J. Appl. Comput. Mech., 7(2) (2021) 970-976
DOI: 10.22055/JACM.2021.35130.2572

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

Solution of the Problem of Analytical Construction of Optimal Regulators for a Fractional Order Oscillatory System in the General Case

Fikret A. Aliev, N.A. Aliev, N.A. Safarova, Y.V. Mamedova

Institute of Applied Mathematics, Baku State University, Z. Khalilov, 23, AZ1148 Baku, Azerbaijan
E-mails: f_aliev@yahoo.com; nihan.aliev@gmail.com; narchis2003@yahoo.com; mamedova_yegana@yahoo.com

Received September 22 2020; Revised January 23 2021; Accepted for publication January 24 2021.
Corresponding author: F.A. Aliev (f_aliev@yahoo.com)

Abstract. An algorithm is proposed for solving the problem of analytical constructing of an optimal fractional-order regulator (OFOR) in the general case. By inscribing the extended functional, the corresponding fractional order Euler-Lagrange equation is determined. Then, using the Mittag-Leffler function, a fundamental solution to the corresponding Hamiltonian system is constructed. It is shown that to obtain an analogue of the analytical construction of AM Letov's regulators, the order of the fractional derivatives must be a rational number, the denominator and numerator of which are odd numbers. Numerical illustrative examples are provided.

Keywords: Fractional derivative, Analytical construction of controllers, Hamiltonian matrix, Fundamental matrix, Mittag-Leffler function, Euler-Lagrange equation.

1. Introduction

Recently, much attention has been paid to solving the Cauchy problem, nonlocal boundary value problems, etc. for solving both ordinary and partial differential equations of fractional order [1-11]. Such problems arise in the mathematical modeling of the memory of metals [12], the motion of oscillatory systems, when the damper is a Newtonian fluid [11, 13], etc. However, to date, the problems of analytical construction of optimal regulators (ACOR) [14], when the motion of an object is described by a system of linear differential equations of fractional order, have not been considered, except for [5,9], where a special case is considered when the fractional order is 1/3 [15].

In this paper, we consider the general case of fractional order ACOR. First, the ACOR problem is formulated for this case, and by introducing an extended functional, the corresponding Euler-Lagrange equation [16-22] is obtained. Using the Mittag-Leffler function [23-25], we construct its fundamental solution matrix for the general case. Then, using the modified ACOR method [14], the matrix feedback coefficient [16-22] of the optimal regulator is constructed. The results are illustrated with various numerical examples.

2. Statement of the problem ACOR with fractional derivative and Euler-Lagrange equations

Let consider the following linear systems of differential equations of fractional order α

$$D^{\alpha}x(t) = Fx(t) + Gu(t), x(t_0) = x_0, \tag{1}$$

where $0 < \alpha < 1$, $x(t)$ is n-dimensional phase vector, whose derivative is of order $\alpha \in (0,1)$, $u(t)$ m-dimensional piecewise continuous vector of control actions, F, G - are given constant matrices of $n \times n$, $n \times m$ dimension, respectively, and stabilizable pairs [14-22], fractional derivative of order α is understood in the sense of Riemann-Liouville [25], x_0 is given nonzero n - dimension vector. The problem is to find such regulation law

$$u(t) = Kx(t), \tag{2}$$

so that the closed system (1), (2)

$$D^{\alpha}x(t) = (F + GK)x(t), x(t_0) = x_0 \tag{3}$$

Published online: January 27 2021

became asymptotically stable [19], and the quadratic functional

$$J=\frac{1}{2}\int_{t_0}^{\infty}\left(x'(t)Qx(t)+u'(t)Ru(t)\right)dt \tag{4}$$

got the minimum value. Here $Q=Q'\geq 0$, $R=R'>0$ are the matrices of corresponding dimensions, the prime denotes the transposition operation.
As in [17, 19, 21-22], let construct the extended functional

$$\bar{J}=\frac{1}{2}\int_{t_0}^{\infty}\left[(x'Qx+u'Ru)+\lambda'\left(Fx+Gu-D^{\alpha}x\right)\right]dt\ , \tag{5}$$

and substitute the expression $D^{\alpha}x$ from [10,22] into (5)

$$D^{\alpha}x=\frac{d}{dt}\int_{t_0}^{t}\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}x(\tau)d\tau, \tag{6}$$

and change the order of the integral in $\bar{J}$. Then we have [21]:

$$\bar{J}=\frac{1}{2}\int_{t_0}^{\infty}\left\{x'(\tau)Qx(\tau)+u'(\tau)Ru(\tau)+\lambda'(\tau)\left[Fx(\tau)+Gu(\tau)\right]-\lim_{0<\varepsilon\to 0}\frac{d}{d\tau}\int_{\tau-\varepsilon}^{\infty}\lambda'(t)\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}dtx(\tau)\right\}d\tau. \tag{7}$$

where $\Gamma(1-\alpha)$ is the Euler Γ function in (5). The last term in (7), after integration by parts and using the fractional derivative, goes over to the form

$$D^{\alpha}\lambda'(\tau)=\lim_{0<\varepsilon\to 0}\frac{d}{d\tau}\int_{\tau-\varepsilon}^{\infty}\lambda'(t)\frac{(\tau-t)^{-\alpha}}{\Gamma(1-\alpha)}dt\ . \tag{8}$$

Now (7) is simplified and we get

$$\bar{J}=\frac{1}{2}\int_{t_0}^{\infty}\left\{x'(\tau)Qx(\tau)+u'(\tau)Ru(\tau)+\lambda'(\tau)\left[Fx(\tau)+Gu(\tau)\right]+(-1)^{1-\alpha}\left[D^{\alpha}\lambda'(\tau)\right]x(\tau)\right\}d\tau. \tag{9}$$

Then similarly [17-19, 21, 23] from (9)

$$\frac{\partial\bar{J}}{\partial x}=x'(t)Q+\lambda'(t)F+(-1)^{1-\alpha}D^{\alpha}\lambda'(t)=0, \tag{10}$$

$$\frac{\partial\bar{J}}{\partial u}=u'(t)R+\lambda'(t)G=0, \tag{11}$$

with conditions [16,26]

$$x(\infty)=0,\quad \lambda(\infty)=0 \tag{12}$$

Thus, the Euler-Lagrange equation for the problem (1) - (4) will have the form [16, 18, 21, 24]

$$\begin{aligned}&D^{\alpha}x=Fx-GR^{-1}G'\lambda,\\&(-1)^{1-\alpha}D^{\alpha}\lambda=-Qx-F'\lambda.\end{aligned} \tag{13}$$

Let $\alpha=q/p$ and both p and q are odd. Then $(-1)^{1-\alpha}$ from (10) becomes unit. Note that the matrix of equations (13)

$$H=\begin{bmatrix}F & -GR^{-1}G'\\ -Q & -F'\end{bmatrix}, \tag{14}$$

is Hamiltonian [17] and has the eigenvalues $\mu_i\left(i=\overline{1,2n}\right)$, which are “mirror-like” symmetric on the complex plane [1],i.e. if μ_i are eigenvalues of matrices (14), then $-\mu_i$ are also eigenvalues of matrices (14). Thus, (13) is a Hamiltonian system with respect to μ. Note that in the general case, the property of “mirror symmetry” of the eigenvalues λ_i^{α} of matrices (13) $(E\lambda_i^{\alpha}-H)$ from [25] not retained for λ_i^{α} due to $\lambda_i=(\mu_i)^{1/\alpha}$. Indeed, if $\alpha=q/p$ and one of p or q are even, then the property of “mirror symmetry” is lost, i.e. either all of them will be on the right half-plane, or will be on the imaginary axis.
Thus, for our case, the ACOR problem requires both p and q to be odd numbers. Let's show it. Let one of p and q be even. Then, when applying the reverse operation, we arrive at a contradiction, i.e. p and q must be odd. For simplicity, consider the case when p and q are odd numbers and all eigenvalues of the matrix H are real numbers. Analogically we can consider the general case.

Comment. Using the results of [5], one can easily show that ACOR is true for any real number α, that is, for each $\alpha \in R$ one can approach the rational number as accurately as desired. Also, each rational number p can be approached to numbers $\alpha = q/p$ where p and q are odd numbers.
Note that the discrete analogue of the ACOR problem [16,27,28] for fractional derivatives has not been studied either, and it makes sense to consider it further.

3. Solution of the ACOR problem for the fractional derivative

Let [9]

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = T \begin{bmatrix} \tilde{x}(t) \\ \tilde{\lambda}(t) \end{bmatrix}, \; T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \tag{15}$$

believing that

$$T^{-1} H T = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}, \tag{16}$$

where T is such that A_+ has eigenvalues on the left half-plane and A_- on the right. Then system (13) with the help of transformation (15), (16) goes to the form

$$D^{\alpha} \begin{bmatrix} \tilde{x}(t) \\ \tilde{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{\lambda}(t) \end{bmatrix}. \tag{17}$$

For simplicity, without losing generality, we accept the notation

$$A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}. \tag{18}$$

Then the solution of the transformed using (15), (16) system (17) has the form [14, 16]

$$\begin{bmatrix} \tilde{x}(t) \\ \tilde{\lambda}(t) \end{bmatrix} = \tilde{X}(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{19}$$

where $\tilde{X}(t)\tilde{X}^{-1}(t_0)$ is the fundamental matrix of the solution to system (17) and и $\tilde{X}(t)$ is determined using the transformed Mittag-Leffler function [4,5,25] and the exponential function [29],

$$\tilde{X}(t) = \sum_{s=0}^{p-1} \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{s+p}{q}} e^{\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p}{q}} t} \frac{1}{\Gamma\left(\frac{s+1}{p}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{pk}{q}} \frac{(t-t_0)^{\frac{s+1}{p}+k}}{\frac{s+1}{p}+k} + \sum_{s=0}^{p-2} \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{s}{q}} e^{\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p}{q}} t_0} \frac{t^{\frac{s+1}{p}-1}}{\Gamma\left(\frac{s+1}{p}\right)} + \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p-1}{q}} e^{\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p}{q}} t_0}, \tag{20}$$

where $[c_1 \;\; c_2]^T$ is the constant unknown column vector of dimension $2n \times 1$.
In the first term $\tilde{X}(t)$ at $s = p-1$ after some transformations we have

$$\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{2p-1}{q}} e^{\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p}{q}} t} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{pk}{q}} \frac{(t-t_0)^{k+1}}{k+1} = -\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p-1}{q}} e^{\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p}{q}} t_0} + \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p-1}{q}} e^{\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p}{q}} t}.$$

Substituting the resulting expression in $\tilde{X}(t)$ we obtain

$$\tilde{X}(t) = \sum_{s=0}^{p-2} \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{s+p}{q}} e^{\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p}{q}} t} \frac{1}{\Gamma\left(\frac{s+1}{p}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{pk}{q}} \frac{(t-t_0)^{\frac{s+1}{p}+k}}{\frac{s+1}{p}+k} + \sum_{s=0}^{p-2} \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{s}{q}} e^{\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p}{q}} t_0} \frac{t^{\frac{s+1}{p}-1}}{\Gamma\left(\frac{s+1}{p}\right)} + \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p-1}{q}} e^{\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{\frac{p}{q}} t}.$$

In order to ensure the asymptotic stability of the solution to system (17), it is necessary to choose $c_2 = 0$, since this follows from the property of the diagonal matrix $\tilde{X}(t)$ (20) and the condition $\mathrm{Re}(A_+) < 0$.
Denoting

$$\tilde{X}(t) = diag\left\{\tilde{X}_1(t), \;\; \tilde{X}_2(t)\right\} \tag{21}$$

we have from (20)*

*Similarly, $\tilde{X}_2(t)$ can be determined from (20).

$$\tilde{X}_1(t)=\sum_{s=0}^{p-2}A_+^{\frac{s+p}{q}}e^{A_+^{\frac{p}{q}}t}\frac{1}{\Gamma\left(\frac{s+1}{p}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}A_+^{\frac{pk}{q}}\frac{(t-t_0)^{\frac{s+1}{p}+k}}{\left(\frac{s+1}{p}+k\right)}+\sum_{s=0}^{p-2}A_+^{\frac{s}{q}}e^{A_+^{\frac{p}{q}}t_0}\frac{t^{\frac{s+1}{p}-1}}{\Gamma\left(\frac{s+1}{p}\right)}+A_+^{\frac{p-1}{q}}e^{A_+^{\frac{p}{q}}t} \tag{22}$$

and after some simple transformations for $\left[\tilde{x}(t)\ \tilde{\lambda}(t)\right]^T$ from (19), taking into account (21) and (22), also from instability of the matrices A_- we obtain

$$\tilde{x}(t)=\tilde{X}_1(t)C_1, \qquad \tilde{\lambda}(t)=0 \tag{23}$$

Here the second relation (23) follows from $X_2(t)\equiv 0$, $\lambda(\infty)=0$ and $C_2=0$.
Based on transformations (15) under condition (23), we have

$$\begin{aligned} x(t)&=T_1\tilde{X}_1(t)C_1,\\ \lambda(t)&=T_3\tilde{X}_1(t)C_1. \end{aligned} \tag{24}$$

Finding from the first equation of (24) $\tilde{X}_1(t)C_1$ (provided that T_1^{-1} exists) in the form

$$\tilde{X}_1(t)C_1=T_1^{-1}x(t), \tag{25}$$

and substituting it into the second equation of (24), we obtain

$$\lambda(t)=T_3T_1^{-1}x(t). \tag{26}$$

If we substitute (25) into (11), we have

$$u(t)=-R^{-1}G'T_3T_1^{-1}x(t), \tag{27}$$

which is a solution to the ACOR problem (1) - (4).
In this case, the closed system (1) + (27) will be as

$$D^{(\alpha)}x(t)=(F-GR^{-1}G'T_3T_1^{-1})x(t)\equiv Bx(t), \quad x(t_0)=x_0, \tag{28}$$

where $B=(F-GR^{-1}G'T_3T_1^{-1})$, the solution of equations (28) using relations (25) has the form

$$x(t)=T_1\tilde{X}_1(t)C_1,$$

which at $t\to\infty$, $\tilde{X}_1(t)\to 0$, where $x(t)\to 0$.
As in the classical case [30], it is possible to write a computational algorithm for solving the ACOR.

4. Examples

1. At $\alpha=\frac{1}{3},(p=3,q=1)$ the matrix solution [22] of system (28) takes the form [3, 20],

$$X(t)=\sum_{s=0}^{1}B^{s+3}e^{B^3t}\frac{1}{\Gamma\left(\frac{s+1}{3}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}B^{3k}\frac{(t-t_0)^{\frac{s+1}{3}+k}}{\left(\frac{s+1}{3}+k\right)}+\sum_{s=0}^{1}B^se^{B^3t_0}\frac{t^{\frac{s+1}{3}-1}}{\Gamma\left(\frac{s+1}{3}\right)}+B^2e^{B^3t}. \tag{29}$$

Then the general solution of equation (28) takes the form:

$$x(t)=X(t)C, \tag{30}$$

where $X(t)$ is defined as (29).
Taking into account the initial condition (1) in (30), we have:

$$X(t_0)C=x(t_0)=x_0, \tag{31}$$

where from (29) we obtain

$$X(t_0)=\sum_{s=0}^{2}B^se^{B^3t_0}\frac{t_0^{\frac{s+1}{3}-1}}{\Gamma\left(\frac{s+1}{3}\right)}. \tag{32}$$

Determining the constant C from (31) in the form

$$C=X^{-1}(t_0)x_0, \tag{33}$$

and substituting in (30) the general solution (30) for the solution of the corresponding Cauchy problem (28), we have:

$$x(t) = X(t)X^{-1}(t_0)x_0 . \tag{34}$$

Thus, from (29) $\lim_{t\to\infty} X(t) = 0$ and (34) we obtain that closed-loop system (28) is asymptotically stable, that is, conditions (12) are satisfied.

2. Consider the following scalar case

$$D^{\alpha}x = x + u$$

with a functional

$$J = \int_0^{\infty} (3x^2 + u^2)dt .$$

In this case, (13) and (14) have the form

$$\begin{bmatrix} D^{\alpha}x \\ D^{\alpha}\lambda \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}, \; H = \begin{bmatrix} 1 & -1 \\ -3 & -1 \end{bmatrix}$$

Here the matrix H has eigenvalues

$$\mu_{1,2i} = (-1)^{i} \pm 2. \tag{35}$$

For simplicity, we choose $\alpha = 1/3$. Then from (18) and (34) we have $\lambda_1 = 8, \lambda_2 = -8$.
As can be seen from these considerations, if one of p and q or both are even, then the property of "mirrority" is lost. The matrix T from (15) has the form

$$T = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{pmatrix},$$

where $T_1 = 1,\ T_3 = 3$, and from (6) $A_+ = -2,\ A_- = 2$.
If to take this into account in (25), then for $\lambda(t)$ we have

$$\lambda(t) = 3x(t) ,$$

and from (26)

$$u(t) = -3x(t) .$$

Thus, the closed system (3) or (27) takes the form

$$D^{1/3}x(t) = -2x(t), \quad x(t_0) = x_0,$$

and its solution from (28) (at $B = -2$) will have the form [16, 18, 19, 22]

$$x(t) = \sum_{s=0}^{1}(-2)^{s+3}e^{-8t}\frac{1}{\Gamma\left(\frac{s+1}{3}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(-2)^{3k}\frac{(t-t_0)^{\frac{s+1}{3}+k}}{\left(\frac{s+1}{3}+k\right)} + \sum_{s=0}^{1}(-2)^{s}e^{-8t_0}\frac{t^{\frac{s-2}{3}}}{\Gamma\left(\frac{s+1}{3}\right)} + 4e^{-8t}.$$

It can be easily shown that at $t \to \infty$ the solution of $x(t)$ tends to zero.

5. Conclusion

For the first time, the ACOR problem is considered in the general case for stabilizable systems of fractional order linear differential equations. A regulation law is given and it is shown that the closed-loop feedback system is asymptotically stable. The results are illustrated with a numerical example. Very interesting is the case considered in pregnancy domain of the problem of (1)-(4), which in special case was solving in [31, 32]. In the fractional case these problems solved in [33] and in general case in [34].

Nomenclature

$x(t)$	n-dimensional phase vector, whose derivative is of order $\alpha \in (0,1)$,	x_0	Given nonzero n - dimension vector
$u(t)$	m-dimensional piecewise continuous vector of control actions	$Q = Q' \geq 0,\ R = R' \geq 0$	The matrices of corresponding dimensions
F,G	Given constant matrices of $n \times n$, $n \times m$ dimension, respectively	$\Gamma(1-\alpha)$	The Euler Γ function

Author Contributions

F.A. Aliev planned the scheme and initiated the project; N.A. Aliev analyzed the results; N.A. Safarova and Y.V. Mamedova developed the mathematical modeling. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

Funding

The authors received no financial support for the research, authorship, and publication of this article.

References

[1] Miller, K.S., Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[2] Odibat, Z., Fractional Power Series Solutions of Fractional Differential Equations by using Generalized Tylor Series, *Appl. Comput. Math.*, 19(1), 2020, 47-58.
[3] Harikrishnan, S., Kanagarajan, K., El Sayed, E.M., Existence and Stability Results for Differential Equations with Complex Order Involving Hilfer Fractional Derivative, *TWMS J. Pure and Applied Mathematics,* 10(1), 2019, 94-101.
[4] Abbas, S., Benchohra, M., Hamidi, N., et al., Hilfer and Hadamard Fractional Differential Equations in Frechet Spaces, *TWMS J. of Pure and Applied Mathematics*, 10(1), 2019, 102-116.
[5] Aliev, F.A., Aliev, N.A., Safarova, N.A., Transformation of the Mittag-Leffler Function to an Exponential Function and of its Applications to Problems with a Fractional Derivative, *Appl. Comput. Math.,* 18(3), 2019, 311-325.
[6] Sweilam, N.H., Nagy, A.M., El-Sayed, A.A., Sinc-Chebyshev Collocation Method for Time-Fractional Order Telegraph Equation, *Appl. Comput. Math.*, 19(2), 2020, 162-174.
[7] Set, E., Akdemir, A.O., Ozata, F., Grüss Type İnequalities for Fractional İntegral Operator İnvolving the Extended Generalized Mittag-Leffler Function, *Appl. Comput. Math.,* 19(3), 2020, 415-420.
[8] Panakhov, E., Ercan, A., Bas, E., et al, Hilfer Fractional Spectral Problem via Bessel Operator, *TWMS J. of Pure and Applied Mathematics*, 10(2), 2019, 199-211.
[9] Aliev, F.A., Aliev, N.A., Mutallimov, M.M., Namazov, A.A., Identification Problem for Determining the Fractional-order Derivative of an Oscillatory System, *Proceedings of IAM.*, 7(2), 2018, 234-246.
[10] Balachandran, K., Govindaraj, V., Rodrıguez-Germa, L., Trujillo, J.J., Stabilizatility of Fractional Dynamical Systems, *Frac. Calc. Appl. Anal.,* 17(2), 2014, 511-532.
[11] Bonilla, B., Rivero, M., Trujillo, J.J., On Systems of Linear Fractional Differential Equations with Constant Coefficients, *Appl. Math. Comput.*, 187, 2007, 68-78.
[12] Rabotnov, Yu.N., *Elements of Hereditary Mechanics of Solids*, Nauka, Moscow, 1977.
[13] Monje, C.A., Chen, Y.Q., Vinagre, B.M., Xue, D., Felik, V., *Fractional-order Systems and Controls. Fundamentals and Applications,* Springer-Verlag, London, 2010.
[14] Letov, A.M., Analytical Design of Controllers, *Automation and Telemechanics*, 21(4), 1960, 436-441.
[15] Aliev, F.A., Aliev, N.A., Safarova, N.A., Gasimova, K.G. Analytical Construction of Regulators for Systems with Fractional Derivatives, *Proceed. of IAM,* 2, 2017, 252-265.
[16] Aliev, F.A., Larin, V.B., Naumenko, K.I., Suntsev, V.I., *Optimization of Linear Time Invariant Control Systems*, Naukova Dumka, Kiev, 1978.
[17] Bryson, A., Ho, Yu.Sh., *Applied Theory of Optimal Control*, Mir, Moscow, 1972.
[18] Aliev, F.A., Larin, V.B., *Optimization of Linear Control Systems*, Gordon Breach, Amsterdam, 1998.
[19] Kalman, R., Falb, P., Arbib, M., *Essays on the Mathematical Theory of Systems,* Mir, Moscow, 1972.
[20] Kvakernaak, H., Sivan, R., *Linear Optimal Control Systems*, Mir, Moscow, 1977.
[21] Aliev, F.A., *Methods for Solving Applied Problems of Optimization of Dynamic Systems*, Elm, Baku, 1989.
[22] Andreev, Yu.I., *Control of Finite-dimensional Linear Objects*, Nauka, Moscow, 1976.
[23] Mittag-Leffler, G., Sur la Representation Analytique d`unebranche Uniformed`une Function Monogene, *Acta Mathematica*, 29, 1904, 101-181.
[24] Aliev, F.A., Aliev, N.A., Safarov,a N.A., New representation of Mittag-Leffler function through the exponential functions with rational derivatives, Proceedings of the 7th International Conference on Control and Optimization with Industrial Applications, 26-28 August, 2020, Baku, Azerbaijan, 80-82.
[25] Samko, S.G, Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives: Theory and applications*, Gordon and Breach Science publishers, Yverdon, Switzerland, 1993.
[26] Larin, V.B., Naumenko, K.I., Suntsev, V.I., *Spectral Methods for the Synthesis of Linear Systems with Feedback*, Naukova Dumka, Kiev, 1973.
[27] Aliev, F.A., Larin, V.B., Synthesis of Optimal Impulse Controllers for Ideal Measurement of Object Coordinates, In the book: *Navigational Gyroscopic Systems*, Kiev, 1973.
[28] Aliev, F.A., Larin, V.B., *Synthesis of Discrete Time Invariant Stabilization Systems*, In the book: *Discrete Control Systems*, Kiev, 1974.
[29] Gantmakher, F.R., *Matrix Theory*, Nauka, Moscow, 1968.
[30] Repin, Yu.A., Tretyakov, V.E., Solution of the Problem of Analytical Construction of Regulators on Electronic Modeling Devices, *Automation and Telemechanics*, 24(6), 1963, 738-743.
[31] Larin, V.B., Suntsev, V.I., On the Problem of Analytical Construction of Regulators, *Automation and Telemechanics,* 12(4), 1968, 142-145.
[32] Youla, D.C., Jabr, H. A., Bongiorno, J.J., Jr. Modern Wiener-Hopf Design of Optimal Controllers-part ii: The multivariable case, *IEEE Transactions on Automatic Control*, 21(3), 1976, 319–338.
[33] Aliev, F.A., Aliev, N.A., Velieva, N.I., Safarova, N.A., Larin Parameterization to Solve the Problem of Analytical Construction of the Optimal Regulator of Oscillatory Systems with Liquid Dampers, *Journal of Applied and Computational Mechanics*, 6, 2020, 1426-1430.
[34] Mahmudov, N.I., Huseynov, G,T., Aliev, N.A., Aliev, F.A., Analytical Approach to a Class of Bagley-Torvik Equations, *TWMS Pure Appl. Math.*, 11(2), 2020, 238-258.

ORCID iD

Fikret A. Aliev https://orcid.org/0000-0001-5402-8920
N.A. Aliev https://orcid.org/0000-0001-8531-8648
N.A. Safarova https://orcid.org/0000-0003-3829-8896
Y.V. Mamedova https://orcid.org/0000-0003-1307-7894

How to cite this article: Aliev F.A., Aliev N.A., Safarova, N.A., Mamedova, Y.V. Solution of the Problem of Analytical Construction of Optimal Regulators for a Fractional Order Oscillatory System in the General Case, *J. Appl. Comput. Mech.*, 7(2), 2021, 970–976. https://doi.org/10.22055/JACM.2021.35130.2572