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Research Paper

The Rank Upgrading Technique for a Harmonic Restoring Force of Nonlinear Oscillators

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Abstract. An enhanced analytical technique for nonlinear oscillators having a harmonic restoring force is proposed. The approach is passed on the change of the auxiliary operator by another suitable one leads to obtain a periodic solution. The fundamental idea of the new approach is based on obtaining an alternative equation free of the harmonic restoring forces. This method is a modification of the homotopy perturbation method. The approach allows not only an actual periodic solution but also the frequency of the problem as a function of the amplitude of oscillation. Three nonlinear oscillators including restoring force, the simple pendulum motion, the cubic Duffing oscillator, the Sine-Gordon equation are offered to clarify the effectiveness and usefulness of the proposed technique. This approach allows an effective mathematical approach to noise and uncertain properties of nonlinear vibrations arising in physics and engineering.

Keywords: Homotopy Perturbation Method, Frequency Expansion, Periodic Solution, Restoring Force, Nonlinear Oscillation.

1. Introduction

The motion of nanoparticles in the capillary fluid moves periodically with an extremely restoring force, and it plays an important role in enhancing mass, energy, and charge transfer in many nano/micro phenomena, from lithium batteries in micro/nanodevices, which are the footstone for nano-industry. The restoring force of capillary vibration plays an important role in both nature and engineering, especially in nano/microdevices [1,2]; the vibration is a balance of the force produced by the capillary's geometric potential [3-5] and the gravity. The relationship between the frequency and the amplitude shows that an extremely restoring force is extremely helpful for mass and heat transfer through the nanofiber membrane [6-9], and it is especially important for nutrition and air transfer in a living body.

Nonlinear oscillations are a significant fact in mechanical structures, engineering problems, and physical science. All differential equations covering physical and engineering phenomena are nonlinear. The techniques of solutions of linear differential equations are relatively available and well determined. On the opposite, in the nonlinear differential equations, the methods of solutions are the lowest available and therefore no exact solution, and, overall, linear approximations are extremely used. A specific type of analytical solution specified nonlinear oscillator with a harmonic restoring force has a great quantity of importance, because, nearly all of the phenomena that appear in mathematical physics and engineering scope can be described by it. Therefore, inspect strongly nonlinear oscillators with cubic and harmonic restoring force is becoming increasingly engaged in nonlinear sciences. Moreover, gain exact solutions for nonlinear oscillatory problems has more difficulties. It is very difficult to gain the solution of nonlinear problems and in general, it is often more complicated to get an analytic approximate solution than a numerical one for a given nonlinear problem. To overcome the shortcomings, many new analytical techniques have been successfully developed. He and Jin [2] supply a short review of analytical methods for the capillary oscillator in a nanoscale deformable tube. In their letter reviews some effective methods to solve analytically the frequency-amplitude relation of the capillary oscillator, including the variational principle, the variational iteration method, the homotopy perturbation method, He's frequency formulation, and Taylor series method.

Yuste and Sánchez [10] used the so-called cubication approach, which consists of replacing the system of restoring force $f(x)$ by an equivalent cubic polynomial expression βx^3, where the value of β is determined by using a weighted mean-square method or by using the principle of harmonic balance [11,12]. Beléndez et al [13] used this idea and replaced the original second-order differential equation with the well-known Duffing equation. Uwe Starossek [14] studied the strongly nonlinear oscillator by assuming that the restoring force has a purely cubic function of the displacement variable. The investigation in nonlinear oscillators with cubic and harmonic restoring force solutions is becoming increasingly attractive in nonlinear sciences [15-17]. Moreover, obtaining exact solutions for nonlinear oscillatory problems has many difficulties. It is very problematic to solve nonlinear problems and overall, it is often more complicated to get an analytic approximation than a numerical one for an offered nonlinear problem. Only analytical approximate solutions are available, many new analytical methods have been successfully developed. Some approximation techniques have been investigated. These include the Akbari-Ganji's [18], the

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cubication technique [13], the pseudo-spectral method [19], the frequency-amplitude formulation [20], the rational variational approach [21], and the closed-form numerical [22] methods and the iteration method [23-25]. Besides, the harmonic balance [26-28,16] has been used to derive periodic solutions to strongly nonlinear oscillatory problems. For some asymptotic approximate solution of nonlinear systems see for example [29-36]. Traditional perturbation methods [37-44] are the most widely used analytical methods for solving nonlinear equations, which is the most flexible tool available for nonlinear analysis of science and engineering problems.

Here in this paper, the main goal is to obtain a periodic solution by an analytical method for the strongly nonlinear oscillation including a harmonic restoring force. We proposed a new technique to relax such a restoring force. This approach is based on obtaining an alternative system free of the trigonometric functions. The outcome system is easier to handle by any analytical perturbation method. Here, we apply the enhanced homotopy perturbation method [45-47], which including the methodology of the expanded parameter [48]. The technology of two homotopy expanded parameters is used [49-51] to construct the homotopy equation. One of these parameters is used to expand the homotopy equation and the other is used to expand the frequency-amplitude equation. To illustrate the effectiveness of the current method, three test examples are considered in this proposal.

2. The proposal method

We aim to apply the enhanced approach to obtain a periodic solution of the simple pendulum equation that has a restoring force. Thus, we consider the following equation:

$$\ddot{\theta} + a\theta = b\sin\theta; \quad \theta(0) = A, \dot{\theta}(0) = 0, \tag{1}$$

where a and b are real parameters. Also, this equation is used to describe the capillary oscillator and a detailed derivation was given in Jin et al.[1]. Several approximate solutions of (1) have been derived by using different techniques [2]. Here, we deal with the approximate periodic solutions to (1) that have been not derived before. Our approach doesn't depend on expanded the sine-function, but to derive an alternative form of it, free of the harmonic function.

By expanding the sine-function reveals that the parameter a is not the full natural frequency. At this end, a suitable primary periodic solution cannot be found. Because the operator ($L \equiv D^2 + a$), which is the highest order derivative and is assumed to be easily invertible, does not the actual auxiliary operator. For convenience, the indeed linear natural frequency must be formulated as

$$\omega_0^2 = a - b. \tag{2}$$

To perform the periodic solution of equation (1), we need to remove the wrong auxiliary operator L and replacing it with the correct auxiliary one. Without resorting to the expanded technique for the harmonic force, equation (1) can be re-arranged in another form. To illustrate the basic concept of the present proposal, let's begin with the integration of equation (1) by applying the operator L^{-1} on both sides, we get

$$\theta(t) = \theta(0) + \frac{b}{D^2 + a}\sin\theta. \tag{3}$$

To obtain an alternative form of equation (1), we re-build it so that the actual natural frequency ω_0^2 is working to get an analytical periodic solution. To illustrate this suggestion, the following process is offered:

Differentiating equation (3) twice to the variable t, we find

$$D^2\theta(t) = \frac{b}{D^2 + a}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right). \tag{4}$$

From the bits of help of the original equation (1) and its first-order derivative, one can remove the harmonic functions $\sin\theta$ and $\cos\theta$ from equation (4). At this end, the pendulum equation (1) is converted to the form

$$D^2\theta(t) = \frac{1}{D^2 + a}\left[\left(\frac{\dddot{\theta}}{\dot{\theta}} + a\right)\ddot{\theta} - \dot{\theta}^2\left(\ddot{\theta} + a\theta\right)\right]. \tag{5}$$

This formulation is free of the restoring forces, the parameter b is disappearing through the process. In formulating the modified equation, one can reset the role of the parameter, b, through the including of the natural frequency ω_0^2. At this end, the alternative form of the pendulum equation (1) is presented having a primary periodic solution when $a > b$. To analyze, such highly nonlinear equation, the perturbation technique is urgent. By applying the homotopy perturbation method [52, 53], the homotopy equation can be constructed in the form

$$D^2\theta + \omega_0^2\theta = \rho\left\{\omega_0^2\theta + \frac{1}{D^2 + a}\left[\left(\frac{\dddot{\theta}}{\dot{\theta}} + a\right)\ddot{\theta} - \dot{\theta}^2\left(\ddot{\theta} + a\theta\right)\right]\right\}; \quad \rho \in [0,1] \tag{6}$$

The additional frequency ω is introduced through the frequency extension technology [47] as follows:

$$\omega^2 = \omega_0^2 + \rho\omega_1 + \rho^2\omega_2 + \dots, \tag{7}$$

where the additional frequency ω is unknown to be determined later. Employing the expansion (7) into the homotopy equation (6), the result is

$$\left(D^2 + \omega^2\right)\theta = \rho\left\{\omega_1\theta + \left(\omega^2 - \rho\omega_1 + \dots\right)\theta + \frac{1}{D^2 + a}\left[\left(\frac{\dddot{\theta}}{\dot{\theta}} + a\right)\ddot{\theta} - \dot{\theta}^2\left(\ddot{\theta} + a\theta\right)\right]\right\}. \tag{8}$$

Consider the solution $\theta(t)$ has been expanded in the form

$$\theta(t)=\theta_0(t)+\rho\theta_1(t)+\rho^2\theta_2(t)+\dots \tag{9}$$

Substituting (9) into the above homotopy equation, equating the identical powers of ρ on both sides, we obtain the first-two unknowns $\theta_0(t)$ and $\theta_1(t)$ in the form

$$\theta_0(t)=A\cos\omega t, \tag{10}$$

$$\left(D^2+\omega^2\right)\theta_1=\left(\omega^2+\omega_1\right)\theta_0+\frac{1}{D^2+a}\left[\left(\frac{\dddot{\theta}_0}{\dot{\theta}_0}+a\right)\ddot{\theta}_0-\dot{\theta}_0^2\left(\ddot{\theta}_0+a\theta_0\right)\right]. \tag{11}$$

Inserting (10) into the first-order equation (11) becomes

$$\left(D^2+\omega^2\right)\theta_1=\left(\omega_1-\frac{1}{4}A^2\omega^2\right)A\cos\omega t-\frac{1}{4}A^3\omega^2\frac{\left(\omega^2-a\right)}{a-9\omega^2}\cos 3\omega t. \tag{12}$$

Avoiding the secular terms, we get

$$\omega_1=\frac{1}{4}A^2\omega^2. \tag{13}$$

Inserting (13) into the expansion (7), letting $\rho\to 1$, we obtain

$$\omega^2=\omega_0^2\left(1-\frac{1}{4}A^2\right)^{-1}. \tag{14}$$

The first-order approximate solution of the pendulum equation (1) is found in the form

$$\theta(t)=A\cos\omega t-\frac{1}{32}A^3\frac{\left(\omega^2-a\right)}{\left(9\omega^2-a\right)}\left(\cos 3\omega t-\cos\omega t\right). \tag{15}$$

It is seen from (14) that the periodic solution is available when the following conditions have been satisfied:

$$a>b \quad \text{and} \quad A^2<4. \tag{16}$$

3. Cubic nonlinear oscillation having the harmonic restoring force

A highly nonlinear oscillator with a cubic and harmonic restoring force is derived in the form

$$\ddot{\theta}+a\theta+Q\theta^3=b\sin\theta; \qquad \theta(0)=A,\ \dot{\theta}(0)=0, \tag{17}$$

Mathematically, this equation is considered as a modification of the equation (1), which is characterized by including the Duffing parameter. Therefore, we follow the same procedure as the previous item. So, the alternative form of the equation (17) is found in the form

$$D^2\theta(t)=\frac{1}{D^2+a}\left[\frac{\dddot{\theta}}{\dot{\theta}}\ddot{\theta}+a\ddot{\theta}-\dot{\theta}^2\left(\ddot{\theta}+a\theta+Q\theta^3\right)-6Q\theta\dot{\theta}^2\right]. \tag{18}$$

The modified homotopy equation includes the frequency expansion (7) with the two small parameters $\rho\in[0,1]$ and $\varepsilon\in[0,1]$, is coming in the form

$$\left(D^2+\omega^2\right)\theta(t)=\rho\left\{\left(\omega_1+\omega^2\right)\theta+\frac{1}{D^2+a}\left[\frac{\dddot{\theta}}{\dot{\theta}}\ddot{\theta}+a\ddot{\theta}-\dot{\theta}^2\left(\ddot{\theta}+a\theta\right)-\varepsilon Q\dot{\theta}^2\theta\left(\theta^2+6\right)\right]\right\}. \tag{19}$$

On using the expansion (9), the zero-order solution (10) is available and the first-order equation has the following configuration:

$$\left(D^2+\omega^2\right)\theta_1(t)=\left[\omega_1-\frac{1}{4}A^2\omega^2-\varepsilon QA^2\omega^2\frac{\left(12+A^2\right)}{8\left(a-\omega^2\right)}\right]A\cos\omega t+\frac{A^3\omega^2}{4\left(9\omega^2-a\right)}\left(\omega^2-a-\frac{1}{4}\varepsilon Q\left(24+A^2\right)\right)\cos 3\omega t+\frac{\varepsilon A^5\omega^2 Q}{16\left(a-25\omega^2\right)}\cos 5\omega t. \tag{20}$$

The requiring condition for the uniform solution is

$$\omega_1(\varepsilon)=\frac{1}{4}A^2\omega^2-\varepsilon QA^2\omega^2\frac{\left(12+A^2\right)}{8\left(\omega^2-a\right)}. \tag{21}$$

The final first-order approximate solution for equation (17), where $\rho\to 1$ and $\varepsilon\to 1$, is performed as

$$\theta(t)=A\cos\omega t-\frac{A^3}{32\left(9\omega^2-a\right)}\left(\omega^2-a-\frac{1}{4}QA^2\left(24+A^2\right)\right)\left(\cos 3\omega t-\cos\omega t\right)+\frac{A^5 Q}{384\left(25\omega^2-a\right)}\left(\cos 5\omega t-\cos\omega t\right). \tag{22}$$

The frequency-amplitude equation can be derived by insert (21) into the expansion (7) yields

$$\omega_0^2 + \left(\frac{1}{4}A^2 - 1\right)\omega^2 - \varepsilon Q A^4 \omega^2 \frac{\left(12 + A^2\right)}{8\left(\omega^2 - a\right)} = 0. \tag{23}$$

This is a complicated frequency-amplitude equation. The absence of the Duffing coefficient Q yields the same frequency formula (14). The perturbation technique is very suitable for the analysis of the above frequency-amplitude equation [54]. To derive an approximate solution of the above equation, we use the following expansion:

$$\omega^2(\varepsilon) = \varpi_0^2 + \varepsilon\varpi_1 + \varepsilon^2\varpi_2 + \ldots \tag{24}$$

Substituting this expansion into the equation (23), equating the identical powers of ε on both sides we get

$$\varpi_0^2 = \omega_0^2\left(1 - \frac{1}{4}A^2\right)^{-1}, \tag{25}$$

$$\varpi_1 = -\frac{\varpi_0^2 Q A^2}{2\left(4 - A^2\right)} \frac{\left(12 + A^2\right)}{\left(\varpi_0^2 - a\right)}. \tag{26}$$

In one iteration operation, we insert (25) and (26) into (24) and setting $\varepsilon \to 1$, yields

$$\omega^2 = \omega_0^2\left(1 - \frac{1}{4}A^2\right)^{-1}\left[1 - \frac{QA^2\left(A^2 + 12\right)}{2\left(aA^2 - 4b\right)}\right]. \tag{27}$$

The periodic solution is available whence the following condition is presented, with $a > b$:

$$\left(1 - \frac{1}{4}A^2\right)\left(1 - \frac{QA^2\left(A^2 + 12\right)}{2\left(aA^2 - 4b\right)}\right) > 0. \tag{28}$$

4. Deriving a periodic solution of the sine-Gordon equation

The sine-Gordon equation is a nonlinear partial differential equation, including the d'Alembert operator and the sine-function of the unknown variable. The equation, as well as several solution techniques, was known the two-century ago in the course of the study of various problems of differential geometry. The Sine-Gordon equation appears in several physical applications [51, 55-57]. The Sine-Gordon equation has attracted wide interest over the years in the depiction of classical and quantum mechanical phenomena [57]. In the current section, we consider the well-known Sine-Gordon equation that has the form

$$y_{tt} - P y_{xx} = \omega_0^2 \sin y, \tag{29}$$

where $y = y(x,t)$, with the initial conditions $y(x,0) = A(x)$, $y_t(x,0) = 0$. where $x-$ coordinate, $t-$ time, and $y-$ the unknown function. The aim is to seek a modified equation free of the harmonic restoring force. To achieve this goal, we first remember that

$$\partial_{tt}\sin y = y_{tt}\cos y - y_t^2 \sin y. \tag{30}$$

Using the fact

$$\partial_x \sin y = y_x \cos y. \tag{31}$$

Thus, one can rewrite (30) in the form

$$\partial_{tt}\sin y = \left(\frac{y_{tt}}{y_x}\partial_x - y_t^2\right)\sin y. \tag{32}$$

from the bits of help of the original equation (29), one can remove the function $\sin y$ from the formula (32), the result is

$$D_t^4 y = P D_t^2 y_{xx} + \left(\frac{y_{tt}}{y_x}\partial_x - y_t^2\right)\left(y_{tt} - P y_{xx}\right). \tag{33}$$

Since the frequency ω_0^2 has disappeared through the replacing process, we restore it, by the addition method, as the auxiliary parameter. The corresponding homotopy equation is formulated as

$$\left(D_t^4 - \omega_0^4\right)y = \rho\left[-\omega_0^4 y + P D_t^2 y_{xx} + \frac{y_{tt}}{y_x}\left(y_{ttx} - P y_{xxx}\right) - y_t^2\left(y_{tt} - P y_{xx}\right)\right];\ \rho \in [0,1] \tag{34}$$

To derive the periodic solution the parameter expansion technology is utilized so that

$$\Omega^2 = \omega_0^2 + \rho\Omega_1 + \rho\Omega_2 + \ldots \tag{35}$$

Employing the expansion (35) into the homotopy equation (34) becomes

$$\left(D_t^4-\Omega^4\right)y=\rho\left[-\Omega^2\left(\Omega^2+2\Omega_1\right)y+PD_t^2y_{xx}+\frac{y_{tt}}{y_x}\left(y_{ttx}-Py_{xxx}\right)-y_t^2\left(y_{tt}-Py_{xx}\right)\right];\ \rho\in[0,1] \tag{36}$$

Consider, as usual, the solution is given by

$$y(x,t)=y_0(x,t)+\rho y_1(x,t)+\rho^2y_2(x,t)+\ldots \tag{37}$$

It is noted that the zero-order solution, when $\rho\to 0,$ is satisfied with

$$y_0=A(x)\cos\Omega t. \tag{38}$$

The first-order of ρ is found to be

$$\left(D_t^4-\Omega^4\right)y_1=\left\{-2\Omega_1A+P\left[\frac{AA_{xxx}}{A_x}-\left(1-\frac{1}{4}A^2\right)A_{xx}\right]+\frac{1}{4}A^3\Omega^2\right\}\Omega^2\cos\Omega t-\frac{1}{4}A^2\Omega^2\left(\Omega^2A+PA_{xx}\right)\cos 3\Omega t. \tag{39}$$

Its solution, without secular terms, has performed in the following configuration:

$$y_1(x,t)=-\frac{A^2\left(\Omega^2A+PA_{xx}\right)}{320\Omega^2}\left(\cos 3\Omega t-\cos\Omega t\right). \tag{40}$$

The first-order approximate solution has been getting in the form

$$y(x,t)=\lim_{\rho\to 1}\left(y_0+\rho y_1\right)=A(x)\cos\Omega t-\frac{A^2\left(\Omega^2A+PA_{xx}\right)}{320\Omega^2}\left(\cos 3\Omega t-\cos\Omega t\right). \tag{41}$$

This solution has been derived under the following condition:

$$\Omega_1=\frac{1}{8}A^2\Omega^2+\frac{1}{2}P\left[\frac{A_{xxx}}{A_x}-\left(1-\frac{1}{4}A^2\right)\frac{A_{xx}}{A}\right]. \tag{42}$$

To formulate the frequency-amplitude equation, we insert (42) into the expansion (35) and letting $\rho\to 1,$ yields

$$\Omega^2(x)=\left(1-\frac{1}{8}A^2\right)^{-1}\left\{\omega_0^2+\frac{1}{2}P\left[\frac{A_{xxx}}{A_x}-\left(1-\frac{1}{4}A^2\right)\frac{A_{xx}}{A}\right]\right\}. \tag{43}$$

The periodic solution is available when the following condition is satisfied:

$$\left(1-\frac{1}{8}A^2\right)\left\{\omega_0^2+\frac{1}{2}P\left[\frac{A_{xxx}}{A_x}-\left(1-\frac{1}{4}A^2\right)\frac{A_{xx}}{A}\right]\right\}>0. \tag{44}$$

4.1 Traveling wave solution for the sine-Gordon equation

In this section, we try to find the traveling wave solution to the above equation (29) assuming that the initial conditions have sought as $y(x,0)=A_0\cos kx$ and $y_t(x,0)=-A_0\omega_0\sin kx$, where the parameter k refers to the wave-number of the traveling wave and A_0 denotes to a constant amplitude. Follow the above procedure, equation (29) can perform as

$$y(x,t)=\frac{\omega_0^2}{\partial_{tt}-P\partial_{xx}}\sin y. \tag{45}$$

As seen, it is a complicated nonlinear equation so we proceed as explained before, the harmonic function $\sin y$ can be relaxed so that the homotopy equation has the following configuration:

$$D_t^2y+\omega_0^2y=\rho\left\{\omega_0^2y+\frac{1}{\partial_{tt}-P\partial_{xx}}\left(\frac{y_{tt}}{y_x}\partial_x-y_t^2\right)\left(y_{tt}-Py_{xx}\right)\right\};\ \rho\in[0,1] \tag{46}$$

Utilizing the approach of the parameter expansion as given by

$$\sigma=\omega_0+\rho\sigma_1+\rho^2\sigma_2+\ldots \tag{47}$$

Employing (47) into the above equation yields:

$$\left(D_t^2+\sigma^2\right)y=\rho\left\{\left(\sigma^2+2\sigma\sigma_1\right)y+\frac{1}{\partial_{tt}-P\partial_{xx}}\left(\frac{y_{tt}}{y_x}\partial_x-y_t^2\right)\left(y_{tt}-Py_{xx}\right)\right\}. \tag{48}$$

In using the expanded solution as given by (37), we have the primary solution as $\rho\to 0,$ is satisfied with

$$y_0(x,t)=A_0\cos(\sigma t+kx). \tag{49}$$

With the help of the above zero-order solution, yields the equation that covered the first-order, has been arranged in the form

$$\left(D_t^2+\sigma^2\right)y_1(x,t)=2\sigma\sigma_1 A_0\cos(\sigma t+kx)-\frac{1}{4}A_0^3\sigma^2\left[\cos(\sigma t+kx)-\frac{1}{9}\cos 3(\sigma t+kx)\right];\qquad y_1(x,0)=0, y_{1t}(x,0)=A_0\sigma_1\sin kx \tag{50}$$

This equation has a bounded solution, because of the initial conditions, in the form

$$y_1=\frac{1}{288}A_0^3\left[2\cos(\sigma t+3kx)-\cos(\sigma t-3kx)-18\cos(\sigma t+kx)+18\cos(\sigma t-kx)-\cos 3(\sigma t+kx)\right]. \tag{51}$$

The above solution is performed under the condition

$$\sigma_1=\frac{1}{8}A_0^2\sigma. \tag{52}$$

The final first-order solution can be derived by insert (51) and (53) in the expansion (37), letting $\rho\to 1$, yields

$$y(x,t)=A_0\cos(\sigma t+kx)+\frac{1}{288}A_0^3\cos 3(\sigma t+kx)+\frac{1}{288}A_0^3\left[\cos(\sigma t-3kx)-2\cos(\sigma t+3kx)-18\cos(\sigma t+kx)+18\cos(\sigma t-kx))\right]. \tag{53}$$

Also, the frequency-amplitude equation can be performed as

$$\sigma=\omega_0\left(1-\frac{1}{8}A_0^2\right)^{-1}. \tag{54}$$

4.2 A periodic solution for a generalized sine-Gordon equation

In the present subsection, a generalized sine-Gordon equation is considered in the form

$$u_{tt}-\left(P+\omega_0^2P_0\cos u\right)u_{xx}=\omega_0^2\left(1-P_0u_x^2\right)\sin u, \tag{55}$$

where the initial conditions are $u(x,0)=A_0\cos kx,\ u_t(x,0)=-A_0\omega_0\sin kx.$ This equation can be rewritten in the form

$$u_{tt}-Pu_{xx}=\omega_0^2\left(1+P_0\partial_{xx}\right)\sin u, \tag{56}$$

where the following formula is used:

$$\partial_{xx}\sin u=u_{xx}\cos u-u_x^2\sin u. \tag{57}$$

Applying the same procedure as in the previous subsection (4.1) to replace the function $\sin u$ with its equivalent linear instruction in (56). Then equation (56) should be transformed into the following form:

$$\partial_{tt}u=\frac{(1+P_0\partial_{xx})}{(\partial_{tt}-P\partial_{xx})}\left(\frac{u_{tt}}{u_x}\partial_x-u_t^2\right)(1+P_0\partial_{xx})^{-1}(u_{tt}-Pu_{xx}). \tag{58}$$

Construct the corresponding homotopy equation with including the parameter ω_0^2 in it as an auxiliary linear part, yields

$$\left(\partial_{tt}+\omega_0^2\right)u=\rho\left[\omega_0^2u+\frac{(1+P_0\partial_{xx})}{(\partial_{tt}-P\partial_{xx})}\left(\frac{u_{tt}}{u_x}\partial_x-u_t^2\right)(1+P_0\partial_{xx})^{-1}(u_{tt}-Pu_{xx})\right];\ \rho\in[0,1] \tag{59}$$

Applying the homotopy perturbation technique to the above equation, yields

$$u_0(x,t)=A_0\cos(\tilde\sigma t+kx), \tag{60}$$

where $\tilde\sigma$ is unknown wave-frequency of the traveling wave solution and will be determined later. This frequency is given similar to the expansion (47) in which each σ is replaced by $\tilde\sigma$. The equation that covers the first-order perturbation in ρ, is given by

$$\left(\partial_{tt}+\tilde\sigma^2\right)u_1=\left(\tilde\sigma^2+2\tilde\sigma\tilde\sigma_1\right)u_0+\frac{(1+P_0\partial_{xx})}{(\partial_{tt}-P\partial_{xx})}\left(\frac{u_{0tt}}{u_{0x}}\partial_x-u_{0t}^2\right)(1+P_0\partial_{xx})^{-1}(u_{0tt}-Pu_{0xx})\ ;\ u_1(x,0)=0,\ u_{1t}(x,0)=A_0\tilde\sigma_1\sin kx. \tag{61}$$

Employing (60) in (61), after simplification, we obtain its solution in the form

$$u_1(x,t)=A_0^3\frac{\left(1-9P_0k^2\right)}{288\left(1-P_0k^2\right)}\left[2\cos(\tilde\sigma t+3kx)-\cos(\tilde\sigma t-3kx)-\cos 3(\tilde\sigma t+kx)\right]+\frac{1}{16}A_0^3\left[\cos(\tilde\sigma t-kx)-\cos(\tilde\sigma t+kx)\right], \tag{62}$$

where the frequency-amplitude formula (54) is still working. The final first-order approximate solution is found in the form

$$u(x,t)=A_0^3\frac{\left(1-9P_0k^2\right)}{288\left(1-P_0k^2\right)}\left[2\cos(\tilde\sigma t+3kx)-\cos(\tilde\sigma t-3kx)-\cos 3(\tilde\sigma t+kx)\right]+\frac{1}{16}A_0^3\left[\cos(\tilde\sigma t-kx)-\cos(\tilde\sigma t+kx)\right]+A_0\cos(\tilde\sigma t+kx). \tag{63}$$

5. Conclusion

The purpose of the article was to employ the modified HPM to find an analytical approximate periodic solution of a nonlinear oscillator with a harmonic restoring force. The approach developed here did not consist of the expanded of the harmonic restoring force, nor used the cubication approach, but to introduce an alternative form free of this force. The alternative equation was solvable by any perturbation method. In this proposal, we presented some examples to illustrate the applicability and to establish the approximate analytical periodic solutions. Also, the traveling wave solution for the Sine-Gordon equation was established. The frequency-amplitude equation was performed in each case. Conditions for the validation of a periodic solution were performed. The method adopted here is a well-established procedure for determining analytical approximations to the periodic solutions of the nonlinear oscillators having a restoring force. The current work suggests an effective modification of the well-known homotopy perturbation method for solving differential equations having a restoring force, and some new findings were obtained. It can be concluded that this article gives an absolute new avenue of research in various fields such as mathematics, vibration theory and engineering. This paper will open up a flood of opportunities for further research.

Author Contributions

Y.O. El-Dib proposed and developed the mathematical modeling of the problem and examined the theory validation. R.T. Matoog introduced a periodic solution for a generalized sine-Gordon equation. The manuscript was written throughout the contribution of all authors. All authors discussed the outcomes, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

The authors declared that there are no competing interests regarding the publication of this paper.

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