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Research Paper

Thermoelastic Memory-dependent Responses to an Infinite Medium with a Cylindrical Hole and Temperature-dependent Properties

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Abstract. The present research discusses a generalized thermoelastic model with variable thermal material properties and derivatives based on memory. Based on this new model, an infinitely long homogeneous, isotropic elastic body with a cylindrical hole is analyzed for thermal behavior analysis. The governing equations are deduced by the application of the principle of memory-dependent derivatives and the generalized law on heat conduction. In a numerical form, the governing differential equations are solved utilizing the Laplace transform technique. Numerical calculations are shown in graphs to explain the effects of the thermal variable material properties and memory dependent derivatives. In addition, the response of the cylindrical hole is studied through the effects of many parameters such as time delay, the kernel function and boundary conditions. The results obtained with those from previous literature are finally verified.

Keywords: Thermoelasticity; Phase-lags; Variable thermal material properties; Cylindrical hole; Memory dependent derivative.

1. Introduction

Thermoelasticity is a discipline of multi-physics that examines the relationship between thermal and deformation fields. This requires thermal conductivity, vibration, strain and stress induced by the heat flow. The various engineering sciences have been affected by thermoelasticity. Significant progress has led to various difficulties in the field of aircraft and computer construction, in which thermal stresses are of prime importance. Thermoelasticity makes it possible, through the action of time dependent forces and heat resources, to identify the stresses generated by the temperature field and measure the temperature dispersion.

Classical uncoupled thermoelectricity theory (UCTE) is considered to depend on Fourier 's thermal conduction law and does not address physical structures and materials, such as amorphous media, glassy, human-made porous materials, polymers and colloids. To prevent this problem, Biot [1] has implemented a Coupled Thermoelasticity Theory (CTE) which predicts an infinite speed for heat spread in accordance with the nature of the parabolic heat equation. Cattaneo [2] suggested the generalization of classical Fourier law in terms of heat conduction in terms of relaxation time in order to obtain the limited velocity of heat propagation waves.

In order to overcome the paradox of the infinite speed of the thermal waves inherent in UCTE and CTE models, both Lord and Shulman [3], Green and Lindsay [4] and Green and Naghdy [5-7] and Tzou [8] formulated generalized thermoelasticity models. Also, Tzou in [9] implemented two different time delays, respectively, in temperature gradient and heat flow in the classical Fourier law, which are called phase lags.

For two decades, numerous researchers have shown that the fractional-order derivatives models have many applications, for example in the viscoelastic mechanics, power-law phenomenon in fluid, complex network, polarization, colored noise, electrode-electrolyte and fractional kinetics, boundary layer effects, and electromagnetic waves.

In the analysis of viscoelastic materials and proofs of the relation between the linear and fractional derivatives theories, Caputo and Mainardi [10-11] found that the practical results are inconsistent. In [12], the author introduced a new mathematical framework, under proposed heat conduction of fractional order by the corresponding theorem of uniqueness, of fractional general thermoelasticity. Povstenko [13] investigated Cattaneo's generalized model with time-fractional derivatives and developed the thermal stress model. Some of these studies also examined in [14-16] with general theories on thermoelasticity with fractional derivatives. Abouelregal [17] has developed a new model of generalized thermoelasticity based on time-fractional multi-relaxation depending on fractional calculations and the expansions of the Taylor series [17].

Likewise, thermoelastic response of a rotating hollow cylinder based on generalized model with higher order derivatives and phase-lags has been investigated in [18]. Recently, Abouelregal et al. [19] proposed a generalized thermoelastic-diffusion model with higher-order fractional time-derivatives and four-phase-lags. Several studies related to the theory of thermal elasticity using different beam theories have also been investigated [20-26].

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In the last few decades, it has become evident that the next state of the physical system depends not only on its present state but also on all its historical ones. The definition of memory-dependent derivatives was introduced in [27] by Wang and Li. This new kind of derivative was a valuable mathematics resource and was a necessary link to other physical difficulties. As of now, memory-dependent derivatives (MDD) are a crucial mathematical tool in explaining many of the real world's phenomenon parallel fractional ordered derivatives. In the Lord-Shulman (LS) generalized theory of thermoelasticity in heat flow speeds, Yu et al. [28] have used the Memory-dependent derivatives (MDD). Several recent findings are reviewed in [29-35] on the general theory of thermoelasticity with MDD.

Thermal conductivity is a significant material parameter that is usually regarded as constant. Nevertheless, several experimental and theoretical investigations have shown that thermal conductivity is strongly related to changes in temperature [36]. In the case of complex high-temperature or high-energy thermal conductivity, the material properties such as elastic modulus, specific heat, and thermal conductivity are no longer constants. The linear or exponential functions of the temperature are assumed in this case [37]. Li et al. [38] have developed the generalized theory of bio-thermoelasticity based on the generalized thermoelastic calculus of modified fractional-order and in the case of varying thermal material properties. Godfrey [39] found that the thermal conductivity of ceramic is decreased by 45% if the temperature rises from 1°C to 400°C which indicates the presence of properties of temperature-dependent thermal materials. Abouelregal [40] addressed a one-dimensional thermoelastic problem based on the fractional-order theory, with a semi-infinite piezoelectric medium with temperature-dependent properties.

To date, few studies have been conducted to analyze thermoelastic problems with memory-dependent derivative and temperature-dependent properties. The present paper is devoted to investigating a theory with variable thermal material properties and derivatives based on memory. Under this model and in limited cases, different traditional and generalized thermoelasticity models can be derived. As an application of this model, we study an isotropic homogeneous cylindrical hole whose inner surface is traction free and subjected to a thermal shock. In addition, the analytical solution for various physical fields, using the Laplace transform procedure, is obtained. We depicted our numerical calculations in figures to explain the influences of the variable thermal material properties and the memory-dependent derivative. Finally, the results obtained in the previous literature are examined in detail and confirmed.

2. Thermoelastic Model with Memory-dependent Derivative

The classical Fourier's law of heat conduction is given by [1]

$$\vec{q}(x,t) = -K\nabla\theta(x,t). \tag{1}$$

Here, $\vec{q}(x,t)$ denotes the heat flux vector, $\theta = T - T_0$ represents the varying temperature in which T is the absolute temperature above the reference temperature T_0 and K denotes the thermal conductivity. Cattaneo [2] has proposed a thermal wave model of heat transfer based on single phase lagging constitutive relation:

$$\vec{q}(x,t+\tau) = -K\nabla\theta(x,t), \tag{2}$$

where τ is the phase lag of the heat flux. Furthermore, a generalized single-phase-lag model has been introduced by Lord and Shulman [3]

$$\left(1+\tau\frac{\partial}{\partial t}\right)\vec{q} = -K\nabla\theta. \tag{3}$$

In fractional order thermoelasticity theory, Sherief et al. [36] investigated a heat conduction equation as follows

$$\left(1+\tau\frac{\partial^\alpha}{\partial t^\alpha}\right)\vec{q} = -K\nabla\theta, \tag{4}$$

where $\partial^\alpha / \partial t^\alpha$ is the Caputo derivative (see [37]).

Yu et al. [27] incorporated the memory-dependent derivatives (MDD) into the generalized thermoelasticity theory of Lord-Shulman (LS) in the rate of heat flux to show the dependence on memory in the following way:

$$\left(1+\tau D_\omega^{(1)}\right)\vec{q} = -K\nabla\theta, \tag{5}$$

where $D_\omega^{(1)}$ is the memory dependent derivatives (MDD) of first order defined by

$$D_\omega^{(1)} f(r,t) = \frac{1}{\omega}\int_{t-\omega}^{t} \kappa(t-\zeta) f'_\zeta(r,\zeta)\, d\zeta, \tag{6}$$

$\omega > 0$ being the time delay and $\kappa(t-\zeta)$ is the kernel function which can be chosen freely with $0 \le \kappa(t-\zeta) \le 1$ for $\zeta \in [t-\omega, t]$. This form of MDD therefore gives a greater probability of capturing the material response.

The energy balance equation without heat sources can be written as

$$\rho C_E \frac{\partial\theta}{\partial t} + \gamma T_0 \frac{\partial}{\partial t}(\operatorname{div}\vec{u}) = -\operatorname{div}\vec{q}, \tag{7}$$

where C_E denotes the specific heat at constant strain, $\gamma = (3\lambda + 2\mu)\alpha_t$ represents the stress temperature modulus, in which α_t denotes the thermal expansion coefficient λ, μ are Lamé's constants, $\vec{u}$ is the displacement vector, ρ is the density of the medium.

By taking the divergence of Eq. (5), using Eq. (7), we obtain the following

$$(1+\tau D_\omega)\left[\rho C_E \frac{\partial\theta}{\partial t} + \gamma T_0 \frac{\partial}{\partial t}(\operatorname{div}\vec{u})\right] = -\operatorname{div}\left[K(\theta)\nabla\theta\right]. \tag{8}$$

The thermal material properties of most materials change with a rise in temperature θ and such increment temperature dependence is constant in a certain temperature range [43,44]. In conjunction with temperature increase, the following linear relations are used for the parameters K and C_E.

$$K = K(\theta) = k_0(1 + k_1\theta), \ \ C_E = C_E(\theta) = k_2(1 + k_1\theta), \tag{9}$$

where k_0 is the value of the thermal conductivity when it independent of temperature and k_1 is a non-positive constant. The thermal diffusivity is defined by $N = K / \rho C_E$, then we have

$$\rho C_E(\theta) = \frac{K(\theta)}{N}. \tag{10}$$

In this case, we use the mapping (Kirchhoff's transformation):

$$\varphi := \frac{1}{k_0}\int_0^{\theta} K(x)dx. \tag{11}$$

Applying the operator ∇ to both sides of Eq. (11), we have

$$k_0\nabla\varphi = K(\theta)\nabla\theta. \tag{12}$$

Again, applying the divergence operator to the above equation, we get

$$k_0\nabla^2\varphi = \mathrm{div}\left[K(\theta)\nabla\theta\right]. \tag{13}$$

In addition, we differentiate both sides of Eq. (12) with respect to time, one gets

$$k_0\frac{\partial\varphi}{\partial t} = K\frac{\partial\theta}{\partial t}. \tag{14}$$

In view of Kirchhoff's transformation (11) and using Eqs. (13) and (14), Eq. (8) becomes

$$(1 + \tau D_\omega)\left[\frac{k_0}{N}\frac{\partial\varphi}{\partial t} + \gamma T_0\frac{\partial}{\partial t}(\mathrm{div}\ \vec{u})\right] = k_0\nabla^2\varphi, \tag{15}$$

where $\varphi = \theta + k_1\theta^2 / 2$. Once φ is known θ is given by

$$\theta = \frac{1}{k_1}\left[\sqrt{2k_1\varphi + 1} - 1\right]. \tag{16}$$

Lastly, more fundamental equations of motion, constitutive equations and strain-displacement relationships for a homogeneous isotropic thermoelastic solid are given by

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}\left[\lambda e_{kk} - \gamma\theta\right], \tag{17}$$

$$2e_{ij} = u_{j,i} + u_{i,j}, \tag{18}$$

$$\sigma_{ij,i} + F_i = \rho\ddot{u}_i, \tag{19}$$

where ρ is the mass density, F_i is the component of the external forces.

The above system is a fully hyperbolic system in the sense that both equations of motion (19) and the equation of heat transport (17) present in the system are of a hyperbolic-type.

Now, Eq. (15) together with Eqs. (9) and (10) describe our generalized thermoelastic model with variable thermal material properties and memory-dependent derivative and denoted by MVLS.

In common practice, the kernel function is considered as the following:

$$\kappa(t, p) = \left(1 + \frac{p - t}{a}\right)^b. \tag{20}$$

In this paper we use the following memory kernel $\kappa(t - \xi)$ which investigated by Ezzat et al [40]:

$$\kappa(t - \xi) := 1 - \frac{2b}{\tau}(t - \xi) + \frac{a^2(t - \xi)}{\tau^2} = \begin{cases} 1, & \text{if } a = b = 0, \\ 1 - \dfrac{t - \xi}{\tau}, & \text{if } a = 1,\ b = \dfrac{1}{2}, \\ \left(1 - \dfrac{t - \xi}{\tau^2}\right)^2, & \text{if } a = b = 1, \end{cases} \tag{21}$$

where a, b are constants.

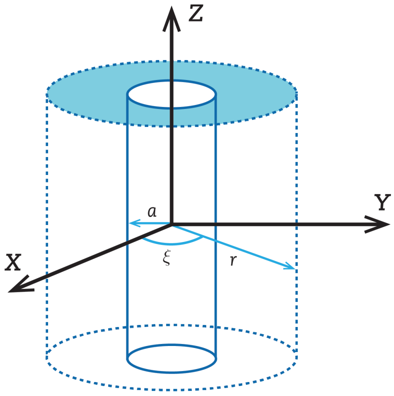

Fig. 1. Schematic diagram for the infinitely cylindrical hole.

3. Maxwell's laws

In the case of slow-moving media, the linear equations of electrodynamics homogeneous, electrically and thermally perfect conducting elastic solid are:

$$\vec{J} = \nabla \times \vec{h} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{h}}{\partial t}, \vec{E} = -\mu_0 \left(\frac{\partial \vec{u}}{\partial t} \times \vec{H} \right), \ \nabla \cdot \vec{h} = 0, \tag{22}$$

where $\vec{E}$ induced electric field, $\vec{J}$ refers to current density, μ_0 is magnetic permeability, $\vec{h}$ is induced magnetic field and $\vec{H}$ is a magnetic field.

Maxwell's stress τ_{ij} can be written as following

$$\tau_{ij} = \mu_o \left[H_i h_j + H_j h_i - H_k h_k \delta_{ij} \right]. \tag{23}$$

For a perfect conductor, The Lorentz force F_i induced by the magnetic field $\vec{H}$ can be described as

$$F_i = \mu_o \left(\vec{J} \times \vec{H} \right)_i. \tag{24}$$

4. Formulation of the problem

Let us study an infinitely long homogeneous, isotropic elastic body with a cylindrical hole of radius, as showed in Figure 1.

We suppose that the internal surface of the hole is traction free and subjected to a thermal shock with an axial magnetic field $\vec{H} = (0,0,H_0)$ effect parallel to the z-axis direction. Also, there are no heat sources or external body forces acting in the medium. Further, we take the cylindrical coordinates (r,ξ,z) where the z-axis and the axis of the cylinder are identical. According to symmetry, all the state functions can be expressed as functions of radial distance r and t.

The displacement vector can be written as

$$\vec{u} = (u(r,t),0,0). \tag{25}$$

The strain-displacement relations

$$\begin{aligned} e_{rr} &= \frac{\partial u}{\partial r}, e_{\xi\xi} = \frac{u}{r}, e_{zz} = 0, \\ e_{rz} &= e_{r\xi} = e_{\xi z} = 0. \end{aligned} \tag{26}$$

The non-vanishing constitutive equations of the generalized Hooke's law can be written as

$$\begin{aligned} \sigma_{rr} &= 2\mu \frac{\partial u}{\partial r} + \lambda e - \gamma\theta, \\ \sigma_{\xi\xi} &= 2\mu \frac{u}{r} + \lambda e - \gamma\theta, \\ \sigma_{zz} &= \lambda e - \gamma\theta, \end{aligned} \tag{27}$$

where

$$e = \frac{1}{r} \frac{\partial (ru)}{\partial r} = \frac{\partial u}{\partial r} + \frac{u}{r}. \tag{28}$$

Then the equation of motion with a magnetic field can be expressed as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\xi\xi}) + F_r = \rho \frac{\partial^2 u}{\partial t^2}. \tag{29}$$

As the constant magnetic field strength H_0 acts in the direction of the z-axis, we consider the magnetic field $\vec{H}=(0,0,H_0)$. Then from Eq. (22), we have

$$\vec{E}=\mu_0 H_0\left(0,\frac{\partial u}{\partial r},0\right),\ \vec{J}=H_0\left(0,\frac{\partial e}{\partial r}-\varepsilon_0\mu_0\frac{\partial^2 u}{\partial t^2},0\right). \tag{30}$$

From which and using Eqs. (23), (24), we get

$$F_r=\mu_0 H_0^2\left(\frac{\partial e}{\partial r}-\varepsilon_0\mu_0\frac{\partial^2 u}{\partial t^2}\right),\ \tau_{rr}=\mu_0 H_0^2 e. \tag{31}$$

Now, from Eqs. (27), (31) and Eq. (29), we obtain

$$(\lambda+2\mu+\mu_0 H_0^2)\frac{\partial e}{\partial r}-\gamma\frac{\partial\theta}{\partial r}=(\rho+\varepsilon_0\mu_0^2 H_0^2)\frac{\partial^2 u}{\partial t^2}. \tag{32}$$

Applying the operator $\frac{1}{r}\frac{\partial}{\partial r}(r)$ affected on Eq. (32) and using Kirchhoff's transformation (11), we have

$$(\lambda+2\mu+\mu_0 H_0^2)\nabla^2 e-\gamma\nabla^2\varphi=(\rho+\varepsilon_0\mu_0^2 H_0^2)\frac{\partial^2 e}{\partial t^2}, \tag{33}$$

where the Laplacian operator ∇^2 has the form

$$\nabla^2\equiv\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}. \tag{34}$$

Furthermore, under Kirchhoff's transformation Eq. (27) becomes

$$\begin{aligned}\sigma_{rr}&=2\mu\frac{\partial u}{\partial r}+\lambda e-\gamma\varphi,\\ \sigma_{\xi\xi}&=2\mu\frac{u}{r}+\lambda e-\gamma\varphi,\\ \sigma_{zz}&=\lambda e-\gamma\varphi.\end{aligned} \tag{35}$$

Also, the heat equation appeared in Eq. (15) turn out to be

$$(1+\tau D_\omega)\left[\frac{k_0}{N}\frac{\partial\varphi}{\partial t}+\gamma T_0\frac{\partial e}{\partial t}\right]=k_0\nabla^2\varphi. \tag{36}$$

Now, by using the next non-dimensional parameters

$$\begin{aligned}&\varphi'=\frac{\gamma\varphi}{\lambda+2\mu},\ \{r',u'\}=c_1\eta\{r,u\},\{t',\tau'\}=c_1^2\eta\{t,\tau\},\\ &\sigma_{ij}'=\frac{\sigma_{ij}}{\lambda+2\mu},\ \eta=\frac{\rho k_2}{k_0}=\frac{1}{N},\ c_1=\sqrt{(\lambda+2\mu)/\rho},\ a_0=\sqrt{\mu_0 H_0^2/\rho}.\end{aligned} \tag{37}$$

Eqs. (33), (35) and (36) become as (dropping the primes for convenience):

$$\nabla^2\varphi=(1+\tau D_\omega)(\frac{\partial\varphi}{\partial t}+\varepsilon\frac{\partial e}{\partial t}), \tag{38}$$

$$B_2\nabla^2 e-\nabla^2\varphi=B_3\frac{\partial^2 e}{\partial t^2}, \tag{39}$$

$$\sigma_{rr}=2\beta^2\frac{\partial u}{\partial r}+(1-2\beta^2)e-\varphi, \tag{40}$$

$$\sigma_{\xi\xi}=2\beta^2\frac{u}{r}+(1-2\beta^2)e-\varphi, \tag{41}$$

$$\sigma_{zz}=(1-2\beta^2)e-\varphi, \tag{42}$$

where

$$\varepsilon=\frac{\gamma^2 T_0}{\rho^2 k_2 c_1^2},\ B_2=1+\frac{a_0^2}{c_1^2},\ B_3=1+\frac{a_0^2}{c^2},\ \beta^2=\frac{\mu}{\lambda+2\mu};\ c=\sqrt{\frac{1}{\varepsilon_0\mu_0}}. \tag{43}$$

5. Boundary and initial conditions

We suppose that the medium initially is at rest so that initial conditions of the problem has the form:

$$u(r,t)\,|_{t=0}=\left.\frac{\partial u(r,t)}{\partial t}\right|_{t=0}=0,\ \theta(r,t)\,|_{t=0}=\left.\frac{\partial \theta(r,t)}{\partial t}\right|_{t=0}=0. \tag{44}$$

We postulated that the internal surface of the cylinder is traction free and subjected to a thermal shock at $r=a$. So it is affected by two types of boundary conditions as following:

$$\begin{aligned}&\theta(a,t)=\theta_0 H(t),\\ &\sigma_{rr}(a,t)=0,\end{aligned} \tag{45}$$

where θ_0 is constant and $H(t)$ is the Heaviside's unit step function.

The boundary conditions (45), taking into account $\varphi=\theta+k_1\theta^2/2$, becomes

$$\varphi(a,t)=\theta_0 H(t)+\frac{1}{2}k_1[\theta_0 H(t)]^2. \tag{46}$$

Moreover, the regularity boundary conditions are

$$\lim_{r\to\infty}\left\{u(r,t),\,\theta(r,t),\,\sigma_{ij}(r,t)\right\}=0. \tag{47}$$

6. The Solution in the Laplace transform domain

Applying the Laplace transform technique which defined by the following equation

$$L[f(r,t)]=\int_0^\infty f(r,t)e^{-st}dt=\bar{f}(r,s);\quad \mathrm{Re}(s)>0, \tag{48}$$

to Eqs. (38)-(42) and taking into account Eq. (9), we obtain

$$\nabla^2\bar{\varphi}=B_4(\bar{\varphi}+B_1\bar{e}), \tag{49}$$

$$B_2\nabla^2\bar{e}-\nabla^2\bar{\varphi}=B_5\,\bar{e}, \tag{50}$$

$$\bar{\sigma}_{rr}=2\beta^2\frac{d\bar{u}}{dr}+(1-2\beta^2)\bar{e}-\bar{\varphi}, \tag{51}$$

$$\bar{\sigma}_{\xi\xi}=2\beta^2\frac{\bar{u}}{r}+(1-2\beta^2)\bar{e}-\bar{\varphi}, \tag{52}$$

$$\bar{\sigma}_{zz}=(1-2\beta^2)\bar{e}-\bar{\varphi}, \tag{53}$$

where

$$B_4=s(1+\frac{\tau}{\omega}\bar{G}(s,\omega)),\ B_5=s^2B_3,\ \nabla^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}. \tag{54}$$

Moreover, one can show that the Laplace transform of any function $f(t)$ with first order MDD is given by

$$\begin{aligned}&L[\omega D_\omega f(t)]=L\left[\int_{t-\omega}^{t}\kappa(t-\zeta)f'(\zeta)d\zeta\right]=L[f(t)]G(s,\omega);\\ &G(s,\omega)=(1-e^{-s\omega})\left[1-\frac{2b}{\omega s}+\frac{2a^2}{\omega^2 s^2}\right]-\left[a^2-2b^2+\frac{2a^2}{\omega s}\right]e^{-s\omega}.\end{aligned} \tag{55}$$

If the kernel function in MDD is constant i.e. when $\kappa(t-\zeta)=1$, then,

$$\bar{G}(s,\omega)=(1-e^{-s\omega}). \tag{56}$$

Solving for $\bar{\varphi}$ or $\bar{e}$ from Eqs. (49) and (50), one obtains

$$(\nabla^4-A\nabla^2+B)\{\bar{\varphi},\bar{e}\}=0, \tag{57}$$

which can be written in the form

$$(\nabla^2-m_1^2)(\nabla^2-m_2^2)\{\bar{\varphi},\bar{e}\}=0, \tag{58}$$

where $m_i^2; i=1,2$ are the square roots of the following characteristic equation

$$\begin{aligned}&m^4-Am^2+B=0;\\ &A=\frac{(B_5+B_2B_4+\varepsilon B_4)}{B_2},\quad B=\frac{B_4B_5}{B_2}.\end{aligned} \tag{59}$$

The solutions of Eq. (57) with taking the conditions regularity condition can be proposed in the form

$$\bar{\varphi} = \sum_{i=1}^{2} A_i K_0(m_i r), \tag{60}$$

where $K_0(.)$ refers to the modified Bessel functions of the second kinds of zero-order and the parameters $A_i, (i = 1,2)$ can be determined from the boundary conditions.

Similarly, we get

$$\bar{e} = \sum_{i=1}^{2} A_i' K_0(m_i r), \tag{61}$$

where $A_i', (i = 1,2)$ are parameters. Substituting the expressions of $\bar{\varphi}$ or $\bar{e}$ into (49), we obtain:

$$A' = \left(\frac{m_i^2 - B_4}{B_4 B_1}\right) A_i. \tag{62}$$

Hence, we get

$$\bar{e} = \sum_{i=1}^{2} L_i A_i K_0(m_i r); \quad L_i = \frac{m_i^2 - B_4}{B_4 B_1} \tag{63}$$

From which together with Eqs. (28), we obtain

$$\bar{u} = -\sum_{i=1}^{2} \frac{1}{m_i} L_i A_i K_1(m_i r). \tag{64}$$

Also, the thermal stresses that appeared in Eqs. (51)-(53) can be expressed as

$$\bar{\sigma}_{rr} = \sum_{i=1}^{2} \left[(L_i - 1) A_i K_0(m_i r) + \frac{2\beta^2 L_i}{m_i r} A_i K_1(m_i r) \right]. \tag{65}$$

$$\bar{\sigma}_{\xi\xi} = \sum_{i=1}^{2} \left[\left((1 - 2\beta^2) L_i - 1\right) A_i K_0(m_i r) - \frac{2\beta^2 L_i}{m_i r} A_i K_1(m_i r) \right]. \tag{66}$$

$$\bar{\sigma}_{zz} = \sum_{i=1}^{2} \left[\left((1 - 2\beta^2) L_i - 1\right) A_i K_0(m_i r) \right]. \tag{67}$$

The boundary conditions (45), taking into account $\varphi = \theta + k_1 \theta^2 / 2$, after using the Laplace transform operator, become

$$\bar{\varphi}(a,s) = \frac{\theta_0}{s} + \frac{k_1 \theta_0^2}{2s} = \bar{g}(s), \qquad \bar{\sigma}_{rr}(a,s) = 0. \tag{68}$$

Therefore, one obtains

$$\sum_{i=1}^{2} A_i K_0(a m_i) = \frac{\theta_0}{s} + \frac{k_1 \theta_0^2}{2s}, \tag{69}$$

$$\sum_{i=1}^{2} \left[(L_i - 1) A_i K_0(a m_i) + \frac{2\beta^2 L_i}{a m_i} A_i K_1(a m_i) \right]. \tag{70}$$

By solving the previous equations, the constants $A_i, (i = 1,2)$ can be determined. Hence, using Eq.(16) we obtain the expressions for the temperature, the stress components, the displacement and other physical quantities of the medium.

It is necessary to apply Laplace inversion on the considered physical quantities obtained in the Laplace transform domain. In this paper, an accurate and efficient numerical method based on a Fourier series expansion [46] is used to obtain the inversion of the Laplace transforms. Using this method, any function in the Laplace domain can be transformed to the time-domain as

$$\psi(R,\tau) = \frac{e^{c\tau}}{\tau} \left[\frac{1}{2} \bar{\psi}(R,c) + \mathrm{Re} \sum_{n=1}^{m} \bar{\psi}(R, c + \frac{in\pi}{\tau})(-1)^n \right], \tag{71}$$

where m is the number of terms, refers to the real part and $i = \sqrt{-1}$. Form the practical experiments, the value of c agrees with the relation $c\tau = 4.7$ [46].

7. Special cases of a generalized thermoelastic model (MVLS)

In section 2, we investigated a generalized thermoelastic model with variable thermal material properties and memory-dependent derivative (MVLS). In limited cases, the proposed model is reduced to several previous models in the presence and absence of variable thermal material properties and memory dependent derivatives. The obtained models are listed as the following:

- Classical thermoelastic model (CTE): $\tau = 0, \quad G(s,\omega) = s\omega, \quad k_1 = 0.$

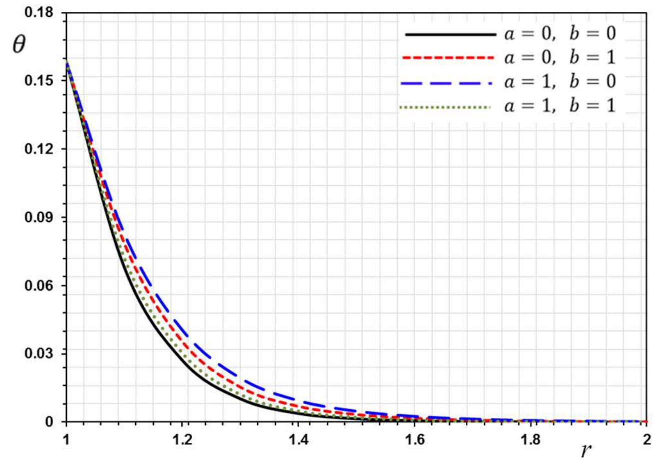

Fig. 2. The effect of the memory kernel κ on the temperature θ.

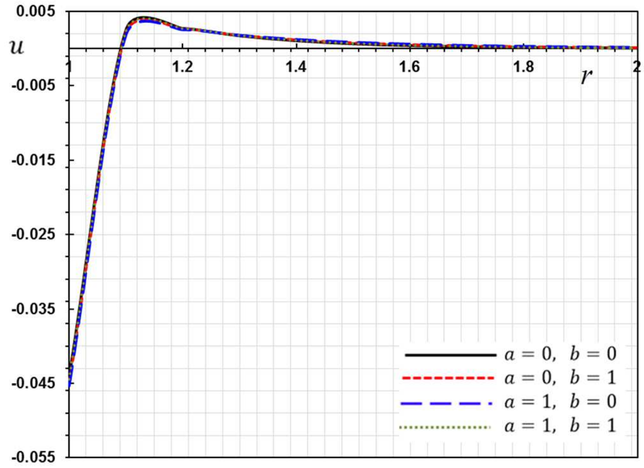

Fig. 3. The effect of the memory kernel κ on the displacement u.

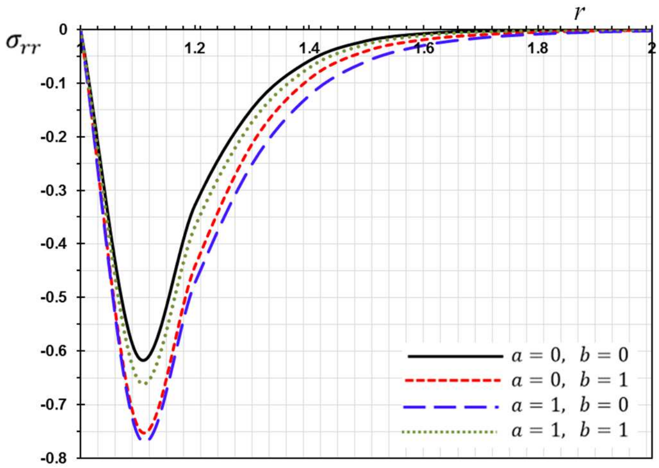

Fig. 4. The effect of the memory kernel κ on the radial stress σ_{rr}.

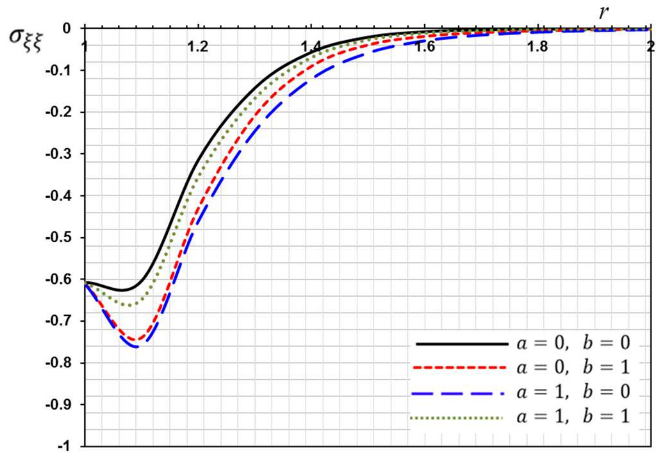

Fig. 5. The effect of the memory kernel κ on the hoop stress $\sigma_{\zeta\zeta}$.

- Lord-Shulman model (LS): $\tau > 0, \quad G(s,\omega) = s\omega, \quad k_1 = 0.$
- Lord-Shulman model with MDD (MLS): $\tau > 0, \quad G(s,\omega) = \text{Eq}(55), \quad k_1 = 0.$
- The classical thermoelastic model with variable thermal material properties (VCTE): $\tau = 0, \quad G(s,\omega) = s\omega, \quad k_1 \neq 0.$
- Lord-Shulman model with variable thermal material properties (VLS): $\tau > 0, \quad G(s,\omega) = s\omega, \quad k_1 \neq 0.$
- Lord-Shulman model with variable thermal material properties and MDD (MVLS): $\tau > 0, \quad G(s,\omega) = \text{Eq}(55), \quad k_1 \neq 0.$

8. Results and Discussion

In this section, we will study the effect of variable thermal material properties and MDD on some different physical models. The analytical results outlined above are represented numerically using Mathematica software. The silicon (Si) material was chosen for purposes of numerical evaluations to investigate the accuracy of the present numerical data. The physical constants for this subject were taken as follows [47]:

$$\lambda = 2.696\times 10^{10}\,\text{kg m}^{-1}\text{s}^{-2}, \qquad \mu = 1.639\times 10^{10}\,\text{kg m}^{-1}\text{s}^{-2}, \qquad \rho = 1740\ \text{kg m}^{-3},$$
$$K = 2.510\ \text{W m}^{-1}\ \text{K}^{-1}, \qquad C_E = 1.07\times 10^{3}\ \text{J kg K}^{-1}, \qquad T_0 = 298\ \text{K}$$

The results are presented graphically in Figures (2-17) at different values of the radius $r\,(1 \le r \le 2)$. The numerical computation was carried out for a single time, $t > 0.15$, when $\tau > 0$. Further for the numerical purpose, we take $H_0 = 100\ \text{Am}^{-1}$, $\varepsilon_0 = 1.2\ \text{Fm}^{-1}$ and $\mu_0 = 1.2\ \text{Hm}^{-1}$. The numerical results of temperature θ, radial displacement u, radial and hoop stresses σ_{rr} and $\sigma_{\zeta\zeta}$ variations are performed along with the radial distance r. Numerical calculations are performed for four cases as follows:

8.1 Influence of the kernel function κ

This section is devoted to discussing how the memory kernel κ acts on the field variables corresponding to the modified model MVLS. The obtained results are represented in Figures (2-5) for the field quantities corresponding to different values of the radius $r\,(1 \le r \le 2)$ at $t > 0.15$ and different values of the constants a, b when the phase-lag $\tau = 0.1$. These figures confirm that the physical quantities depend not only on the radius r but also on the kernel of memory dependent derivative.

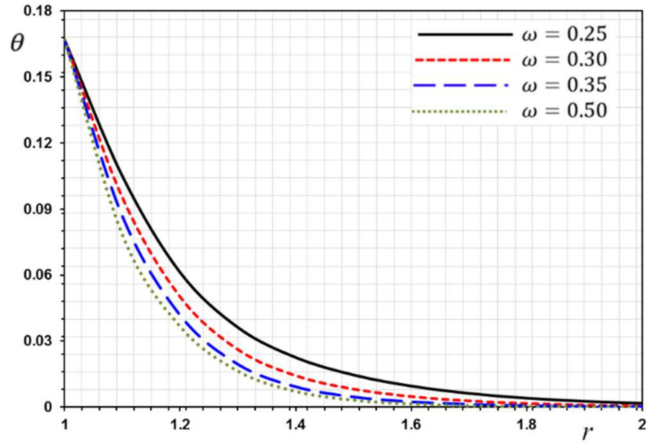

Fig. 6. The effect of the memory time delay ω on the temperature θ.

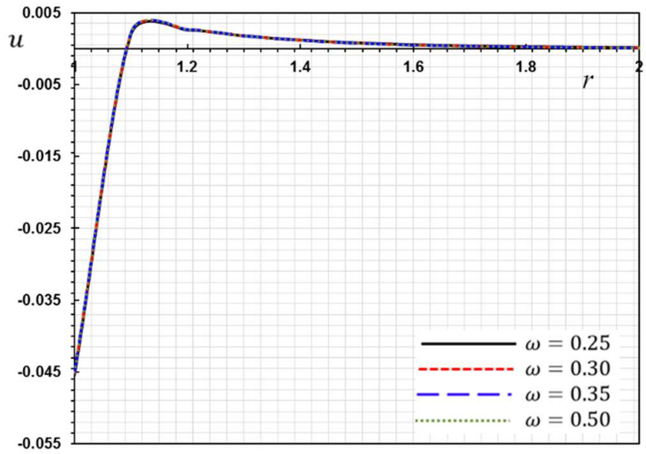

Fig. 7. The effect of the memory time delay ω on the displacement u.

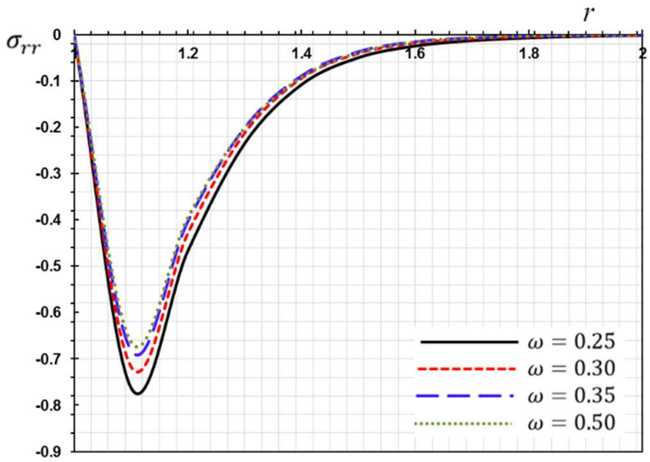

Fig. 8. The effect of the memory time delay ω on the radial stress σ_{rr}.

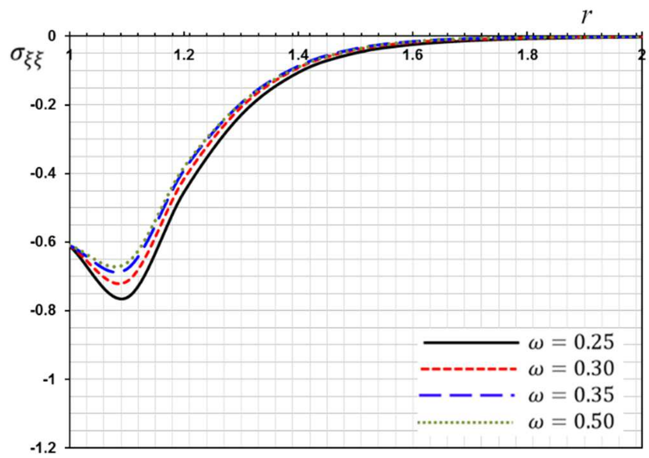

Fig. 9. The effect of the memory time delay ω on the hoop stress $\sigma_{\zeta\zeta}$.

Figure 2 depicts that the variation of temperature θ with different values of the constants a, b (the kernel κ of MDD) decreases with increasing the radius r for $1 < r < 2$. Also, we conclude that the magnitude of the temperature curve for MVLS in the case $(a = 1, b = 0)$ is greater than that for the other cases of the kernel, although they coincide to a constant value as we move away from the hole. It is manifested from the figure that the values of the temperature converge to zero when the radius r tends to 2, which agrees with the regularity boundary conditions. From Figure 3, we observe that the memory kernel κ has a weak effect on the displacement u. Moreover, we find that the values of the displacement converge to zero at r tends to 2, which corresponds to the regularity boundary conditions.

It is evident from Figures 4 and 5, that the different values of the constants a, b (the memory kernel κ), has clearly effect on the radial stress σ_{rr} and the hoop stress $\sigma_{\zeta\zeta}$, respectively. Also, we conclude that the depth of the stress curves for MVLS in the case $(a = 0, b = 0)$ is smaller than that for the other cases of the kernel. It is displayed from the figure that the values of the stresses converge to zero when the radius r tends to 2, which is in quite good agreement with the regularity boundary conditions.

Finally, we conclude that the kernel of MDD has a significant effect on all the fields except the displacement u (a weak effect).

8.2 Influence of the memory time delay ω on the physical fields

Here we study the effect of the changing of the memory time delay ω on the field variables. The obtained results are shown in Figures (6-9) for the field quantities corresponding to different values of the radius $r\,(1 \le r \le 2)$ at $t > 0.15$ and different values of the time delay ω of MDD in the case $a = 1, b = 1$, when the phase-lag $\tau = 0.2$. These figures emphasize that the physical quantities depend not only on the radius r but also on the time delay ω of memory dependent derivative.

As shown in Figure 6 by increasing the amount of the memory time delay ω, the variation of temperature θ deceased in the interval $1 < r < 2$. The memory time delay ω has a weak effect on the displacement u as in Figure 7. Also, the memory time delay ω has clearly effect on the radial stress σ_{rr} and the hoop stress $\sigma_{\zeta\zeta}$, respectively see Figures 8,9. Also, we conclude that the depth of the stress curves decreases with increasing the values of ω, although they coincide to a constant value as we move away from the hole. Finally, It is displayed from the above figures that the values of all physical fields converge to zero when the radius r tends to 2, which is in quite good agreement with the regularity boundary conditions. Moreover, these figures confirm that the memory kernel κ of MDD has a significant effect on all the fields except the displacement u (a weak effect).

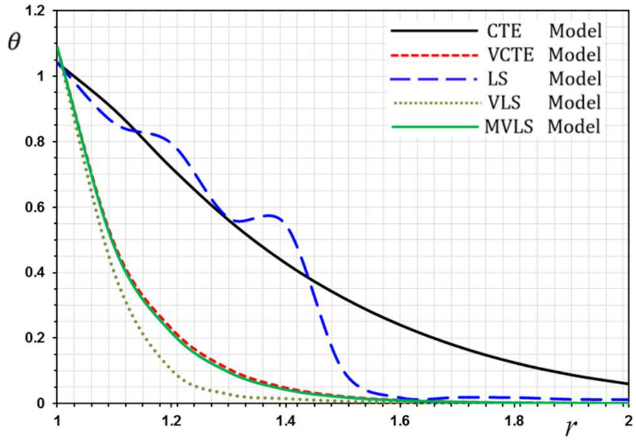

Fig. 10. Comparison between different models of thermoelasticity with respect to the temperature θ.

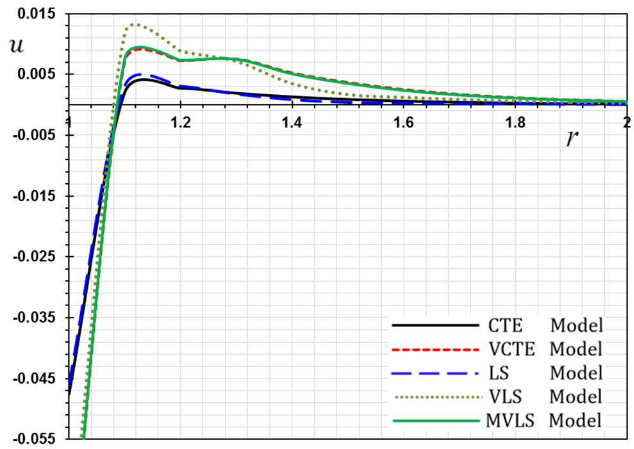

Fig. 11. Comparison between different models of thermoelasticity with respect to the displacement u.

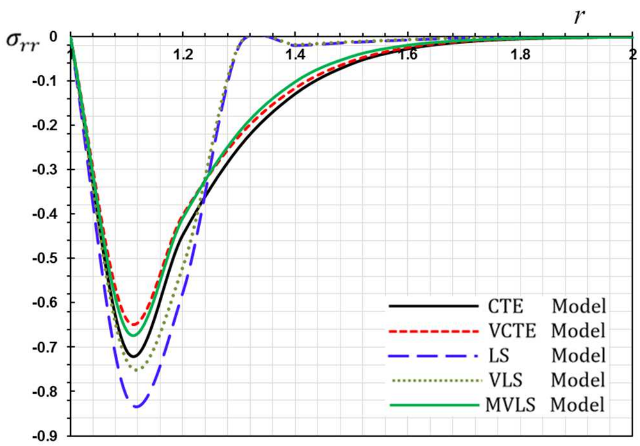

Fig. 12. Comparison between different models of thermoelasticity with respect to the radial stress σ_{rr}.

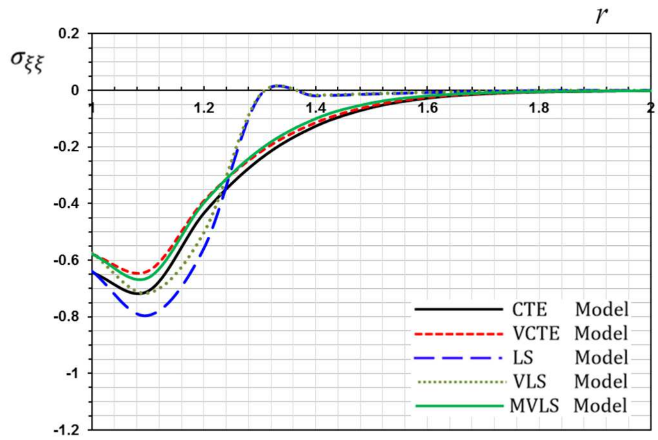

Fig. 13. Comparison between different models of thermoelasticity with respect to the hoop stress $\sigma_{\zeta\zeta}$.

8.3 Comparison between different models of thermoelasticity

This section is dedicated to studying the distributions of the physical fields for different models of thermoelasticity mentioned in the previous section (CTE, VCTE, LS, VLS and MVLS). The obtained results are represented in Figures (10-13) for the field quantities corresponding to different values of the radius $r(1 \le r \le 2)$ at $t = 0.15$, $k_1 = -0.2$ and the time delay $\omega = 0.3$ of MDD with $a = 1$, $b = 1$, when the phase-lag $\tau = 0.1$. These figures assure that the variable thermal material properties and MDD have a significant effect on all the physical quantities.

The graphs in Figures 10-13 represent the curves predicted by five different theories of thermoelasticity. Figure 10 shows that the variations of temperature for the CTE model are larger in comparison with the VCTE, MVLS and VLS, and the values of the temperature converge to zero when the radius u tends to more than 2. Also, the LS model started small than the CTE but soon becomes larger and then decreasing again. Clearly, the variations of the displacement u for the CTE theory are small in comparison with the LS theory as in Figure 11. From Figures 12 and 13 we see that the values in the classical theory of thermoelasticity (LS model) are different compared to those of other theories. Finally, the above figures emphasize that the results of our study (MVLS) differ from the classical thermoelastic model (CTE) and the Lord-Shulman model (LS) of the phenomenon of limited velocities of the propagation of heat waves. Hence, our modified model is the best.

8.4 Influence of the variable thermal material properties on the physical fields

This section is devoted to discussing how the variable thermal parameter k_1 affects the field variables of the modified model MVLS. The obtained results are shown in Figures (14-17) for the field quantities corresponding to different values of the radius $r(1 \le r \le 2)$ at $t = 0.15$ and different values of the variable thermal parameter k_1, when the phase-lag $\tau = 0.1$, the time delay $\omega = 0.3$ of MDD with $a = 1$, $b = 1$.

In Figure 14 we can see that the variable thermal parameter k_1 has a significant effect on the temperature θ in the interval $1 \le r \le 2$. Also when decreasing k_1 then the temperature θ is increased, although the temperature curves coincide to a constant value as we move away from the hole. Figure 15, it is shown that the displacement u increases as k_1 decreases in the interval $1 \le r \le 1.1$ and increases in the interval $1.1 \le r \le 2$. The graphs in Figures 16,17 represent the figures the effect of the variable thermal parameter k_1 on The radial stress σ_{rr} and the hoop stress $\sigma_{\zeta\zeta}$ and show that the curves increase with a decrease in the value of k_1. Finally, these figures assure that the physical quantities depend not only on the radius r but also on the variable thermal parameter k_1. Also, it is noticed that the variable thermal parameter k_1 has a clear effect on all the fields.

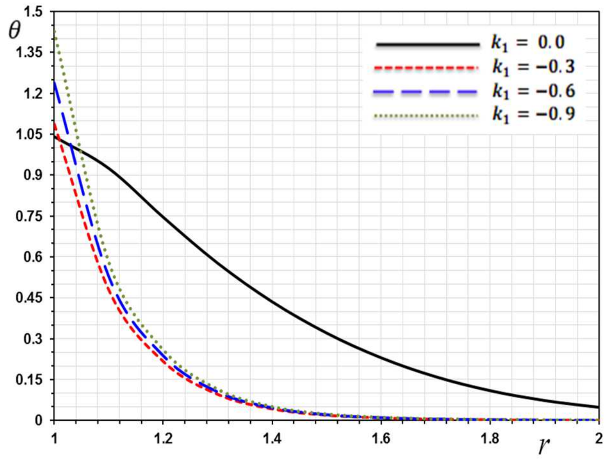

Fig. 14. The effect of the variable thermal parameter k_1 on the temperature θ.

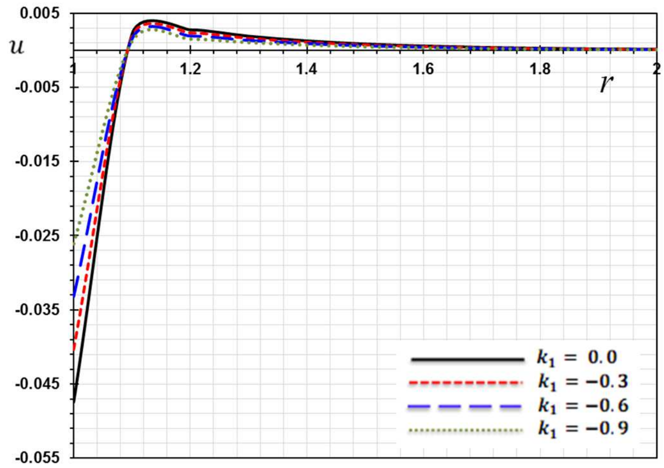

Fig. 15. The effect of the variable thermal parameter k_1 on the displacement u.

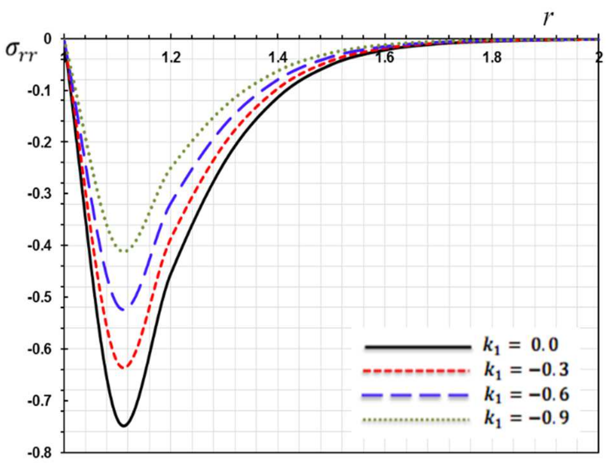

Fig. 16. The effect of the variable thermal parameter k_1 on the radial stress σ_{rr}.

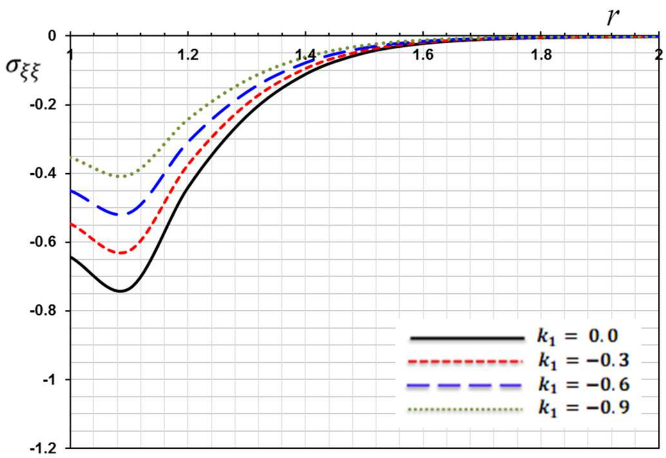

Fig. 17. The effect of the variable thermal parameter k_1 on the hoop stress $\sigma_{\xi\xi}$.

9. Conclusion

In the context of this paper, a generalized thermoelastic model with variable thermal material properties and memory-dependent derivative (MVLS) is investigated. In limited cases, the proposed model reduces to various classical, generalized thermoelasticity models (see section 7). According to this model, the distributions of the physical quantities for an isotropic homogeneous cylindrical hole whose inner surface is traction free and subjected to a thermal shock, are discussed. Numerical simulation results yield the following conclusions:

The effects of the variable thermal parameter k_1 and the memory kernel K on all the physical fields under consideration are very obvious. The results of our study (MVLS) differ from the classical thermoelastic model (CTE) and the Lord-Shulman model (LS) of the phenomenon of limited velocities of the propagation of heat waves. The obtained results are very useful for the material science researchers and material designers who are working on the development of the thermo-viscoelasticity models. The technique introduced in this study is important in real-life engineering problems and mathematical biology models according to the memory-dependent derivative.

Author Contributions

The authors contributed equally to this work. All authors discussed the results, reviewed and approved the final version of the manuscript.

Conflict of Interest

The authors declare no conflict of interest.

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