Solution of the Problem of Analytical Construction of Optimal Regulators for a Fractional Order Oscillatory System in the General Case

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Abstract. An algorithm is proposed for solving the problem of analytical constructing of an optimal fractional-order regulator (OFOR) in the general case. By inscribing the extended functional, the corresponding fractional order Euler-Lagrange equation is determined. Then, using the Mittag-Leffler function, a fundamental solution to the corresponding Hamiltonian system is constructed. It is shown that to obtain an analogue of the analytical construction of AM Letov’s regulators, the order of the fractional derivatives must be a rational number, the denominator and numerator of which are odd numbers. Numerical illustrative examples are provided.

Keywords: Fractional derivative, Analytical construction of controllers, Hamiltonian matrix, Fundamental matrix, Mittag-Leffler function, Euler-Lagrange equation.

1. Introduction

Recently, much attention has been paid to solving the Cauchy problem, nonlocal boundary value problems, etc. for solving both ordinary and partial differential equations of fractional order [1-11]. Such problems arise in the mathematical modeling of the memory of metals [12], the motion of oscillatory systems, when the damper is a Newtonian fluid [11, 13], etc. However, to date, the problems of analytical construction of optimal regulators (ACOR) [14], when the motion of an object is described by a system of linear differential equations of fractional order, have not been considered, except for [5,9], where a special case is considered when the fractional order is 1/3 [15].

In this paper, we consider the general case of fractional order ACOR. First, the ACOR problem is formulated for this case, and by introducing an extended functional, the corresponding Euler-Lagrange equation [16-22] is obtained. Using the Mittag-Leffler function [23-25], we construct its fundamental solution matrix for the general case. Then, using the modified ACOR method [14], the matrix feedback coefficient [16-22] of the optimal regulator is constructed. The results are illustrated with various numerical examples.

2. Statement of the problem ACOR with fractional derivative and Euler-Lagrange equations

Let consider the following linear systems of differential equations of fractional order α

\[ D^\alpha x(t) = Fx(t) + Gu(t), x(t_0) = x_0, \]  

where \( 0 < \alpha < 1 \), \( x(t) \) is \( n \)-dimensional phase vector, whose derivative is of order \( \alpha \in (0,1) \), \( u(t) \) \( m \)-dimensional piecewise continuous vector of control actions, \( F,G \) - are given constant matrices of \( n \times n \), \( n \times m \) dimension, respectively, and stabilizable pairs [14-22], fractional derivative of order \( \alpha \) is understood in the sense of Riemann-Liouville [25], \( x_0 \) is given nonzero \( n \)-dimensional vector. The problem is to find such regulation law

\[ u(t) = Kx(t), \]  

so that the closed system (1), (2)

\[ D^\alpha x(t) = (F + GK)x(t), x(t_0) = x_0 \]
became asymptotically stable [19], and the quadratic functional

\[ J = \frac{1}{2} \int_{0}^{\infty} \left[ x'(t)Qx(t) + u'(t)Ru(t) \right] dt \]  

(4)

got the minimum value. Here \( Q \geq 0, R = R' > 0 \) are the matrices of corresponding dimensions, the prime denotes the transposition operation.

As in [17, 19, 21-22], let construct the extended functional

\[ \mathcal{J} = \frac{1}{2} \int_{0}^{\infty} \left[ x'(t)Qx(t) + u'(t)Ru(t) + \lambda'(t) \left( Fx + Gu - D'x \right) \right] dt, \]

(5)

and substitute the expression \( D'x \) from [10,22] into (5)

\[ \alpha = \lim_{\substack{t \to -\infty \\Gamma(1-\alpha)}} \int_{t}^{\infty} \frac{d}{d\tau} \left( \Gamma(1-\alpha) \right) dx(\tau) d\tau. \]

(6)

and change the order of the integral in \( \mathcal{J} \). Then we have [21]:

\[ \mathcal{J} = \frac{1}{2} \int_{0}^{\infty} \left[ x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau) + \lambda'(\tau) \left( Fx(\tau) + Gu(\tau) \right) - \lim_{\substack{\tau \to -\infty}} \frac{d}{d\tau} \int_{\tau}^{\infty} \frac{(t-\tau)^{\alpha}}{\Gamma(1-\alpha)} dx(\tau) d\tau. \]

(7)

where \( \Gamma(1-\alpha) \) is the Euler \( \Gamma \) function in (5). The last term in (7), after integration by parts and using the fractional derivative, goes over to the form

\[ D'\lambda(\tau) = \lim_{\substack{\tau \to -\infty}} \frac{d}{d\tau} \int_{\tau}^{\infty} \frac{(t-\tau)^{\alpha}}{\Gamma(1-\alpha)} dt. \]

(8)

Now (7) is simplified and we get

\[ \mathcal{J} = \frac{1}{2} \int_{0}^{\infty} \left[ x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau) + \lambda'(\tau) \left( Fx(\tau) + Gu(\tau) \right) + (-1)^{1-\alpha} \left| D'\lambda(\tau) \right| x(\tau) \right] d\tau. \]

(9)

Then similarly [17,19,21,23] from (9)

\[ \frac{\partial \mathcal{J}}{\partial x} = x'(t)Q + \lambda'(t)F + (-1)^{1-\alpha} D'\lambda(t) = 0, \]

(10)

\[ \frac{\partial \mathcal{J}}{\partial u} = u'(t)R + \lambda'(t)G = 0, \]

(11)

with conditions [16,26]

\[ x(\infty) = 0, \quad \lambda(\infty) = 0 \]

(12)

Thus, the Euler-Lagrange equation for the problem (1) - (4) will have the form [16, 18, 21, 24]

\[ D'x = Fx - GR^{-1}G', \]

\[ (-1)^{1-\alpha} D'\lambda = -Qx - F'\lambda. \]

(13)

Let \( \alpha = q/p \) and both \( p \) and \( q \) are odd. Then \( (-1)^{1-\alpha} \) from (10) becomes unit. Note that the matrix of equations (13)

\[ H = \begin{bmatrix} F & -GR^{-1}G' \\ -Q & -F' \end{bmatrix}, \]

(14)

is Hamiltonian [17] and has the eigenvalues \( \mu_i (i = 1,2n) \), which are “mirror-like" symmetric on the complex plane [1], i.e. if \( \mu_i \) are eigenvalues of matrices (14), then \( -\mu_i \) are also eigenvalues of matrices (14). Thus, (13) is a Hamiltonian system with respect to \( \mu \).

Note that in the general case, the property of “mirror symmetry” of the eigenvalues \( \lambda_i \) of matrices (13) \( (E\lambda^i - H) \) from [25] not retained for \( \lambda_i^\alpha \) due to \( \lambda_i = (\mu_i)^{\lambda_i/\alpha} \). Indeed, if \( \alpha = q/p \) and one of \( p \) or \( q \) are even, then the property of “mirror symmetry” is lost, i.e. either all of them will be on the right half-plane, or will be on the imaginary axis.

Thus, for our case, the ACOR problem requires both \( p \) and \( q \) to be odd numbers. Let’s show it. Let one of \( p \) and \( q \) be even. Then, when applying the reverse operation, we arrive at a contradiction, i.e. \( p \) and \( q \) must be odd. For simplicity, consider the case when \( p \) and \( q \) are odd numbers and all eigenvalues of the matrix \( H \) are real numbers. Analogically we can consider the general case.
Comment. Using the results of [5], one can easily show that ACOR is true for any real number $\alpha$, that is, for each $\alpha \in \mathbb{R}$ one can approach the rational number as accurately as desired. Also, each rational number $p$ can be approached to numbers $\alpha = q/p$ where $p$ and $q$ are odd numbers. Note that the discrete analogue of the ACOR problem [16,27,28] for fractional derivatives has not been studied either, and it makes sense to consider it further.

3. Solution of the ACOR problem for the fractional derivative

Let [9]

$$
\begin{align*}
\dot{x}(t) &= A \dot{x}(t), \\
\dot{\lambda}(t) &= T \dot{\lambda}(t),
\end{align*}
$$

believing that

$$
T^{-1}HT = 
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix},
$$

where $T$ is such that $A_\lambda$ has eigenvalues on the left half-plane and $A_\mu$ on the right. Then system (13) with the help of transformation (15), (16) goes to the form

$$
D^\alpha \begin{bmatrix}
\dot{x}(t) \\
\dot{\lambda}(t)
\end{bmatrix} = 
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix} \begin{bmatrix}
\dot{x}(t) \\
\dot{\lambda}(t)
\end{bmatrix},
$$

For simplicity, without losing generality, we accept the notation

$$
A = 
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}.
$$

Then the solution of the transformed using (15), (16) system (17) has the form [14, 16]

$$
\begin{align*}
\dot{x}(t) &= X(t) \left[ \begin{array}{c} c_1 \\ c_2 \end{array} \right], \\
\dot{\lambda}(t) &= X(t) \left[ \begin{array}{c} c_1 \\ c_2 \end{array} \right],
\end{align*}
$$

where $X(t)$ is the fundamental matrix of the solution to system (17) and $X(t)$ is determined using the transformed Mittag-Leffler function [4,5,25] and the exponential function [29],

$$
X(t) = \sum_{k=0}^{\infty} \begin{bmatrix}
0 & 0 \\
A & 0
\end{bmatrix}^{\frac{\alpha^k}{p}} t^\frac{\alpha^k}{p} \frac{T^{\alpha^k} - 1}{\Gamma(\alpha^k + 1)} - \begin{bmatrix}
0 & 0 \\
0 & A
\end{bmatrix}^{\frac{\alpha^k}{p}} t^\frac{\alpha^k}{p} \frac{T^{\alpha^k} - 1}{\Gamma(\alpha^k + 1)},
$$

where $[c_1, c_2]^T$ is the constant unknown column vector of dimension $2n \times 1$. In the first term $X(t)$ at $s = p - 1$ after some transformations we have

$$
\begin{align*}
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}^{\frac{\alpha^k}{p}} t^\frac{\alpha^k}{p} \frac{T^{\alpha^k} - 1}{\Gamma(\alpha^k + 1)} = & \\
\begin{bmatrix}
0 & A \\
A & 0
\end{bmatrix}^{\frac{\alpha^k}{p}} t^\frac{\alpha^k}{p} \frac{T^{\alpha^k} - 1}{\Gamma(\alpha^k + 1)}
\end{align*}
$$

Substituting the resulting expression in $X(t)$ we obtain

$$
\begin{align*}
\dot{X}(t) = & \\
\end{align*}
$$

In order to ensure the asymptotic stability of the solution to system (17), it is necessary to choose $c_2 = 0$, since this follows from the property of the diagonal matrix $X(t)$ (20) and the condition $\text{Re}(A_\mu) < 0$.

Denoting

$$
\dot{X}(t) = \text{diag}\left\{ \dot{X}_1(t), \dot{X}_2(t) \right\}
$$

we have from (20)"
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and after some simple transformations for \( \hat{x}(t) \) from (19), taking into account (21) and (22), also from instability of the matrices \( A_\alpha \) we obtain

\[
\hat{x}(t) = \tilde{X}_0(t)C_1, \quad \hat{\lambda}(t) = 0
\]  

Here the second relation (23) follows from \( X_0(t) \equiv 0 \), \( \lambda(\infty) = 0 \) and \( C_1 = 0 \).

4. Examples

1. At \( \alpha = \frac{1}{3} \) \((p = 3, q = 1)\) the matrix solution [22] of system (28) takes the form [3, 20],

\[
X(t) = \sum_{k=0}^{\infty} B^k e^{Bk} \frac{1}{\Gamma(\frac{s+1}{3})} \sum_{k=0}^{\infty} (-1)^k \frac{t^{t-k}}{k!} \sum_{k=0}^{\infty} B^k e^{Bk} \left( \frac{s+1}{3} + k \right) + \sum_{k=0}^{\infty} B^k e^{Bk} \frac{t^{s+1-k}}{\Gamma \left( \frac{s+1}{3} \right)} + B^k e^{Bk}.
\]  

Then the general solution of equation (28) takes the form:

\[
x(t) = X(t)C_1, 
\]  

where \( X(t) \) is defined as (29).

Taking into account the initial condition (1) in (30), we have:

\[
X(t_0)C = x(t_0) = x_0, 
\]  

where from (29) we obtain

\[
X(t_0) = \sum_{k=0}^{\infty} B^k e^{Bk} \frac{t^{s+1-k}}{\Gamma \left( \frac{s+1}{3} \right)} 
\]  

Determining the constant \( C_1 \) from (31) in the form

\[
C = X^{-1}(t_0)x_0,
\]  

and substituting in (30) the general solution (30) for the solution of the corresponding Cauchy problem (28), we have:
Thus, from (29) \( \lim_{t \to \infty} X(t) = 0 \) and (34) we obtain that closed-loop system (28) is asymptotically stable, that is, conditions (12) are satisfied.

2. Consider the following scalar case

\[ D'x = x + u \]

with a functional

\[ J = \int_0^\infty (3x^2 + u^2)dt . \]

In this case, (13) and (14) have the form

\[
\begin{bmatrix}
D'x \\
D'\lambda
\end{bmatrix} =
\begin{bmatrix}
1 & -1 \\
-3 & -1
\end{bmatrix} \lambda,
H = \begin{bmatrix}
1 & -1 \\
-3 & -1
\end{bmatrix}
\]

Here the matrix \( H \) has eigenvalues

\[
\mu_{1,2} = (-1)^i \pm \sqrt{2}.
\]

For simplicity, we choose \( \alpha = 1/3 \). Then from (18) and (34) we have \( \lambda_i = 8, \lambda_{i'} = -8 \).

As can be seen from these considerations, if one of \( p \) and \( q \) or both are even, then the property of “mirrority” is lost. The matrix \( T \) from (15) has the form

\[
T = \begin{bmatrix}
1 & 1 \\
3 & -1
\end{bmatrix},
T^{-1} = \begin{bmatrix}
1 & 1 \\
3 & 1
\end{bmatrix},
\]

where \( T_1 = 1, T_2 = 3 \), and from (6) \( A_1 = -2, A_2 = 2 \).

If to take this into account in (25), then for \( \lambda(t) \) we have

\[ \lambda(t) = 3x(t) , \]

and from (26)

\[ u(t) = -3x(t) . \]

Thus, the closed system (3) or (27) takes the form

\[ D^{1/2}x(t) = -2x(t), \quad x(t_0) = x_0, \]

and its solution from (28) (at \( B = -2 \)) will have the form [16, 18, 19, 22]

\[
x(t) = \sum_{k=0}^{\infty} (-2)^{k} e^{-x_0 \frac{t}{3}} \sum_{i=0}^{\infty} \left( \begin{array}{c}
\frac{s+1}{3} \\
\frac{s+1}{3} + i
\end{array} \right) \left( \begin{array}{c}
\frac{t - t_0}{3} \\
\frac{t - t_0}{3} + i
\end{array} \right) + \sum_{i=0}^{\infty} (-2)^i e^{-x_0 \frac{t}{3}} \sum_{k=0}^{\infty} \left( \begin{array}{c}
\frac{s+1}{3} \\
\frac{s+1}{3} + k
\end{array} \right) \left( \begin{array}{c}
\frac{t - t_0}{3} \\
\frac{t - t_0}{3} + k
\end{array} \right) + 4e^{-x_0}.
\]

It can be easily shown that at \( t \to \infty \) the solution of \( x(t) \) tends to zero.

5. Conclusion

For the first time, the ACOR problem is considered in the general case for stabilizable systems of fractional order linear differential equations. A regulation law is given and it is shown that the closed-loop feedback system is asymptotically stable. The results are illustrated with a numerical example. Very interesting is the case considered in pregnancy domain of the problem of (1)-(4), which in special case was solving in [31, 32]. In the fractional case these problems solved in [33] and in general case in [34].

**Nomenclature**

- \( x(t) \): n-dimensional phase vector, whose derivative is of order \( \alpha \in (0,1) \),
- \( x_0 \): Given nonzero n-dimensional vector
- \( u(t) \): m-dimensional piecewise continuous vector of control actions
- \( F, G \): Given constant matrices of \( n \times n \), \( n \times m \) dimension, respectively
- \( Q = Q' \geq 0, R = R' \geq 0 \): The matrices of corresponding dimensions
- \( \Gamma(1-\alpha) \): The Euler \( \Gamma' \) function
Author Contributions
F.A. Aliev planned the scheme and initiated the project; N.A. Aliev analyzed the results; N.A. Safarova and Y.V. Mamedova developed the mathematical modeling. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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