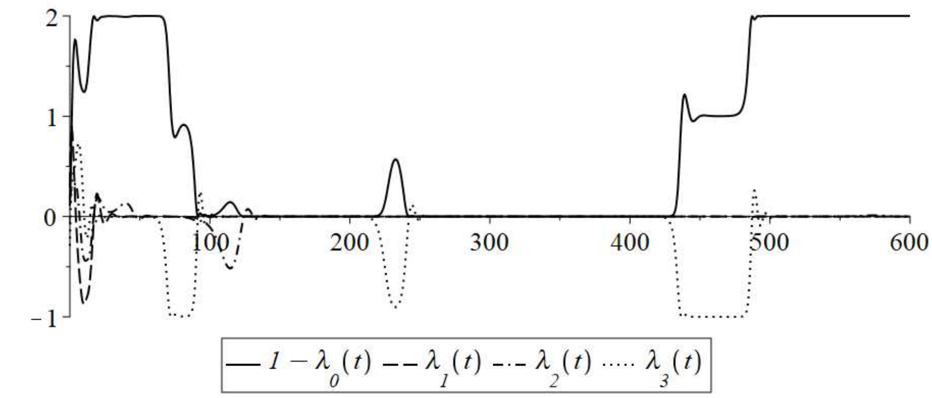

Fig. 2. Quaternion components time history, $\nu = 0$.

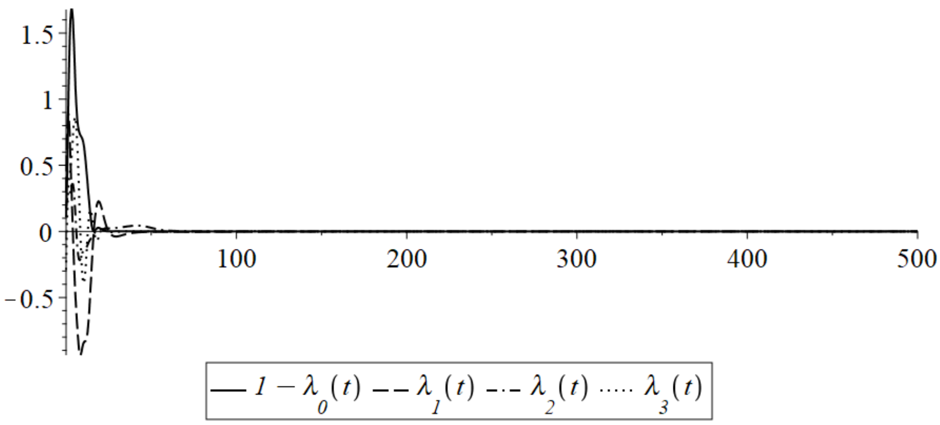

Fig. 3. Quaternion components time history, $\nu = 1/3$.

5. Conclusion

In this paper, we provide a constructive approach to robustness analysis in the problem of attitude control for a rigid body subjected to nonstationary disturbing torques with zero mean values. The suggested approach is based on the decomposition method, the comparison principle and the averaging technique for the system governing the rigid body attitude dynamics. The proved theorem provides us conditions under which disturbing torques do not violate asymptotic stability of the programmed attitude motion of the body. The suggested approach is applicable in the problems where disturbing torques are not small in magnitude. For this reason, the results obtained in the paper seems to be attractive for various engineering applications from nanomechanical oscillator [31], to large space telescope [32] since the potential of these devices is significantly affected by their increased sensitivity to external perturbations. In particular, the results of the paper can be exploited for the problem of satellite attitude stabilization with the use of electrodynamic attitude control system [29, 33, 34]. As is known, satellite that moves in the Earth's gravitational and magnetic fields [35, 36] is subjected to a lot of disturbing torques [1, 37-40]. The majority of these torques can be modelled by almost periodic functions of time with zero mean values. The magnitudes of these torques are often close to each other and, generally, they are not negligibly small [5]. For this reason, the usage of well-known perturbation techniques faces difficulties in such problems, whereas the methods based on the Lyapunov functions seem to be effective [41]. It was also demonstrated that the averaging technique, known as a powerful tool for investigation of various dynamical systems [42, 43], can be successfully applied for construction of Lyapunov functions for nonstationary differential systems.

Author Contributions

All authors contributed equally in preparation of this manuscript. All authors discussed the results, reviewed and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article..

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ORCID iD

A. Yu. Aleksandrov https://orcid.org/0000-0001-7186-7996
A. A. Tikhonov https://orcid.org/0000-0003-0838-5876

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