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Research Paper

A novel approach to compute the numerical solution of variable coefficient fractional Burgers' equation with delay

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Abstract. In this article, we come up with a novel numerical scheme based on Haar wavelet (HW) along with non-standard finite difference (NSFD) scheme to solve time fractional Burgers' equation with variable diffusion coefficient and time delay. In the solution process, we discretize the fractional time derivative by NSFD $L1$ formula and spatial derivative by HWs series expansion. We use the quasilinearisation process to linearize the nonlinear term. Also, the convergence of the scheme is discussed. The efficiency and correctness of the proposed scheme are assessed by L_∞-error and L_2-error norms.

Keywords: Fractional Burgers' equation; Nonstandard finite difference; Haar wavelets; Variable coefficient; Time delay.

1. Introduction

Recently, the study of fractional calculus has been given considerable attention because of its wide applications in modeling of signal processing, electrochemistry, mathematical biology, fluid dynamics, and other scientific areas. Several books on fractional calculus have been published by the researchers such as Miller and Ross [1], Oldham and Spainer [2], Podlubny [3] and Kilbas et al. [4], Agarwal et al. [5]. Also, a delay differential equations have numerous applications in chemical, biological and transportation systems which can be found in [6–9, 11].

In this article, we consider the variable coefficient partial differential equation with time delay, given by

$$D_t^\sigma \omega(x,t) - (\mu(x)\omega_x(x,t))_x + \omega^p(x,t)\omega_x(x,t) = g\big(x,t,\omega(x,t-\tau)\big), \qquad (x,t)\in(0,1)\times(0,T], \tag{1}$$

with the boundary conditions (BCs)

$$\omega(0,t)=\beta_0(t), \quad \omega(1,t)=\beta_1(t), \quad t\in(0,T] \tag{2}$$

and initial condition (IC)

$$\omega(x,t)=\varphi(x,t), \qquad (x,t)\in[0,1]\times[-\tau,0]. \tag{3}$$

Here

A1 $\omega(x,t)\in C^{2,1}\Big([0,1]\times[-\tau,T]\Big)$ is unknown function with time variable t and space variable x.

A2 p is any positive integer.

A3 $\mu(x)$ is space variable coefficient which satisfies $0<c_0\le\mu(x)\le c_1$.

A4 $\varphi(x,t)$ is sufficiently smooth prehistory function.

A5 $g(x,t,\omega(x,t-\tau))$ stands for nonlinear delay source term with $\tau>0$ and the fractional derivative of order σ $(0<\sigma\le 1)$ is defined as

$$D_t^\sigma\omega(x,t)=\begin{cases}\frac{1}{\Gamma(1-\sigma)}\int_0^t (t-\xi)^{-\sigma}\frac{\partial\omega(x,\xi)}{\partial\xi}d\xi, & 0<\sigma<1,\\ \frac{\partial\omega(x,t)}{\partial t}, & \sigma=1,\end{cases} \tag{4}$$

where $\Gamma()$ represents gamma function (see [4]).

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For $\sigma = 1$, Eq.(1) becomes classical semilinear convection reaction-diffusion equation with time delay. Several numerical schemes have been studied for $\sigma = 1$. For instant, Zang et al. [12] discussed a linearized compact multi-splitting scheme to solve nonlinear partial differential equations (PDEs) with time delay. Pao [13] proposed a monotone iterative technique for the numerical solutions of a convection reaction-diffusion equation with delay. Zang et al. [14] proposed implicit-explicit multistep finite-element methods for nonlinear convection-diffusion-reaction equations with time delay. Xie et al. [15] proposed a compact finite difference scheme for variable coefficient PDEs with time delay.

Meanwhile, fractional differential equations with delay have drawn the attention of researchers due to their broad utilization in economics, population dynamics, automatic control, etc. (see [16–24] and the references therein). Also, with variable coefficient more complicated natural phenomena can be modeled (See [25–29]). It is not easy to solve fractional partial differential equations (FPDEs) with delay accurately and effectively. Because the evolution of dependent variable of FPDEs with delay at time t not only depends at $t - \tau$, but also depends on all previous solutions due to the character of history dependence of a fractional derivative. There are only few cases, when analytical solutions of the fractional differential equations with delay can be found. For instant, Zigen Ouyang [30] discussed the existence and uniqueness of the solution for a class of nonlinear fractional-order partial differential equations with delay. Rihan [31] proposed an unconditionally stable implicit difference scheme to solve time-fractional PDEs with time delay. Pimenov et al. [32] discussed an unconditionally stable difference scheme for fractional diffusion equations with nonlinear delay terms. Zang et al. [33] proposed a compact finite difference scheme for semilinear FPDEs with time delay. Mohebbi [34] proposed finite difference and spectral collocation method for the numerical solution of semilinear FPDEs with time delay. Sweilam et al. [35] proposed a nonstandard weighted average finite difference method for the numerical study of variable order Burgers' equation with proportional delay. Jaradat et al. [36] proposed two numerical scheme based on fractional power series and homotopy perturbation technique to study the propagation of population growth model.

Numerical solutions of fractional differential equations using the wavelets method can be found in [37–44, 58]. Most of the numerical methods aforementioned, are based on HWs operational matrix of fractional order integration.

In this work, we establish a numerical technique for time-fractional Burgers' equation with variable diffusion coefficient and time delay. In the solution process, we discretize the fractional time derivative by NSFD $L1$ formula and spatial derivative by HWs series. Also, we use the quasilinearisation process to deal with the nonlinear term. As far as we know, the study of fractional differential equations with the help of HWs along with NSFD $L1$ scheme does not exist in literature. The proposed method provides the implicit form of the approximate solution and has the following advantages:

(i) Compared to other methods, present method is conceptually simple, accurate, fast, memory-efficient, and has minimal computation cost.

(ii) In the solution process, a few sparse matrices are obtained and there is no complex methodology or integration required.

(iii) The present method is convenient to use for nonlinear boundary value problems as the BCs are automatically used in the algorithm.

This paper is organized as follows: In Section 2., we discuss the preliminaries of NSFD and HWs. In Section 3., we establish the numerical scheme for the Eq.(1) with the help of HWs approximation. Section 4., illustrates the convergence of the method with the help of L_2-error norm estimation. In Section 5., we give an algorithm for the entire methodology. In Section 6., we test the method to some examples and report its L_∞-error and L_2-error norms in tables. We also illustrate its accuracy with the help of several graphs. In Section 7., we discuss the conclusion of the paper.

2. Methodology

In this part we discuss about NSFD scheme and HWs.

2.1 NSFD Scheme

The foundation of the NSFD scheme is based on exact finite difference schemes. Originally, it was established by Mickens [45, 46] to analyze ordinary differential equations (ODEs), and successively, their uses have been investigated in several fields. The formulation of NSFD scheme depends on two issues: (i) how derivative is discretized and (ii) how the nonlinear term is treated.
One of the simplest discretization is the forward difference scheme. In the finite difference method, first derivative term $\frac{d\omega}{dt}$ is approximated by $\frac{\omega(t+\Delta t)-\omega(t)}{\Delta t}$. However, in Mickens type scheme $\frac{d\omega}{dt} \approx \frac{\omega(t+\Delta t)-\omega(t)}{\phi(\Delta t)}$, i.e., the step size Δt is replaced by an increasing continuous function $\phi(\Delta t)$ which satisfy the following properties:

$$\phi(\Delta t) = \Delta t + O(\Delta t^2), \quad 0 < \phi(\Delta t) < 1, \quad \text{as} \quad \Delta t \to 0.$$

Examples of the denominator functions $\phi(\Delta t)$ that follows the above conditions are [45, 47–49]

$$\sin \Delta t, \qquad e^{\Delta t} - 1, \qquad 1 - e^{-\Delta t}, \qquad \frac{1 - e^{-\lambda \Delta t}}{\lambda}, \qquad etc.$$

A method is termed as nonstandard if at least one of the following conditions satisfy [50]:

1. In the difference approximation of the derivative, the denominator is replaced by a non-trivial function of the step size which is more complicated than those conventionally used.

2. A nonlocal approximation is used.

The literature review reveals that there is no general method to find $\phi(\Delta t)$ and which nonlinear terms are to be placed instead. Some special techniques can be found in [45, 47–49, 51, 52].

2.2 Haar Wavelets

Definition 2..1. *Suppose x belongs to the interval $[0, 1]$. We define*

$$\psi_{\jmath\kappa}(x) = \psi(2^\jmath x - \kappa), \quad 0 \le \kappa \le 2^\jmath - 1, \quad \jmath = 0, 1, 2, ..., J. \tag{5}$$

where

$$\psi(x) = \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1)}(x), \tag{6}$$

and J is the maximum level of resolution. Then, the HWs are defined by

$$\hbar_1(x) = \chi_{[0,1]}(x), \qquad \hbar_\imath(x) = \psi_{\jmath\kappa}(x) \qquad \imath = 2^\jmath + \kappa + 1. \tag{7}$$

Dividing the interval $[0,1]$ into sub-intervals having equal length $\Delta x = 1/2M$. In this case the Haar wavelet is given as:

$$\hbar_\imath(x) = \chi_{[\zeta_1,\zeta_2)} - \chi_{[\zeta_2,\zeta_3)}, \tag{8}$$

where

$$\zeta_1 = 2\kappa\rho\Delta x, \quad \zeta_2 = (2\kappa+1)\rho\Delta x, \qquad \zeta_3 = 2(\kappa+1)\rho\Delta x \quad \text{and} \quad \rho = M/m. \tag{9}$$

In the above relations, J represents maximum level of resolution of HWs and the integers $m = 2^\jmath,\ 0 \leq \jmath \leq J$ is parameter related to dilation and $\kappa,\ 0 \leq \kappa \leq m-1$, is associated with the translation. Using the relation $\imath = m+\kappa+1$, we can evaluate the index "$\imath$" of the Haar function. The minimal value of $\imath$ is 2 which is obtained by taking the least values κ and m, i.e., 0 and 1 respectively. Whereas the largest value of $\imath$ is 2^J+1, which is obtained by taking the largest value of $\jmath$ and κ, i.e., $\jmath = J$ and $\kappa = m-1$ respectively. For the case $\imath = 1$, we have $\hbar_1(x) = \chi_{[0,1]}(x)$.

To analyze the differential equation of q^{th}-order, the following integrals of Haar functions are needed.

$$\mathcal{I}_{\imath,\vartheta}(x) = \frac{1}{(\vartheta-1)!}\int_0^x (x-\xi)^{\vartheta-1}\hbar_\imath(\xi)\,d\xi, \tag{10}$$

where ϑ and $\imath$ are integers such that $1 \leq \vartheta \leq q$, $1 \leq \imath \leq 2M$. The integral defined by (10) can be determined easily and is given by

$$\mathcal{I}_{\imath,\vartheta} = \begin{cases} 0, & x < \zeta_1, \\ \frac{(x-\zeta_1)^\vartheta}{\vartheta!}, & x \in [\zeta_1,\zeta_2), \\ \frac{(x-\zeta_1)^\vartheta}{\vartheta!} - 2\frac{(x-\zeta_2)^\vartheta}{\vartheta!}, & x \in [\zeta_2,\zeta_3), \\ \frac{(x-\zeta_1)^\vartheta}{\vartheta!} - 2\frac{(x-\zeta_2)^\vartheta}{\vartheta!} + \frac{(x-\zeta_3)^\vartheta}{\vartheta!}, & x \geq \zeta_3. \end{cases} \tag{11}$$

Formula (11) works with $\imath > 1$. In case $\imath = 1$, we have $\zeta_1 = 0, \zeta_2 = \zeta_3 = 1$ and

$$\mathcal{I}_{1,\vartheta} = \frac{x^\vartheta}{\vartheta!}. \tag{12}$$

We choose the collocation points $x_c = \frac{c-0.5}{2M}$, $c = 1,2,3,\cdots,2M$ and $M = 2^J$. We denote the Haar matrix $\mathscr{H}_x$ and operational matrix of integration by $\mathbb{P}_\vartheta$, $\vartheta = 1,2,3,\cdots$ are defined as

$$\mathscr{H}_x = [\hbar(\imath,c)]_{2M\times 2M},$$

and

$$\mathbb{P}_\vartheta = [\mathcal{I}_\vartheta(\imath,c)]_{2M\times 2M},$$

respectively. Here $\hbar(\imath,c) = \hbar_\imath(x_c)$, $\mathcal{I}_\vartheta(\imath,c) = \mathcal{I}_{\imath,\vartheta}(x_c)$, and ϑ denotes the order of integration.

3. Discretization

3.1 Semi Discretization at Temporal Grid

Let us consider fractional differential equation (1) and denote $D_t^\sigma\omega(x,t) = \frac{\partial^\sigma\omega(x,t)}{\partial t^\sigma}$, we have

$$\frac{\partial^\sigma\omega}{\partial t^\sigma} - \mu'(x)\omega_x - \mu(x)\omega_{xx} + \omega^p\omega_x = g\big(x,t,\omega(x,t-\tau)\big), \qquad (x,t) \in (0,1)\times(0,T], \tag{13}$$

with the BCs

$$\omega(0,t) = \beta_0(t), \qquad \omega(1,t) = \beta_1(t), \qquad t \in [0,T] \tag{14}$$

and IC

$$\omega(x,t) = \varphi(x,t), \qquad x \in [0,1]\times[-\tau,0]. \tag{15}$$

Suppose Δt is the size of time step such that $\Delta t = \frac{\tau}{n}$, where n is a positive integer. Also, $t_\ell = \ell\Delta t\ \ (-n \leq \ell \leq N)$, where $N = [\frac{T}{\Delta t}]$. We discretize the time derivative in Eq.(13) in the Caputo's sense by $L1$ formula [53]

$$\frac{\partial^\sigma\omega}{\partial t^\sigma}\Big|_{t_\ell} = \frac{(\Delta t)^{-\sigma}}{\Gamma(2-\sigma)}\Big[\omega_\ell - \omega_{\ell-1} + \sum_{q=1}^{\ell-1}\Theta_q^\sigma\Big(\omega_{\ell-q} - w_{\ell-q-1}\Big)\Big] + O(\Delta t), \tag{16}$$

where $\Theta_q^\sigma = (q+1)^{1-\sigma} - q^{1-\sigma}$.

Now, we discretize Eq.(13) according to $\theta-$weighted scheme [54], we get

$$\begin{aligned} &\frac{(\Delta t)^{-\sigma}}{\Gamma(2-\sigma)}\Big[\omega_{\ell+1} - \omega_\ell\Big] + \frac{(\Delta t)^{-\sigma}}{\Gamma(2-\sigma)}\sum_{q=1}^{\ell}\Theta_q^\sigma\Big[\omega_{\ell-q+1} - w_{\ell-q}\Big] - \theta\mu'(x)(\omega_{\ell+1})_x - (1-\theta)\mu'(x)(\omega_\ell)_x \\ &\quad - \theta\mu(x)(\omega_{\ell+1})_{xx} + (1-\theta)\mu(x)(\omega_\ell)_x + \theta(\omega_{\ell+1})^p(\omega_{\ell+1})_x + (1-\theta)(\omega_\ell)^p(\omega_\ell)_x = \\ &\qquad \theta g\big(x,t_{\ell+1},\omega(x,t_{\ell+1}-\tau)\big) + (1-\theta)g\big(x,t_\ell,\omega(x,t_\ell-\tau)\big). \end{aligned} \tag{17}$$

Taking $\theta = \frac{1}{2}$, i.e., average of two successive time levels ℓ and $\ell+1$ and replacing Δt by $\phi(\Delta t)$, we get

$$\frac{(\phi(\Delta t))^{-\sigma}}{\Gamma(2-\sigma)}\Big[\omega_{\ell+1}-\omega_\ell\Big]+\frac{(\phi(\Delta t))^{-\sigma}}{\Gamma(2-\sigma)}\sum_{q=1}^{\ell}\Theta_q^\sigma\Big[\omega_{\ell-q+1}-\omega_{\ell-q}\Big]-\frac{\mu'(x)}{2}\Big((\omega_{\ell+1})_x+(\omega_\ell)_x\Big)$$
$$-\frac{\mu(x)}{2}\Big((\omega_{\ell+1})_{xx}+(\omega_\ell)_{xx}\Big)+\frac{1}{2}\Big((\omega_{\ell+1})^p(\omega_{\ell+1})_x+(\omega_\ell)^p(\omega_\ell)_x\Big)=$$
$$\frac{1}{2}\Big(g\big(x,t_{\ell+1},\omega(x,t_{\ell+1}-\tau)\big)+g\big(x,t_\ell,\omega(x,t_\ell-\tau)\big)\Big), \quad (18)$$

with the BCs

$$\omega_{\ell+1}(0)=\beta_0(t_{\ell+1}), \qquad \omega_{\ell+1}(1)=\beta_1(t_{\ell+1}), \qquad 0\le\ell\le N-1, \quad (19)$$

and IC

$$\omega_{\ell+1}(x)=\varphi_{\ell+1}(x), \qquad -n\le\ell<0. \quad (20)$$

Equation (18) can be rearranged at the $(\ell+1)$th time level as follows:

$$-\frac{\mu(x)}{2}(\omega_{\ell+1})_{xx}+\frac{1}{2}(\omega_{\ell+1})^p(\omega_{\ell+1})_x-\frac{\mu'(x)}{2}(\omega_{\ell+1})_x+G\omega_{\ell+1}=-G\sum_{q=1}^{\ell}\Theta_q^\sigma\Big[\omega_{\ell-q+1}-\omega_{\ell-q}\Big]$$
$$+G\omega_\ell+\frac{\mu'(x)}{2}(\omega_\ell)_x+\frac{\mu(x)}{2}(\omega_\ell)_{xx}-\frac{1}{2}(\omega_\ell)^p(\omega_\ell)_x$$
$$+\frac{1}{2}\Big(g\big(x,t_{\ell+1},\omega(x,t_{\ell+1}-\tau)\big)+g\big(x,t_\ell,\omega(x,t_\ell-\tau)\big)\Big), \quad (21)$$

with the BCs

$$\omega_{\ell+1}(0)=\beta_1(t_{\ell+1}), \qquad \omega_{\ell+1}(1)=\beta_0(t_{\ell+1}), \qquad 0\le\ell\le N-1, \quad (22)$$

and IC

$$\omega_{\ell+1}(x)=\varphi_{\ell+1}(x), \qquad -n\le\ell<0, \quad (23)$$

where $G=\frac{(\phi(\Delta t))^{-\sigma}}{\Gamma(2-\sigma)}$. At every $(\ell+1)$th time step, Eq.(21) represents a nonlinear ODE. There are several techniques to linearize the nonlinearity term in Eq.(21). We use the quasilinearisation technique [55] which has quadratic rate of convergence, i.e., $|\omega_{\ell+1}^{r+1}-\omega_{\ell+1}^{r}|\le|\omega_{\ell+1}^{r}-\omega_{\ell+1}^{r-1}|^2$, where r is the iteration index and $\omega_{\ell+1}^{r+1}$, $\omega_{\ell+1}^{r}$ and $\omega_{\ell+1}^{r-1}$ are the approximated solutions at $(r+1)^{th}$, $(r)^{th}$ and $(r-1)^{th}$ steps, respectively. Implementing the quasilinearisation process, we get

$$(\omega_{\ell+1}^{r+1})^p(\omega_{\ell+1}^{r+1})_x=\Big[(\omega_{\ell+1}^{r})^p(\omega_{\ell+1}^{r})_x+p(\omega_{\ell+1}^{r})^{p-1}\Big(\omega_{\ell+1}^{r+1}-\omega_{\ell+1}^{r}\Big)(\omega_{\ell+1}^{r})_x+(\omega_{\ell+1}^{r})^p\Big((\omega_{\ell+1}^{r+1})_x-(\omega_{\ell+1}^{r})_x\Big)\Big]. \quad (24)$$

Using Eq.(24) into the Eq.(21) and simplifying, we get

$$-\frac{\mu(x)}{2}(\omega_{\ell+1}^{r+1})_{xx}+\frac{1}{2}\Big((\omega_{\ell+1}^{r})^p-\frac{\mu'(x)}{2}\Big)(\omega_{\ell+1}^{r+1})_x+\Big(\frac{p}{2}(\omega_{\ell+1}^{r})^{p-1}(\omega_{\ell+1}^{r})_x+G\Big)\omega_{\ell+1}^{r+1}=\frac{\mu(x)}{2}(\omega_\ell^{r+1})_{xx}$$
$$+\Big(\frac{\mu'(x)}{2}-\frac{1}{2}(\omega_\ell^{r+1})^p\Big)(\omega_\ell^{r+1})_x+\frac{p}{2}(\omega_{\ell+1}^{r})^p(\omega_{\ell+1}^{r})_x-G\sum_{q=1}^{\ell}\Theta_q^\sigma\Big[\omega_{\ell-q+1}^{r+1}-\omega_{\ell-q}^{r+1}\Big]$$
$$+G\omega_\ell^{r+1}+\frac{1}{2}\Big(g\big(x,t_{\ell+1},\omega^{r+1}(x,t_{\ell+1}-\tau)\big)+g\big(x,t_\ell,\omega^{r+1}(x,t_\ell-\tau)\big)\Big), \quad (25)$$

with the BCs

$$\omega_{\ell+1}^{r+1}(0)=\beta_0(t_{\ell+1}), \qquad \omega_{\ell+1}^{r+1}(1)=\beta_1(t_{\ell+1}), \qquad 0\le\ell\le N-1, \quad (26)$$

and IC

$$\omega_{\ell+1}^{r+1}(x)=\varphi_{\ell+1}(x), \qquad -n\le\ell<0. \quad (27)$$

3.2 Space Discretization by Haar Wavelets

Let us approximate higher order derivative $(\omega_{\ell+1}^{r+1})_{xx}(x)$ by HWs finite series as

$$(\omega_{\ell+1}^{r+1})_{xx}(x)=\sum_{\imath=1}^{2M}c_\imath\hbar_\imath(x). \quad (28)$$

Now, integrating Eq.(28) twice and using BCs, we get

$$(\omega_{\ell+1}^{r+1})_x(x)=\sum_{\imath=1}^{2M}c_\imath\Big[\mathcal{I}_{\imath,1}(x)-\mathcal{I}_{\imath,2}(1)\Big]+\Big(\beta_1(t_{\ell+1})-\beta_0(t_{\ell+1})\Big), \quad (29)$$

$$(\omega_{\ell+1}^{r+1})(x)=\sum_{\imath=1}^{2M}c_\imath\Big[\mathcal{I}_{\imath,2}(x)-x\mathcal{I}_{\imath,2}(1)\Big]+x\Big(\beta_1(t_{\ell+1})-\beta_0(t_{\ell+1})\Big)+\beta_0(t_{\ell+1}). \quad (30)$$

Using Eqs.(28)-(30) in Eq.(25), we get

$$\sum_{\imath=1}^{2M} c_\imath\Big[-\frac{\mu(x)}{2}\hbar_\imath(x)+\Big(\frac{1}{2}(\omega_{\ell+1}^r)^p-\frac{\mu'(x)}{2}\Big)\Big(\mathcal{I}_{\imath,1}(x)-\mathcal{I}_{\imath,2}(1)\Big)+\Big(\frac{p}{2}(\omega_{\ell+1}^r)^{p-1}(\omega_{\ell+1}^r)_x+G\Big)\Big(\mathcal{I}_{\imath,2}(x)-x\mathcal{I}_{\imath,2}(1)\Big)\Big]$$
$$=-\Big(\frac{1}{2}(\omega_{\ell+1}^r)^p-\frac{\mu'(x)}{2}\Big)\Big(\beta_1(t_{\ell+1})-\beta_0(t_{\ell+1})\Big)-\Big(\frac{p}{2}(\omega_{\ell+1}^r)^{p-1}(\omega_{\ell+1}^r)_x+G\Big)+\Big(\frac{\mu'(x)}{2}-\frac{1}{2}(\omega_\ell^{r+1})^p\Big)(\omega_\ell^{r+1})_x$$
$$\Big(x\big(\beta_1(t_{\ell+1})-\beta_0(t_{\ell+1})\big)+\beta_0(t_{\ell+1})\Big)+\frac{p}{2}(\omega_{\ell+1}^r)^p(\omega_{\ell+1}^r)_x+\frac{\mu(x)}{2}(\omega_\ell^{r+1})_{xx}-G\sum_{q=1}^{\ell-1}\Theta_q^\gamma\Big[\omega_{\ell-q+1}^{r+1}-\omega_{\ell-q}^{r+1}\Big]$$
$$+G\omega_\ell^{r+1}+\frac{1}{2}\Big(g\big(x,t_{\ell+1},\omega^{r+1}(x,t_{\ell+1}-\tau)\big)+g\big(x,t_\ell,\omega^{r+1}(x,t_\ell-\tau)\big)\Big). \quad (31)$$

Now, discretizing the Eq.(31) by taking collocation points $x_c=\frac{c-0.5}{2M},\ c=1(1)2M$, we obtain the following linear system which can be written in matrix form as

$$AC=b, \quad (32)$$

where $C=[c_1,c_2,c_3,\cdots,c_{2M}]^T$, $b=[b_1,b_2,b_3,\cdots,b_{2M}]^T$ and $A=[a_{\jmath,\imath}]_{2M\times 2M}$. The elements of the matrix A and column vector b are

$$a_{\jmath,\imath}=-\frac{\mu(x_\jmath)}{2}\hbar_\imath(x_\jmath)+\Big(\frac{1}{2}(\omega_{\ell+1}^r)^p-\frac{\mu'(x_\jmath)}{2}\Big)\Big(\mathcal{I}_{\imath,1}(x_\jmath)-\mathcal{I}_{\imath,2}(1)\Big)+\Big(\frac{p}{2}(\omega_{\ell+1}^r)^{p-1}(\omega_{\ell+1}^r)_x+G\Big)\Big(\mathcal{I}_{\imath,2}(x_\jmath)-x_\jmath\mathcal{I}_{\imath,2}(1)\Big),$$

$$b_\jmath=-\Big(\frac{1}{2}(\omega_{\ell+1}^r)^p-\frac{\mu'(x_\jmath)}{2}\Big)\Big(\beta_1(t_{\ell+1})-\beta_0(t_{\ell+1})\Big)-\Big(\frac{p}{2}(\omega_{\ell+1}^r)^{p-1}(\omega_{\ell+1}^r)_x+G\Big)+\Big(\frac{\mu'(x_\jmath)}{2}-\frac{1}{2}(\omega_\ell^{r+1})^p\Big)(\omega_\ell^{r+1})_x$$
$$\Big(x_\jmath\big(\beta_1(t_{\ell+1})-\beta_0(t_{\ell+1})\big)+\beta_0(t_{\ell+1})\Big)+\frac{p}{2}(\omega_{\ell+1}^r)^p(\omega_{\ell+1}^r)_x+\frac{\mu(x_\jmath)}{2}(\omega_\ell^{r+1})_{xx}-G\sum_{q=1}^{\ell-1}\Theta_q^\gamma\Big[\omega_{\ell-q+1}^{r+1}-\omega_{\ell-q}^{r+1}\Big]$$
$$+G\omega_\ell^{r+1}+\frac{1}{2}\Big(g\big(x_\jmath,t_{\ell+1},\omega^{r+1}(x_\jmath,t_{\ell+1}-\tau)\big)+g\big(x_\jmath,t_\ell,\omega^{r+1}(x_\jmath,t_\ell-\tau)\big)\Big), 1\le\jmath\le 2M,$$

respectively. Wavelet coefficients $c_\jmath$ $(1\le\jmath\le 2M)$ can be calculated successively by solving the linear system (32). To start the iteration, we choose the initial condition as the initial guess. Then, at each time level, we can obtain approximate solutions using the wavelet coefficients into the Eqs.(28),(29), and (30).

4. Convergence analysis

We consider the asymptotic expression of the Eq.(30) for the investigation of convergence of the proposed scheme as follows

$$\omega(x)=\sum_{\imath=1}^{\infty}c_\imath\Big[\mathcal{I}_{\imath,2}(x)-x\mathcal{I}_{\imath,2}(1)\Big]+x\Big(\beta_1(t_{\ell+1})-\beta_0(t_{\ell+1})\Big)+\beta_0(t_{\ell+1}). \quad (33)$$

Theorem 4..1. *Let us suppose that* $\mathcal{F}(x)=\frac{d^2\omega(x)}{dx^2}\in L^2(\mathbb{R})$ *be continuous on* $[0,1]$ *and its first order derivative is bounded, i.e.,*

$$\forall x\in[0,1]\ \ \exists\mathcal{K}:\ \Big|\frac{d\mathcal{F}(x)}{dx}\Big|<\mathcal{K}. \quad (34)$$

Then $||E_J||_2=||\omega(x)-\omega_{2M}(x)||_2$*, where* $\omega_{2M}(x)$ *is the approximated solution, converges with the second order convergence rate.*

Proof. Consider $||E_J||_2$. Hence we have,

$$||E_J||_2^2=\int_0^1\Big[\sum_{\jmath=J+1}^{\infty}\sum_{\kappa=0}^{2^\jmath-1}c_{2^\jmath+\kappa+1}\Big(\mathcal{I}_{2^\jmath+\kappa+1,2}(x)-x\mathcal{I}_{2^\jmath+\kappa+1,2}(1)\Big)\Big]^2dx$$
$$=\sum_{\jmath=J+1}^{\infty}\sum_{\kappa=0}^{2^\jmath-1}\sum_{l=J+1}^{\infty}\sum_{s=0}^{2^l-1}c_{2^\jmath+\kappa+1}c_{2^l+s+1}\int_0^1\Big[\Big(\mathcal{I}_{2^\jmath+\kappa+1,2}(x)-x\mathcal{I}_{2^\jmath+\kappa+1,2}(1)\Big)$$
$$\Big(\mathcal{I}_{2^l+s+1,2}(x)-x\mathcal{I}_{2^l+s+1,2}(1)\Big)\Big]dx. \quad (35)$$

Firstly, we show that the function $\mathcal{I}_{\imath,2}(x)$ is bounded over each subinterval of $[0,1]$. It is clear that $\mathcal{I}_{\imath,2}(x)=0$ for $x\in[0,\zeta_1]$. Also the function $\mathcal{I}_{\imath,2}(x)$ is increasing monotonically over the interval $[\zeta_1,\zeta_2]$ and therefore at the end point $x=\zeta_2$, its maximum value can be obtained. Thus

$$\mathcal{I}_{\imath,2}(x)=\mathcal{I}_{2^\jmath+\kappa+1,2}(x)\le\frac{(\zeta_2-\zeta_1)^2}{2}=\frac{1}{2}\Big(\frac{1}{2^{(\jmath+1)}}\Big)^2,\ \ \forall x\in[\zeta_1,\zeta_2]. \quad (36)$$

In the interval $[\zeta_2,\zeta_3]$, the function $\mathcal{I}_{\imath,2}(x)$ can be simply shown to be monotonically increasing with the help of Eqs. (9), (10), and $\frac{\partial\mathcal{I}_{\imath,2}(x)}{\partial x}>0$ if $x<\zeta_3$. Thus at the end point $x=\zeta_3$, the function $\mathcal{I}_{\imath,2}(x)$ attains its maximum value and we have

$$\mathcal{I}_{\imath,2}(x)=\mathcal{I}_{2^\jmath+\kappa+1,2}(x)\le\Big(\frac{1}{2^{(\jmath+1)}}\Big)^2,\ \ \forall x\in[\zeta_2,\zeta_3]. \quad (37)$$

For the subinterval $[\zeta_3, 1]$, $\mathcal{I}_{\imath,2}(x)$ can be expressed as ([59, Equation (22)])

$$\mathcal{I}_{\imath,2}(x) = \mathcal{I}_{2^\jmath+\kappa+1,2}(x) = \Big(\frac{1}{2^{(\jmath+1)}}\Big)^2, \ \ \forall x \in [\zeta_3, 1]. \tag{38}$$

Thus the function $\mathcal{I}_{\imath,2}(x)$ is bounded and its upper bound is given by

$$\mathcal{I}_{\imath,2}(x) = \mathcal{I}_{2^\jmath+\kappa+1,2}(x) \leq \Big(\frac{1}{2^{(\jmath+1)}}\Big)^2, \ \ \forall x \in [0, 1]. \tag{39}$$

Now, we have

$$\begin{aligned} \Big(\mathcal{I}_{\imath,2}(x) - x\mathcal{I}_{\imath,2}(1)\Big) &\leq |\mathcal{I}_{\imath,2}(x)| + |x||\mathcal{I}_{\imath,2}(1)|, \\ &\leq |\mathcal{I}_{\imath,2}(x)| + |\mathcal{I}_{\imath,2}(1)|, \\ &\leq 2\Big(\frac{1}{2^{(\jmath+1)}}\Big)^2. \end{aligned} \tag{40}$$

Similarly, we have

$$\Big(\mathcal{I}_{l,2}(x) - x\mathcal{I}_{l,2}(1)\Big) \leq 2\Big(\frac{1}{2^{(l+1)}}\Big)^2. \tag{41}$$

Also $c_\imath$ $(\imath = 2^\jmath + \kappa + 1)$ can be evaluated as

$$\begin{aligned} c_\imath &= 2^\jmath \int_0^1 \mathcal{F}(x)h_\imath(x) = 2^\jmath\Big[\int_{\zeta_1}^{\zeta_2} \mathcal{F}(x)dx - \int_{\zeta_2}^{\zeta_3} \mathcal{F}(x)dx\Big], \\ &= 2^\jmath\Big[(\zeta_2 - \zeta_1)\mathcal{F}(\eta_1) - (\zeta_3 - \zeta_2)\mathcal{F}(\eta_2)\Big], \end{aligned} \tag{42}$$

where $\eta_1 \in (\zeta_1, \zeta_2)$ and $\eta_2 \in (\zeta_2, \zeta_3)$. It follows from Eq. (9) that $\zeta_2 - \zeta_1 = \zeta_3 - \zeta_2 = \frac{1}{2m} = \frac{1}{2^{\jmath+1}}$. Equation (42) becomes

$$c_\imath = \frac{1}{2}\Big[\mathcal{F}(\eta_1) - \mathcal{F}(\eta_2)\Big] = \frac{1}{2}(\eta_1 - \eta_2)\frac{d\mathcal{F}(\xi)}{d\xi}, \ \xi \in (\eta_1, \eta_2). \tag{43}$$

From assumption (34) and (43), it follows that

$$c_\imath \leq \mathcal{K}\frac{1}{2^{\jmath+1}}. \tag{44}$$

Using (36) to (44) in Eq.(35), we get

$$\begin{aligned} ||E_J||_2^2 &\leq \sum_{\jmath=J+1}^{\infty}\sum_{\kappa=0}^{2^\jmath-1}\sum_{l=J+1}^{\infty}\sum_{s=0}^{2^l-1} \frac{\mathcal{K}^2}{2^{\jmath+l+2}}\Big(\frac{1}{2^{(\jmath+1)}}\Big)^2\Big(\frac{1}{2^{(l+1)}}\Big)^2, \\ &\leq \mathcal{K}^2 \sum_{\jmath=J+1}^{\infty}\sum_{l=J+1}^{\infty} \frac{1}{2^{\jmath+l+2}}2^\jmath\Big(\frac{1}{2^{(\jmath+1)}}\Big)^2 2^l\Big(\frac{1}{2^{(l+1)}}\Big)^2, \\ &\leq 3\mathcal{K}^2\Big(\frac{1}{2^{J+1}}\Big)^4. \end{aligned}$$

Hence

$$||E_J||_2 \leq \sqrt{3}\mathcal{K}\Big(\frac{1}{2^{J+1}}\Big)^2. \tag{45}$$

□

From Eq.(45), it can be seen that the error bound directly depends on J-th level of resolution of HW and also $||E_J||_2 \to 0$ as J moves towards infinity. Thus, if J tends to infinity our scheme approaches to the exact solution.

5. Algorithm

In this part, we describe the algorithm of the proposed methodology. Following are the key steps:

Step 1: In Eq.(13)**, the time derivative of fractional order is replaced by NSFD $L1$ formula and an average of other terms taken to arrive at second order nonlinear ODE in** (21).

Step 2: The non-linear term in (21) **is linearised by quasilinearisation** (24) **to arrive at second order linear ODE** (25).

Step 3: Haar wavelet collocation method is applied on (25) **to get system of linear equations** (32).

Step 4: The linear system (32) **gives wavelet coefficients for each iteration at each time level.**

Step 5: To get the solution of each iteration at each time level, computed wavelet coefficients are used in Eq. (30).

Step 6: We repeat steps 4, 5 to achieve better accuracy by quasilinearisation.

6. Computational Experiment

In order to assess the competency and accuracy of the proposed scheme, we implement the scheme over some benchmark problems and report errors by using
(i) Mean root square error norm (L_2)

$$L_2\text{-error} = \sqrt{\Delta x \sum_{j=0}^{2M} \mid \omega_j^{exact} - (\omega_{2M})_j \mid^2}, \tag{46}$$

(ii) Maximum error norm (L_∞)

$$L_\infty\text{-error} = \|\omega^{exact} - \omega_{2M}\|_\infty = \max_j \mid \omega_j^{exact} - (\omega_{2M})_j \mid, \tag{47}$$

where w_{2M} is the approximated solution by the present scheme. Additionally, we calculate ROC (rate of convergence) which is defined as [56]

$$\text{ROC} = \frac{\log[E_\omega(2M)/E_\omega(4M)]}{\log 2}, \tag{48}$$

where E_ω is any of the error norm with different J. All the computations are performed for number of iterations (for quasilinearization) $r = 5$ by using the Mathematica software on a computer with processor Intel (R) Core(TM) i5-7200U CPU @ 2.50GHz 2.70GHz and RAM 4.00GB. Throughout the computations, we have taken the denominator function $\phi(\Delta t) = e^{\Delta t} - 1$.

6.1 Example 1

Here we investigate the following nonlinear fractional differential equation with time delay given by

$$\frac{\partial^\sigma \omega}{\partial t^\sigma} - (\mu(x)\omega_x)_x + \omega^5 w_x = \omega^2(x, t-0.1) + H(x,t), \qquad (x,t) \in (0,1) \times (0,1], \tag{49}$$

with homogenous BCs

$$\omega(0,t) = 0 = \omega(1,t), \quad t \in (0,1], \tag{50}$$

and IC

$$\omega(x,t) = t^{2+\sigma} \sin(2\pi x), \qquad x \in [0,1] \times [-0.1, 0], \tag{51}$$

where $\mu(x) = x^2 + 1$ and

$$H(x,t) = \frac{\Gamma(3+\sigma)}{\Gamma(3)} t^2 \sin(2\pi x) - 2\pi t^{2+\sigma}[2x\cos(2\pi x) - 2\pi(x^2+1)\sin(2\pi x)] + 2\pi \cos(2\pi x) \sin^5(2\pi x) t^{12+6\sigma} - \sin^2(2\pi x)(t-0.1)^{4+2\sigma}. \tag{52}$$

It is observed that the function [57]

$$\omega(x,t) = t^{2+\sigma} \sin(2\pi x), \tag{53}$$

is exact solution of the problem 6.1.

In Table 1, for the comparison purpose, we take $\sigma = 0.25,\ 0.50,\ 0.75$, $T = 1$, number of collocation points $2M = 64$ and compute L_2-error and L_∞-error norms for different time steps Δt. From Table 1, we can see that the present scheme provides better numerical results as compared to the method in [57]. Also, we see that the accuracy improves on decreasing the time step. In Table 2, we take $\sigma = 0.25,\ 0.50,\ 0.75$, $T = 1$, $\Delta t = 0.001$ and compute L_2-error and L_∞-error norms for different J. We can see that our results are better than the results presented in [57]. Also, we can see that both L_∞-error and L_2-error norms decrease with an increase of J. It is well known that HW method converges quadratically. In Table 3, we list the ROC of the present scheme and see that it is very close to its theoretical value. From Table 3, we can also see that the present scheme provides acceptable results for $\sigma = 1$ and the ROC approaches its theoretical value. Moreover, Fig. 1 represents the absolute error of the problem 6.1 for $\sigma = 0.25$ (left) and $\sigma = 0.75$ (right) at $T = 1$ with time step $\Delta t = 0.01$ for $J = 2, 3, 4, 5$. We can see that the solutions provided by our scheme are bounded throughout the domain and the absolute error becomes smaller and smaller on increasing J. Figure 2 shows the analytical solutions surface (left) and approximated solutions surface (right) for $\sigma = 0.60$, $2M = 32$ and step size of time $\Delta t = 0.001$. From the Fig. 2, we can see that the analytical solutions surface and the approximated solutions surface by our method look nearly the same.

Table 1. Comparison of results with existing results for $J = 5,\ \sigma = 0.25,\ 0.50,\ 0.75$ at $T = 1$ for different time step Δt of the problem 6.1.

	[57] L_∞-error			Present method L_∞-error			Present method L_2-error		
Δt	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$
0.05	2.908E-03	3.257E-03	4.035E-03	1.70349E-03	1.65777E-03	1.03992E-03	1.02973E-03	1.00502E-03	6.60411E-04
0.025	1.638E-03	1.826E-03	2.206E-03	1.00799E-03	1.01348E-03	7.51685E-04	6.45837E-04	6.48957E-04	5.06063E-04
0.0125	8.709E-04	9.679E-04	1.149E-03	6.52747E-04	6.65386E-04	5.94737E-04	4.57053E-04	4.63107E-04	4.04856E-04

Table 2. Comparison with existing results at $T = 1$ for $\sigma = 0.25,\ 0.50,\ 0.75$ and $\Delta t = 0.001$ of the problem 6.1.

	[57] ($\Delta t = 0.001$) L_∞-error				Present method ($\Delta t = 0.001$) L_∞-error			Present method ($\Delta t = 0.001$) L_2-error		
N	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$	J	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$
10	3.554E-02	3.413E-02	3.494E-02	2	2.90154E-02	2.87638E-02	2.83680E-02	1.71543E-02	1.70207E-02	1.68181E-02
20	8.997E-03	8.621E-03	8.832E-03	3	7.47062E-03	7.41068E-03	7.31401E-03	4.48617E-03	4.45259E-03	4.39958E-03
40	2.259E-03	2.182E-03	2.224E-03	4	1.90919E-03	1.89472E-03	1.86838E-03	1.14277E-03	1.13523E-03	1.12079E-03

Table 3. L_2 and L_∞-error norms and convergence rate for $T = 0.5$, $\Delta t = 0.001$

	$\sigma = 0.5$				$\sigma = 1$			
J	L_∞-error	ROC	L_2-error	ROC	L_∞-error	ROC	L_2-error	ROC
1	1.50845E-2		9.50961E-3		9.55952E-3		6.16729E-3	
2	4.95726E-3	1.60545	2.98425E-3	1.67201	3.18107E-3	1.58743	1.94040E-3	1.66828
3	1.30375E-3	1.92689	7.86108E-3	1.92457	8.31188E-4	1.93626	5.04348E-4	1.94386
4	3.36283E-4	1.95491	2.04648E-4	1.94158	2.05780E-4	2.01407	1.24324E-4	2.02032

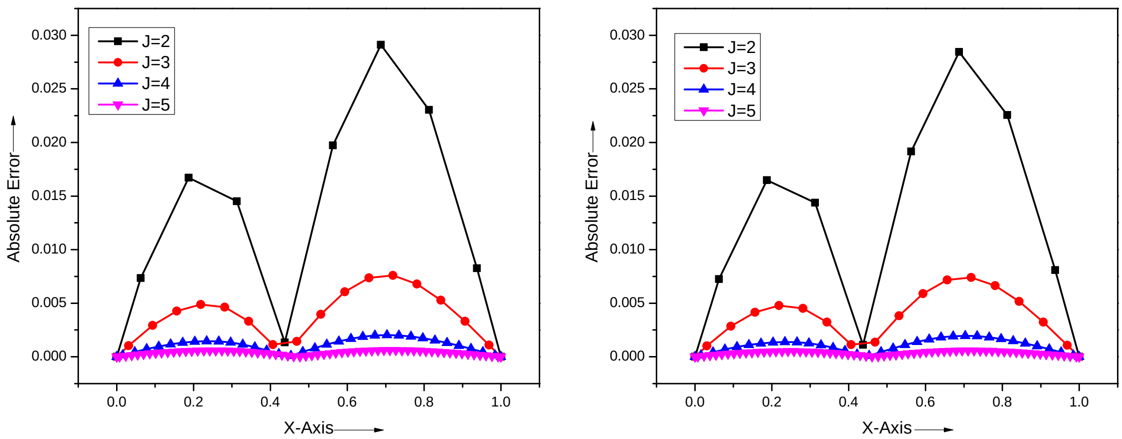

Fig. 1. Absolute error for $\sigma = 0.25$ (left) and $\sigma = 0.75$ (right) with $\Delta t = 0.01$, for different J at time $T = 1$ for the problem 6.1.

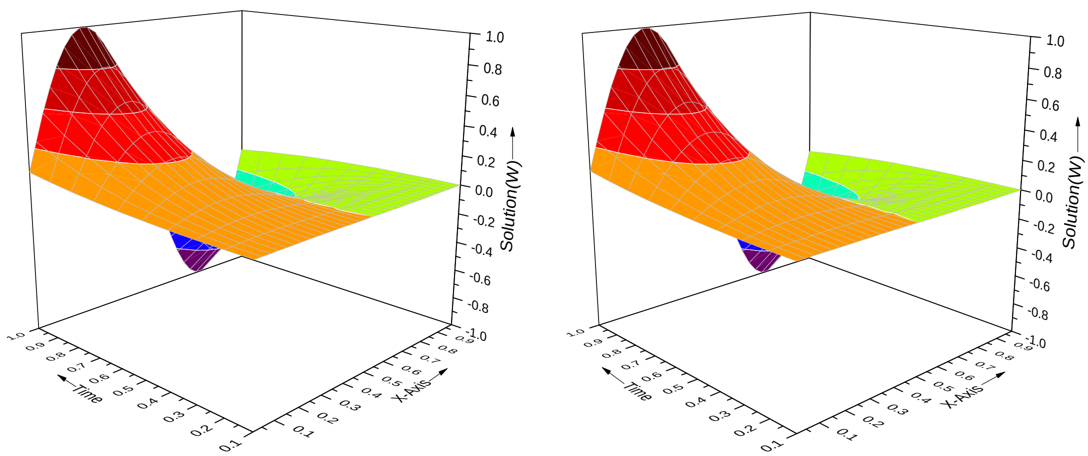

Fig. 2. $3D$ surface of analytical solutions (left) and numerical solutions (right) with $\sigma = 0.60$, $\Delta t = 0.001$ and $J = 4$ for the problem 6.1.

6.2 Example 2

Here we investigate the following nonlinear fractional differential equation with time delay given by

$$\frac{\partial^\sigma \omega}{\partial t^\sigma} - (\mu(x)\omega_x)_x + \omega^2\omega_x = \omega^2(x, t - 0.1) + H(x,t), \qquad (x,t) \in (0,1) \times (0,1], \tag{54}$$

with homogenous BCs

$$\omega(0,t) = 0 = \omega(1,t), \quad t \in (0,1], \tag{55}$$

and IC

$$\omega(x,t) = t^{2+\sigma}\sin(2\pi x), \qquad x \in [0,1] \times [-0.1, 0], \tag{56}$$

where $\mu(x) = \cos x$ and

$$H(x,t) = \frac{\Gamma(3+\sigma)}{\Gamma(3)} t^2 \sin(2\pi x) - 2\pi t^{2+\sigma}[\cos(2\pi x)\cos(x) - 2\pi \sin(2\pi x)\sin(x)] + 2\pi \cos(2\pi x)\sin^2(2\pi x) t^{6+3\sigma} - \sin^2(2\pi x)(t-0.1)^{4+2\sigma}. \quad (57)$$

We can simply verify that the function

$$\omega(x,t) = t^{2+\sigma}\sin(2\pi x), \quad (58)$$

is satisfying the problem 6.2.

In this test problem, we take $p = 2$ and the delay $\tau = 0.1$ with the trigonometric function as a variable coefficient. In Table 4, we list L_2-error and L_∞-error norms for $\sigma = 0.30,\ 0.60,\ 0.90$, $\Delta t = 0.001$ at $T = 1$ for different J. From the Table 4, we can see that both L_2-error and L_∞-error norms decrease with an increase of J. In Table 6, we list the ROC of the present scheme and see that it is very close to its theoretical value. From Table 6, we can also see that the present scheme provides acceptable results for $\sigma = 1$ and the ROC approaches its theoretical value. In Table 5, we take $\sigma = 0.25,\ 0.50,\ 0.75$, $T = 0.5$ and compute L_2-error and L_∞-error norms for different step size of time Δt and we observe that on refining temporal step, accuracy improves. Moreover, Fig. 3 represents the absolute error of the problem 6.2 for $\sigma = 0.25$ (left) and $\sigma = 0.75$ (right) at $T = 1$ for $J = 2, 3, 4, 5$ and $\Delta t = 0.01$. It is clear that the solutions provided by our scheme are bounded throughout the domain and the errors become smaller and smaller on increasing J. Figure 4 shows the analytical solutions surface (left) and approximated solutions surface (right) for $\sigma = 0.40$, $2M = 32$ and step size of time $\Delta t = 0.001$. From the Fig. 4, we can see that the analytical solutions surface and the approximated solutions surface by our method look nearly the same.

Table 4. L_∞-error and L_2-error norms of the example 6.2 for $\sigma = 0.30,\ 0.60,\ 0.90$ with time step size $\Delta t = 0.001$ at time $T = 1$.

	L_∞-error			L_2-error		
J	$\sigma = 0.30$	$\sigma = 0.60$	$\sigma = 0.90$	$\sigma = 0.30$	$\sigma = 0.60$	$\sigma = 0.90$
2	3.38918E-02	3.23104E-02	3.00905E-02	1.97125E-02	1.89886E-02	1.80144E-02
3	8.73449E-03	8.35108E-03	7.79198E-03	5.13766E-03	4.95101E-03	4.68521E-03
4	2.25167E-03	2.15218E-03	1.98516E-03	1.31252E-03	1.26681E-03	1.18616E-03

Table 5. L_∞-error and L_2-error norms of the problem 6.2 for $J = 5$ at $T = 0.5$, $\sigma = 0.25,\ 0.50,\ 0.75$ for different time step Δt.

	L_∞-error			L_2-error		
Δt	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$
0.05	1.08496E-03	1.10371E-03	7.79740E-04	6.71480E-04	6.80235E-04	4.81670E-04
0.025	5.86196E-04	6.15676E-04	4.82923E-04	3.69085E-04	3.84127E-04	3.01464E-04
0.0125	3.30623E-04	3.48835E-04	2.93638E-04	2.14695E-04	2.22647E-04	1.86777E-04

Table 6. L_∞-error and L_2-error norms and rate of convergence for problem 6.2 at $T = 0.5$ and $\Delta t = 0.001$.

	$\sigma = 0.5$				$\sigma = 1$			
J	L_∞-error	ROC	L_2-error	ROC	L_∞-error	ROC	L_2-error	ROC
1	1.33402E-2		9.86484E-3		8.06318E-3		6.20404E-3	
2	4.68023E-3	1.51113	3.02875E-3	1.70357	2.86148E-3	1.49458	1.89454E-3	1.71135
3	1.26213E-3	1.89071	7.99162E-4	1.92216	7.60675E-4	1.91141	4.87908E-4	1.95717
4	3.31431E-4	1.92908	2.12018E-4	1.91389	1.86088E-4	2.03129	1.18024E-4	2.04753

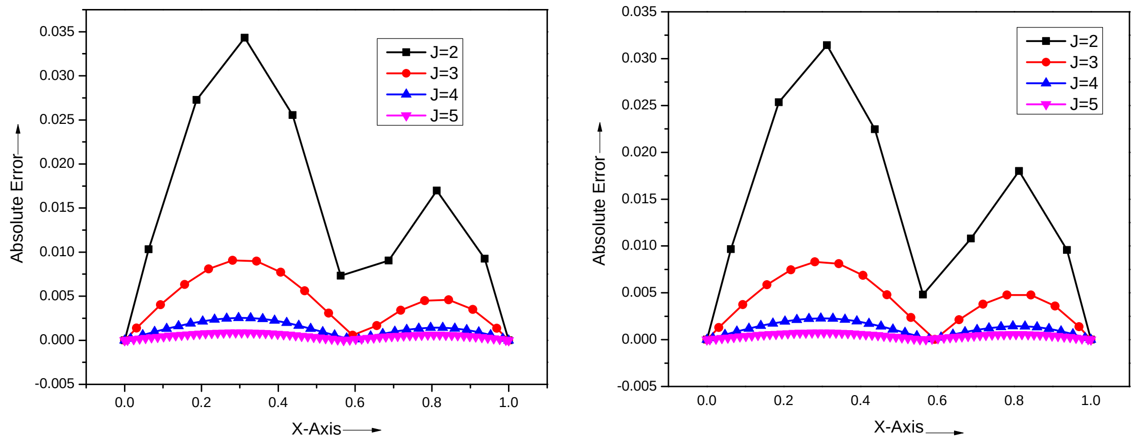

Fig. 3. Absolute errors for $\sigma = 0.25$ (left) and $\sigma = 0.75$ (right) with $\Delta t = 0.01$, for different J at time $T = 1$ for the problem 6.2.

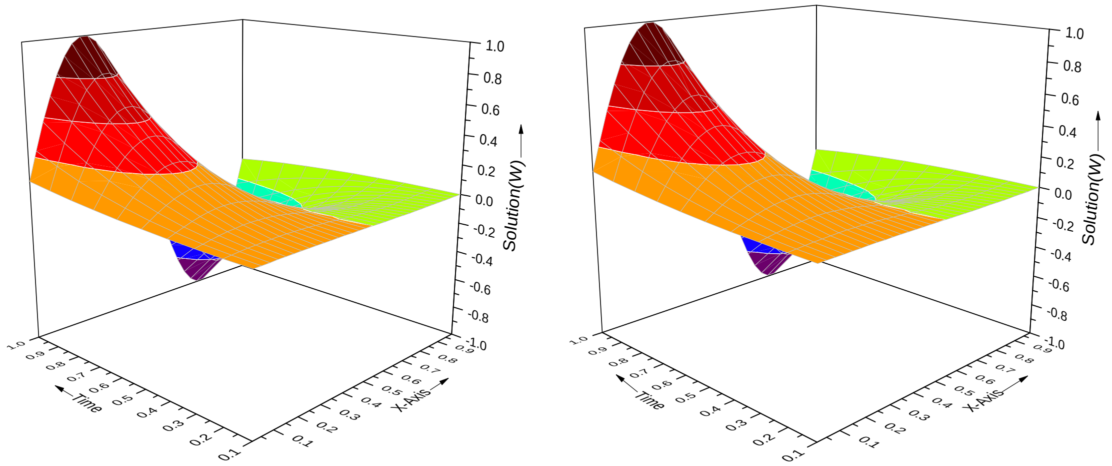

Fig. 4. $3D$ surface of analytical solutions (left) and numerical solutions (right) with $\sigma = 0.40$, $\Delta t = 0.001$ and $J = 4$ for the problem 6.2.

6.3 Example 3

Here we implement our scheme to the following fractional differential equation with time delay, given as

$$\frac{\partial^\sigma \omega}{\partial t^\sigma} - (\mu(x)\omega_x)_x + \omega^3 w_x = \omega^2(x, t - 0.2) + H(x,t), \qquad (x,t) \in (0,1) \times (0,1], \tag{59}$$

with the non-homogenous BCs

$$\omega(0,t) = t^{2+\sigma} = \omega(1,t), \quad t \in (0,1], \tag{60}$$

and IC

$$\omega(x,t) = t^{2+\sigma} \cos(2\pi x), \qquad x \in [0,1] \times [-0.2, 0], \tag{61}$$

where $\mu(x) = e^x + \cos(x)$. We can simply verify that the function

$$\omega(x,t) = t^{2+\sigma} \cos(2\pi x), \tag{62}$$

is the exact solution. From Eq. (59), the forcing term $H(x,t)$ can be easily obtained. In this test problem, we take $p = 3$ and the delay $\tau = 0.2$ with the non-homogeneous BCs and variable coefficient as the function of a sum of the exponential function and trigonometric function. In Table 7, we list L_∞-error and L_2-error norms for $\sigma = 0.30,\ 0.50,\ 0.70$, $\Delta t = 0.001$ at $T = 1$ for different J. From the table 7, we can see that both L_2-error and L_∞-error norms decrease with an increase of J. In Table 8, we take $\sigma = 0.25,\ 0.50,\ 0.75$, $T = 0.5$ and compute L_2-error and L_∞-error norms for different time steps Δt and we observe that on refining the size of time step, errors become smaller and smaller. In Table 9, we list the ROC of the present scheme and see that it is very close to its theoretical value. From Table 9, we can also see that the present scheme provides acceptable results for $\sigma = 1$ and the ROC approaches its theoretical value. Moreover, Fig 5 shows the absolute error of the problem 6.3 for $\sigma = 0.25$ (left) and $\sigma = 0.75$ (right) at $T = 1$ with the time step $\Delta t = 0.01$ for $J = 2, 3, 4, 5$. We can see that the solutions provided by our scheme are bounded throughout the domain and the absolute error becomes smaller and smaller on increasing number of collocation points. Figure 6 represents the analytical solutions surface (left) and approximated solutions surface (right) for $\sigma = 0.60$, $2M = 32$ and step size of time $\Delta t = 0.001$. From the Fig. 6, we can see that the analytical solutions surface and the approximated solutions surface by our method look nearly the same.

Table 7. L_2-error and L_∞-error norms with $\sigma = 0.30,\ 0.50,\ 0.70$ with time step size $\Delta t = 0.001$ for the problem 6.3 at $T = 1$.

	L_∞-error			L_2-error		
J	$\sigma = 0.30$	$\sigma = 0.50$	$\sigma = 0.70$	$\sigma = 0.30$	$\sigma = 0.50$	$\sigma = 0.70$
2	4.80040E-02	4.75432E-02	4.69048E-02	2.97316E-02	2.94184E-02	2.89867E-02
3	1.20974E-02	1.19872E-02	1.18320E-02	7.33877E-03	7.26364E-03	7.15842E-03
4	3.04304E-03	3.01658E-03	2.97696E-03	1.83869E-03	1.82078E-03	1.79430E-03

Table 8. L_2-error and L_∞-error norms with $J = 5,\ \sigma = 0.25,\ 0.50,\ 0.75$ having different step size Δt at $T = 0.5$ for the problem 6.3.

	L_∞-error			L_2-error		
Δt	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$	$\sigma = 0.25$	$\sigma = 0.50$	$\sigma = 0.75$
0.05	6.97678E-04	6.85794E-04	4.93929E-04	4.23641E-04	4.15639E-04	2.99024E-04
0.025	4.32752E-04	4.25740E-04	3.35597E-04	2.62849E-04	2.58228E-04	2.03169E-04
0.0125	2.97120E-04	2.84200E-04	2.35135E-04	1.80394E-04	1.72265E-04	1.42163E-04

Table 9. L_2-error and L_∞-error norms and rate of convergence at $T = 0.5$, $\Delta t = 0.001$, for the problem 6.3.

	$\sigma = 0.5$				$\sigma = 1$			
J	L_∞-error	ROC	L_2-error	ROC	L_∞-error	ROC	L_2-error	ROC
1	3.04307E-2		2.20260E-2		1.88139E-2		1.36016E-2	
2	8.14624E-3	1.90132	5.10164E-3	2.11017	5.09046E-3	1.88593	3.14986E-3	2.11041
3	2.08078E-3	1.96900	1.26481E-3	2.01204	1.29339E-3	1.97664	7.75356E-4	2.02236
4	5.30599E-4	1.97143	3.21354E-4	1.97669	3.18945E-4	2.01978	1.90421E-4	2.02567

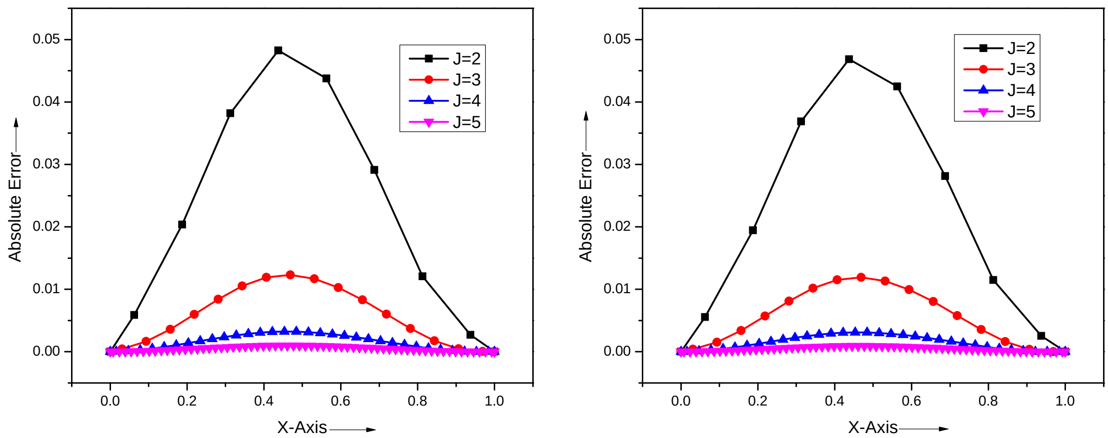

Fig. 5. Absolute error for $\sigma = 0.25$ (left) and $\sigma = 0.75$ (right) with $\Delta t = 0.01$, at time $T = 1$ with different J for the problem 6.3.

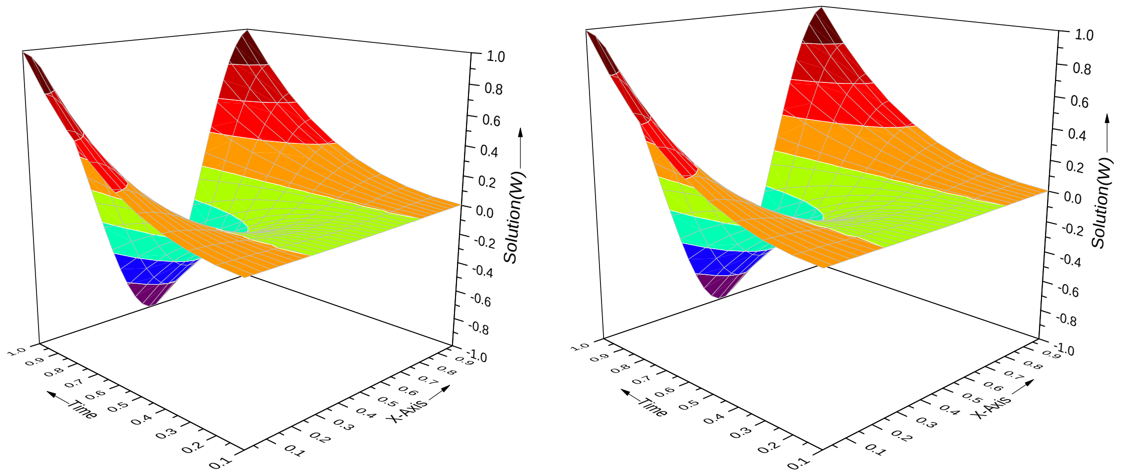

Fig. 6. $3D$ surface of analytical solutions (left) and numerical solutions (right) with $J = 4$, $\Delta t = 0.001$ and $\sigma = 0.60$ for the problem 6.3.

6.4 Example 4

Here we implement our scheme to the following time delay differential equation, given by

$$\frac{\partial \omega}{\partial t} - (\mu(x)\omega_x)_x + \omega^5 w_x = \omega^2(x, t-0.1) + H(x,t), \qquad (x,t) \in (0,1) \times (0,1], \tag{63}$$

with the homogenous BCs

$$\omega(0,t) = 0 = \omega(1,t), \quad t \in (0,1], \tag{64}$$

and IC

$$\omega(x,t) = t\sin(x), \qquad x \in [0,1] \times [-0.1, 0], \tag{65}$$

where $\mu(x) = e^x + x^2 + 1$. We can simply verify that the function

$$\omega(x,t) = t\sin(x), \tag{66}$$

is the exact solution. From Eq.(63), the forcing term $H(x,t)$ can be easily obtained. In this test problem, we take $p = 5$ and the delay $\tau = 0.1$ with the homogeneous BCs and variable coefficient as the function of a sum of the exponential function and algebraic function. In Table 10, we list L_∞ and L_2-error norms and the ROC of the present scheme. From the Table 10, we can see both L_2-error and L_∞-error norms decrease with an increase of J and the ROC of the scheme is very close to its theoretical value. Moreover, Fig. 7 shows the absolute error of the problem 6.4 for $T = 0.5$ with the time step $\Delta t = 0.001$ for $J = 2, 3, 4, 5$. We can see that the solutions provided by our scheme are bounded throughout the domain and the absolute error becomes smaller and smaller on increasing number of collocation points. Figure 8 represents the analytical solutions surface (left) and approximated solutions surface (right) for $J = 4$ and step size of time $\Delta t = 0.001$. From the Fig. 8, we can see that the analytical solutions surface and the approximated solutions surface by our method look nearly the same.

Table 10. L_2-error and L_∞-error norms and rate of convergence at $T = 1$, $\Delta t = 0.001$, for the problem 6.4

J	L_∞-error	ROC	L_2-error	ROC
1	2.16869E-2		1.57635E-2	
2	5.91776E-3	1.87370	4.15136E-3	1.92494
3	1.48787E-3	1.99180	1.04129E-3	1.99521
4	3.59759E-4	2.04814	2.51806E-4	2.04799

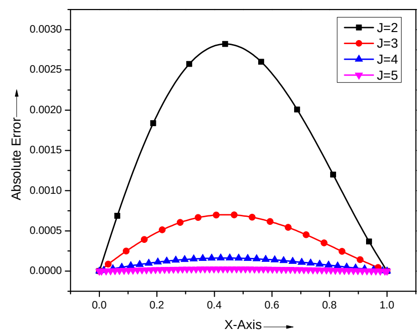

Fig. 7. Absolute error with $\Delta t = 0.001$, at time $T = 0.5$ with different J for the problem 6.4.

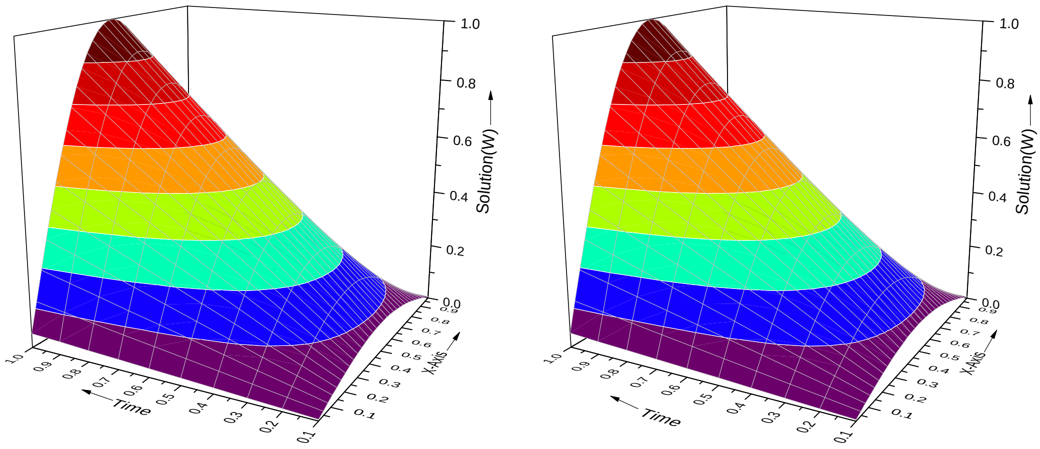

Fig. 8. $3D$ surface of analytical solutions (left) and numerical solutions (right) with $\Delta t = 0.001$, $J = 4$ for the problem 6.4.

7. Conclusion

In this article, a novel scheme based on HWs coupled with NSFD scheme for the numerical solution of fractional order time derivative Burgers' equation with variable coefficient and time delay is proposed. The boundedness and convergence of the method are discussed. The proposed scheme is tested over several examples and the accuracy of the numerical scheme is assessed by computing errors and comparing with existing results. Also, theoretical rate of convergence of the proposed method is verified with the discrete data. We have recognized that the proposed scheme requires lower computational cost and easy to implement compare to others method. The computed results are quite reasonable and comparable with the results existing in the literature. The key concept used in this method is to look ahead to be implemented over the similar types of PDEs having variable coefficient with time delay which model different kinds of real-life problems.

Author Contributions

Conceptualization, A.K.V.; methodology, A.K.V., software, M.K.R.; validation R.P.A.; formal analysis, M.K.R., and R.P.A.; investigation, R.P.A.; resources, A.K.V. and R.P.A.; writing-original draft preparation, A.K.V., M.K.R., and R.P.A.; writing-review and editing, A.K.V., M.K.R, and R.P.A., supervision, R.P.A. All authors have read and agreed the published version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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