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Research Paper

Dynamically Consistent NSFD Methods for Predator-prey System

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Abstract. In this paper, we introduce two nonstandard finite difference (NSFD) methods for solving the mathematical model of the Rosenzweig-MacArthur predator-prey system. These new proposed numerical methods have important features such as positivity and elementary stability. Numerical comparisons between the proposed methods and the other methods such as second-order and forth order Runge-Kutta methods (we refer them RK2 and RK4, respectively), Euler method, and NSFD method presented in [1] indicate that the new methods have better accuracy and convergence.

Keywords: Predator-Prey, Stability, Positivity, Elementary stable, Nonstandard Finite Difference.

1. Introduction

Since predator-prey systems are related to ecological life and human activities such as forestry, fishery and so on, they have been studied by a great many ecologists. These types of systems don't have analytical solutions, so the numerical techniques to obtain approximate solutions close to the analytical solution must be used. One of these methods involves using the finite difference to construct a discrete model of predator-prey systems. But the disadvantage of this method is that sometimes the qualitative properties (such as positivity, elementary stability, etc.) of the exact solution are not transferred to the numerical solution. To overcome these deficiencies, NSFD methods have been proposed in accordance with Mickens' rules [1-8].

In fact, NFSD methods not only do guarantee conventional features i.e, consistency, stability, and consequently convergence but also they preserve the qualitative features of the system such as positivity, elementary stability, and conservation law [9-16]. In this paper, we present two NSFD methods for solving the predator-prey system. Reasons for presenting these methods are to guarantee positivity property and elementary stability of the solution for the predator-prey system. Also, new methods have better accuracy and convergence.

The paper is organized as follows: In Section 2, we give some definitions and preliminary setting. In Section 3, we present the mathematical model of Rosenzweig-MacArthur predator-prey system [17] with logistic intrinsic growth rate and its equilibrium points. In Section 4, we introduce new schemes. We investigate the positivity property in Section 5. In Section 6, the elementary stability of schemes are investigated. Comparison of results obtained by the new methods and standard methods are presented in Section 7. In Section 8, general results and discussion of the paper are given.

2. Definitions and Preliminaries

Let us to consider

$$\frac{d}{dt}X(t) = F\big(X(t)\big), \quad \geq 0, \quad X(0) = X_0 \in R^k, \tag{1}$$

where $X = [X^1, X^2, \dots, X^k]$: $[t_0, T] \to R^k$and $F = [F^1, F^2, \dots, F^k]^T$: $R^k \to R^k$. In this paper, we solve the system (1) with nonstandard discretization methods. Suppose that $t_n = t_0 + nh$, h is step-size and $X_n \approx X(t_n)$, then discrete form of system (1) is as follows:

$$D_h X_n = \mathcal{F}_n(F, X_n), \tag{2}$$

where $D_h X_n \approx \frac{d}{dt}X(tn)$, and $\mathcal{F}_n(F, X_n)$ approximates $F(X(t_n))$ at time t_n.
Class of NSFDs and their formulations are determined around two issues: first, how to formulate discretization of and second, what are the appropriate ways to approximate nonlinear terms. The forward finite difference approximation for the first-order derivative is one of the most common methods for discretization. In standard mode, derivative $\frac{dx}{dt}$ is approximated by $\frac{x(t+h)-x(t)}{h}$, which h indicates the step-size. While in the methods presented by Mickens, this term is approximated by $\frac{x(t+h)-x(t)}{\psi(h)}$, where $\psi(h)$ is an increasing continuous function of h, which satisfies the following condition

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$$\psi(h) = h + O(h^2), \qquad 0 < \psi(h) < 1, \quad h \to 0. \tag{3}$$

Note, when $h \to 0$ we must obtain the first derivative whatever the $\phi(h)$ taken. In fact, it must be

$$\frac{dx}{dt} = \lim_{h\to 0} \frac{x\left(t+\psi_1(h)\right)-x(t)}{\psi_2(h)},$$

where $\psi_1(h)$ and $\psi_2(h)$ are continuous functions of the step-size h verifying (3).

Definition 1. A scheme is called nonstandard if at least one of the following conditions is satisfied:
if a nonlocal approximation of nonlinear terms is used in it or a more complex function $\varphi(h)$ is replaced by a non-negative function φ such that

$$\varphi(h) = h + O(h^2) \quad as \quad 0 < h \to 0.$$

For instance,

$$x(1-2x) \approx x_k(1-2x_{k+1}), \quad x^2 \approx x_k x_{k+1},$$

$$\varphi(h) = 1 - e^{-h}, \quad \varphi(h) = \frac{1-e^{-qh}}{q}, \quad q > 0$$

Definition 2. Any constant-vector $\tilde{X}$ is called equilibrium point of the differential equation in (1) if

$$F(\tilde{X}) = 0.$$

3. Mathematical Model

Let consider mathematical model of the general Rosenzweig-MacArthur predator-prey model [1, 17, 18, 19, 20, 21, 22] with logistic intrinsic growth rate of the prey as follows:

$$\begin{cases} \dfrac{dx}{dt} = bx(1-x) - ag(x)xy, \quad x(t_0) = x_0 \geq 0 \\ \dfrac{dy}{dt} = g(x)xy - cy, \quad y(t_0) = y_0 \geq 0 \end{cases}, \tag{4}$$

where

- x is the prey population size.
- y is the predator population size.
- $b > 0$ is the logistic intrinsic growth rate of the prey.
- $a > 0$ is the capture rate.
- $c > 0$ is the mortality rate of the predator.
- $xg(x)$ as $x \longrightarrow \infty$ is bounded.

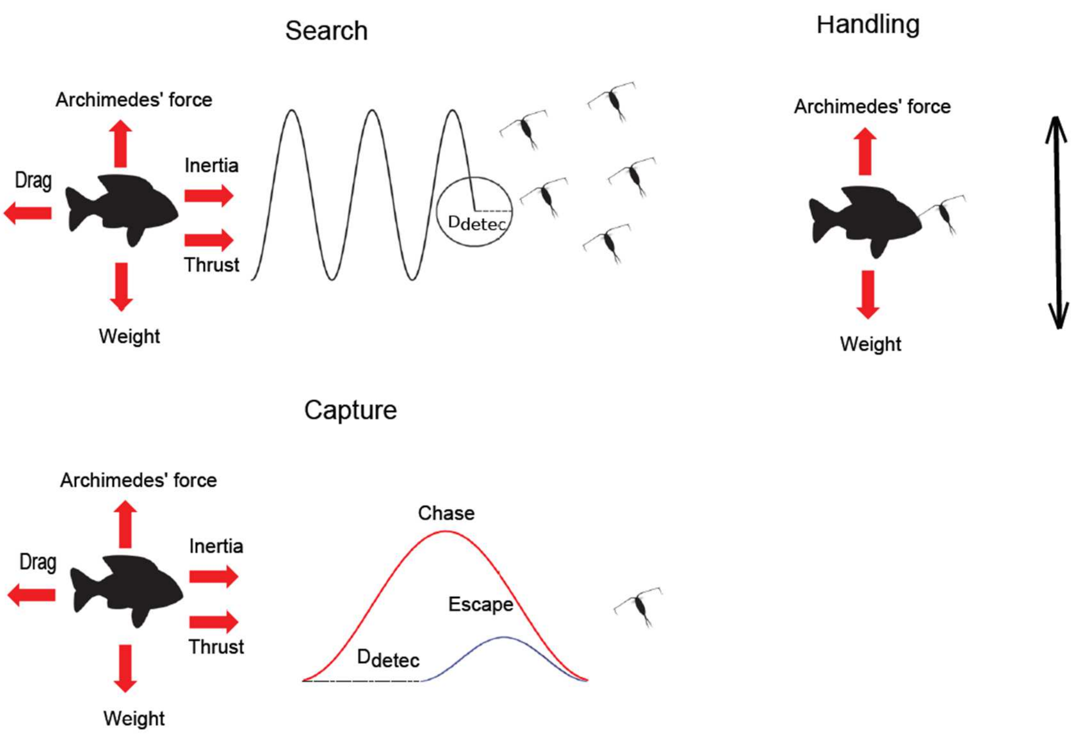


Fig. 1. The main properties of the model.

Furthermore

$$g(x) \geq 0, \quad g'(x) \leq 0, \quad [xg(x)]' \geq 0, \quad x > 0. \tag{5}$$

Fig. 1 shows that predation is by essence a mechanical interaction. For more details see[28]. The system (2) has two equilibrium points as follows:

- $E_0 = (0,0)$ is the trivial equilibrium point and always linearly stable.
- $E_1 = (1,0)$, that if $g(1) < c$ in this case is linearly stable and if $g(1) > c$ then is linearly unstable .
- The equilibrium point $E^* = (x^*, y^*)$ exist where x^* is the solution of $xg(x) = c$ and $y^* = \frac{bx^*(1-x^*)}{ac}$.

It is worth mentioning that the equilibrium point E^* exists if and only if $g(1) > c$. If $b + ay^*g'(x^*) > 0$, then E^* is linearly stable, and if $b + ay^*g'(x^*) < 0$ then E^* is linearly unstable.

The predator has to use energy to keep hovering, but it makes use of Archimedes' force. We have three sequences in predation. The first sequence involves the predator searching for its prey, an action which happens in a cyclic pattern. Motion suggests interaction between mechanical thrust, inertia and drag. Prey profusion and predator's detection distance or Ddect restricts encounter, which – if successful – will lead to the next sequence – i.e., capture. Here the predator moves to capture the prey, but the prey strives to escape. If the predator succeeds, it will need to keep hovering (i.e. taking the prey) while consuming it.

The main properties of the model

4. Construction of New Schemes

We present two new methods for solving system 4 as follows:

4.1 Scheme 1

The first positive and elementary stable nonstandard (PESN) scheme is presented as:

$$\begin{cases} \dfrac{x_{k+1} - x_k}{\varphi(h)} = bx_k(1 - x_{k+1}) - a[2x_{k+1} - x_k]y_k g(x_k), \\[2ex] \dfrac{y_{k+1} - y_k}{\varphi(h)} = \dfrac{x_k + x_{k+1}}{2} y_k g(x_k) - c[2y_{k+1} - y_k], \end{cases} \tag{6}$$

The above relation can be rewritten as follows:

$$\begin{cases} x_{k+1} = \dfrac{x_k(1 + b\varphi(h) + a\varphi(h)y_k g(x_k))}{1 + bx_k\varphi(h) + 2a\varphi(h)g(x_k)y_k}, \\[2ex] y_{k+1} = \dfrac{y_k(2 + 2c\varphi(h) + g(x_k)\varphi(h)(x_k + x_{k+1}))}{2 + 4c\varphi(h)}. \end{cases} \tag{7}$$

4.2 Scheme 2

Our second PESN scheme is proposed as:

$$\begin{cases} \dfrac{x_{k+1} - x_k}{\varphi(h)} = bx_k - bx_k x_{k+1} - ag(x_k)x_{k+1}y_k, \\[2ex] \dfrac{y_{k+1} - y_k}{\varphi(h)} = g(x_k)x_{k+1}y_k - cy_{k+1}, \end{cases} \tag{8}$$

Scheme 2 can be written in the following form:

$$\begin{cases} x_{k+1} = \dfrac{x_k(1 + b\varphi(h))}{1 + bx_k\varphi(h) + a\varphi(h)g(x_k)y_k}, \\[2ex] y_{k+1} = \dfrac{y_k(1 + g(x_k)\varphi(h)x_{k+1})}{1 + c\varphi(h)}. \end{cases} \tag{9}$$

5. Positivity

In this section, we present the positivity property of the constructed methods in the previous section. The positivity of the numerical methods is important in the study of population biology models because state variables that represent subpopulations should not take negative values [16].

Definition 5.1 For the step size h, and initial condition $y_0 \in \mathbb{R}_+^n$ the numerical method is called positive if the solution $y_k \in \mathbb{R}_+^n$ for all $k \in \boldsymbol{N}$ is positive.

The new schemes preserve the positivity property as is stated in the following theorem.

Theorem 5.2 Suppose the function $g(x)$ in (5) holds and (x^*, y^*) is the interior equilibrium of system (4). If $0 < x^* < min\{1, -g(x^*)/2g'(x^*)\}$, then the new schemes (7) and (9) are positive.

Proof. Since the constants a, b, c and the function g are always positive, the schemes (7) and (9) are unconditionally non-negative.

6. Elementary Stability

In this section, we examine the elementary stability of the proposed methods and the sufficient conditions for their establishment. The construction of difference methods is importance in the problems of ordinary differential equations that are locally stable at equilibrium points. Difference methods with this feature are called elementary stable methods [1, 15, 23].

Definition 2. For the given step size h, the finite difference method is called elementary stabel, only its equilibrium points are those of the original differential system, and the properties of linear stability are the same for both the system of equations and the numerical method.

To present the main result of this paper, we use the following lemma to prove the main resul [24, 25].

Lemma 1. For the quadratic equation $\lambda^2 - \alpha\lambda + \beta = 0$, by using the well-known Jury condition, both roots satisfy $|\lambda_i| < 1, i = 1,2$, if we have:

- $1 + \alpha + \beta > 0$,
- $1 - \alpha + \beta > 0$ and
- $\beta < 1$. ■

Theorem 6.1 The schemes in (7) and (9) are elementary stables.
Proof. The equilibrium points of (7) are precisely the equilibria E_0^* and E^* of system (4). The Jacobian J of (7) has the form $J(x_k, y_k) = [j_{mn}(x_k, y_k)]_{2\times 2}$, where

$$j_{11}(x_k,y_k) = \frac{(1+b\varphi(h)+a\varphi(h)y_kg(x_k)+a\varphi(h)y_kg'(x_k)x_k)(1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k)}{(1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k)^2}$$

$$-\frac{(b\varphi(h)+2a\varphi(h)g'(x_k)y_k)x_k(1+b\varphi(h)+a\varphi(h)g(x_k)y_k)}{(1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k)^2},$$

$$=\frac{(1+b\varphi(h)+a\varphi(h)y_kg(x_k))(1+2a\varphi(h)y_k(g(x_k)-g'(x_k)x_k)}{(1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k)^2}$$

$$+\frac{a\varphi(h)x_ky_kg'(x_k)}{1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k},$$

$$j_{12}(x_k,y_k) = \frac{x_ka\varphi(h)g(x_k)((1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k))}{(1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k)^2}$$

$$-\frac{2a\varphi(h)g(x_k)x_k(1+b\varphi(h)+a\varphi(h)g(x_k)y_k)}{(1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k)^2},$$

$$=\frac{x_ka\varphi(h)g(x_k)(-1+b\varphi(h)(x_k-2))}{(1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k)^2}$$

$$j_{21}(x_k,y_k) = \frac{g'(x_k)y_k\varphi(h)(x_k+x_{k+1})+y_kg(x_k)\varphi(h)(1+j_{11}(x_k,y_k))}{2+4c\varphi(h)},$$

$$=\frac{g'(x_k)y_k\varphi(h)(x_k+\frac{x_k(1+b\varphi(h)+a\varphi(h)y_kg(x_k))}{1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k})+y_kg(x_k)\varphi(h)(1+j_{11}(x_k,y_k))}{2+4c\varphi(h)}$$

$$j_{22}(x_k,y_k) = \frac{2+2c\varphi(h)+g(x_k)\varphi(h)(x_k+x_{k+1})+g(x_k)\varphi(h)y_kj_{12}(x_k,y_k)}{2+4c\varphi(h)},$$

$$=\frac{2+2c\varphi(h)+g(x_k)\varphi(h)\left(x_k+\frac{x_k(1+b\varphi(h)+a\varphi(h)y_kg(x_k))}{1+bx_k\varphi(h)+2a\varphi(h)g(x_k)y_k}\right)+y_k\varphi(h)g(x_k)j_{12}(x_k,y_k)}{2+4c\varphi(h)}.$$

Substituting E_0 in $J(x_k, y_k)$, we derive

$$J(0,0) = \begin{pmatrix} 1+b\varphi(h) & 0 \\ 0 & \dfrac{1+c\varphi(h)}{1+2c\varphi(h)} \end{pmatrix},$$

the eigenvalues of $J(0,0)$ are

$$\lambda_1 = 1+b\varphi(h), \qquad \lambda_2 = \frac{1+c\varphi(h)}{1+2c\varphi(h)},$$

since $|\lambda_1| > 1$, the equilibrium point E_0 is always unstable.

Now, for the equilibrium point E_1 we have

$$J(1,0) = \begin{pmatrix} \dfrac{1}{1+b\varphi(h)} & -\dfrac{a\varphi(h)g(1)}{1+b\varphi(h)} \\ 0 & \dfrac{1+c\varphi(h)+g(1)\varphi(h)}{1+2c\varphi(h)} \end{pmatrix},$$

whose eigenvalues are:

$$\lambda_1 = \frac{1}{1+b\varphi(h)},$$

$$\lambda_2 = \frac{1+c\varphi(h)+g(1)\varphi(h)}{1+2c\varphi(h)}.$$

It is clear that $|\lambda_1| < 1$ and if $g(1) < c$ then $|\lambda_2| < 1$ too, and $E_1 = (1,0)$ is stable. Otherwise, the equilibrium E_1 is unstable. Finally for the equilibrium $E^* = (x^*, y^*)$ the Jacobian matrix is:

$$J(x^*,y^*) = \begin{pmatrix} 1 - \dfrac{b\varphi(h)x^* + a\varphi(h)y^*x^*g'(x^*)}{1+2b\varphi(h) - b\varphi(h)x^*} & -\dfrac{a\varphi(h)c}{1+2b\varphi(h)-b\varphi(h)x^*} \\ \dfrac{2\varphi(h)y^*(x^*g'(x^*)+g(x^*))}{2+4c\varphi(h)} - \dfrac{(\varphi(h))^2cy^*[b+ay^*g'(x^*)]}{(2+4c\varphi(h))(1+2b\varphi(h)-b\varphi(h)x^*)} & 1 - \dfrac{\varphi(h)^2cb(1-x^*)}{(2+4c\varphi(h))(1+2b\varphi(h)-b\varphi(h)x^*)} \end{pmatrix},$$

eigenvalues of $J(x^*, y^*)$ are roots of the quadratic equation $\lambda^2 + \alpha\lambda + \beta = 0$ where

$$\alpha = \frac{\varphi(h)x^*B}{A} + \frac{\varphi(h)^2cb(1-x^*)}{AC} - 2,$$

$$\beta = 1 - \frac{\varphi(h)x^*B}{A} + \frac{\varphi(h)^2b(1-x^*)[c+2x^{*2}g'(x^*)]}{AC},$$

and

$$A = 1 + 2b\varphi(h) - b\varphi(h)x^* > 0, \quad 0 < x^* < 1,$$

$$B = b + ay^*g'(x^*), \quad g'(x^*) < 0,$$

$$C = 2 + 4c\varphi(h) > 0.$$

Since A and C are positive for all $0 < x^* < 1$, therefore, if $B > 0$ then

$$1 + \alpha + \beta = \frac{2\varphi(h)^2b(1-x^*)x[g(x^*) + x^*g'(x^*)]}{AC} > 0,$$

$$1 - \alpha + \beta = \frac{C(4 + 8b\varphi(h) - 6b\varphi(h)x^*) - 4x^*ay^*g'(x^*)\varphi(h) - 6cx^*ay^*g'(x^*)\varphi(h)^2}{AC} > 0,$$

$$\beta = 1 - \frac{\varphi(h)x^*B}{A} + \frac{\varphi(h)^2b(1-x^*)[c+2x^{*2}g'(x^*)]}{AC},$$

now, if $x^* < \frac{-g(x^*)}{2g'(x^*)}$ then $\beta < 1$ and following Lemma 1, E^* is stable; otherwise, E^* is unstable. These results show that between the system (4) and the numerical scheme (7) around all equilibria exist dynamical consistency. Therefore, the new proposed scheme is elementary stable, and this completes the proof.

Similarly, the equilibrium points of (9) are exactly the same equilibria as those of the original system. The Jacobian J of (9) has the form $J(x_k, y_k) = [j_{mn}(x_k, y_k)]_{2\times2}$, where

$$\begin{aligned} j_{11}(x_k, y_k) &= \frac{(1+b\varphi(h))(1+bx_k\varphi(h)+a\varphi(h)g(x_k)y_k)}{(1+bx_k\varphi(h)+a\varphi(h)g(x_k)y_k)^2} \\ &\quad - \frac{(b\varphi(h)+a\varphi(h)g'(x_k)y_k)x_k(1+b\varphi(h))}{(1+bx_k\varphi(h)+a\varphi(h)g(x_k)y_k)^2}, \\ &= \frac{(1+b\varphi(h))[1+a\varphi(h)y_k(g(x_k)-x_kg'(x_k))]}{(1+bx_k\varphi(h)+a\varphi(h)g(x_k)y_k)^2} \end{aligned}$$

$$j_{12}(x_k, y_k) = -\frac{x_ka\varphi(h)g(x_k)(1+b\varphi(h))}{(1+bx_k\varphi(h)+a\varphi(h)g(x_k)y_k)^2}$$

$$\begin{aligned} j_{21}(x_k, y_k) &= \frac{1}{1+c\varphi(h)}[g'(x_k)y_k\varphi(h)(\frac{x_k(1+b\varphi(h))}{1+bx_k\varphi(h)+a\varphi(h)g(x_k)y_k}) \\ &\quad + g(x_k)y_k\varphi(h)(\frac{(1+b\varphi(h))[1+a\varphi(h)y_k(g(x_k)-x_kg'(x_k))]}{(1+bx_k\varphi(h)+a\varphi(h)g(x_k)y_k)^2})], \end{aligned}$$

$$j_{22}(x_k,y_k)=\frac{1}{1+c\varphi(h)}[1+g(x_k)\varphi(h)(\frac{x_k(1+b\varphi(h))}{1+bx_k\varphi(h)+a\varphi(h)g(x_k)y_k})$$

$$+y_k\varphi(h)g(x_k)(-\frac{x_k a\varphi(h)g(x_k)(1+b\varphi(h))}{(1+bx_k\varphi(h)+a\varphi(h)g(x_k)y_k)^2})].$$

Substituting E_0 in $J(x_k,y_k)$, we derive

$$J(0,0)=\begin{pmatrix}1+b\varphi(h) & 0\\ 0 & \frac{1}{1+c\varphi(h)}\end{pmatrix},$$

the eigenvalues of $J(0,0)$ are

$$\lambda_1=1+b\varphi(h),\qquad \lambda_2=\frac{1}{1+c\varphi(h)},$$

since $|\lambda_1|>1$, the equilibrium E_0 is always unstable.

Now, for the equilibrium E_1 we have

$$J(1,0)=\begin{pmatrix}\frac{1}{1+b\varphi(h)} & -\frac{a\varphi(h)g(1)}{1+b\varphi(h)}\\ 0 & \frac{1+g(1)\varphi(h)}{1+c\varphi(h)}\end{pmatrix},$$

whose eigenvalues are:

$$\lambda_1=\frac{1}{1+b\varphi(h)},$$

$$\lambda_2=\frac{1+g(1)\varphi(h)}{1+c\varphi(h)},$$

it is clear that $|\lambda_1|<1$ and if $g(1)<c$ then $|\lambda_2|<1$ too, and $E_1=(1,0)$ is stable. Otherwise, the equilibrium E_1 is unstable.

Finally, for equilibrium $E^*=(x^*,y^*)$ the Jacobian matrix is:

$$J(x^*,y^*)=\begin{pmatrix}1-\frac{b\varphi(h)x^*+a\varphi(h)y^*x^*g'(x^*)}{1+b\varphi(h)} & -\frac{a\varphi(h)c}{1+b\varphi(h)}\\ \frac{\varphi(h)y^*(x^*g'(x^*)+g(x^*))}{1+c\varphi(h)}-\frac{(\varphi(h))^2cy^*[b+ay^*g'(x^*)]}{(1+c\varphi(h))(1+b\varphi(h))} & 1-\frac{\varphi(h)^2cb(1-x^*)}{(1+c\varphi(h))(1+b\varphi(h))}\end{pmatrix}$$

eigenvalues of $J(x^*,y^*)$ are roots of the quadratic equation $\lambda^2+\alpha\lambda+\beta=0$ where

$$\alpha=\frac{\varphi(h)x^*B}{A}+\frac{\varphi(h)^2cb(1-x^*)}{AC}-2,$$

$$\beta=1-\frac{\varphi(h)x^*B}{A}+\frac{\varphi(h)^2b(1-x^*)[c+2x^{*2}g'(x^*)]}{AC},$$

and

$$A=1+b\varphi(h)>0,\quad 0<x^*<1,$$

$$B=b+ay^*g'(x^*),\quad g'(x^*)<0,$$

$$C=1+c\varphi(h)>0.$$

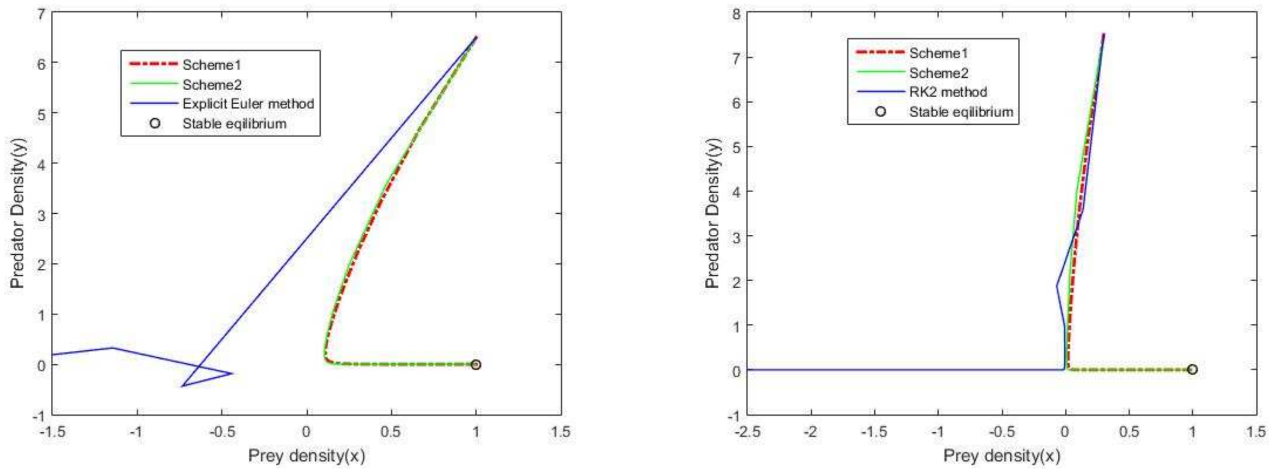
A and C are positive for all $0<x^*<1$; therefore, if $B>0$ then

$$1+\alpha+\beta=\frac{2\varphi(h)^2b(1-x^*)x[g(x^*)+x^*g'(x^*)]}{AC}>0,$$

$$1-\alpha+\beta=\frac{C(4+4b\varphi(h)-2b\varphi(h)x^*)-2x^*ay^*g'(x^*)\varphi(h)}{AC}>0,$$

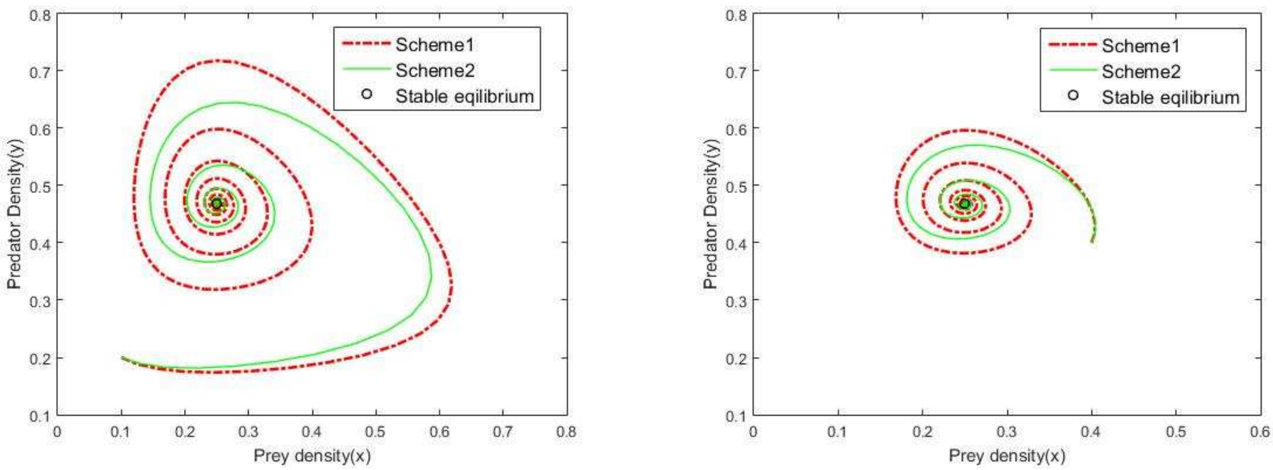
$$\beta=1-\frac{\varphi(h)x^*B}{A}+\frac{\varphi(h)^2b(1-x^*)[c+2x^{*2}g'(x^*)]}{AC},$$

now, if $x^*<\frac{-g(x^*)}{2g'(x^*)}$ then $\beta<1$ and following Lemma 1, E^* is stable; otherwise, E^* is unstable. These results show that between the system (4) and the numerical scheme (9) around all equilibria exist dynamical consistency. Therefore, the new proposed scheme is elementary stable, and this completes the proof.∎



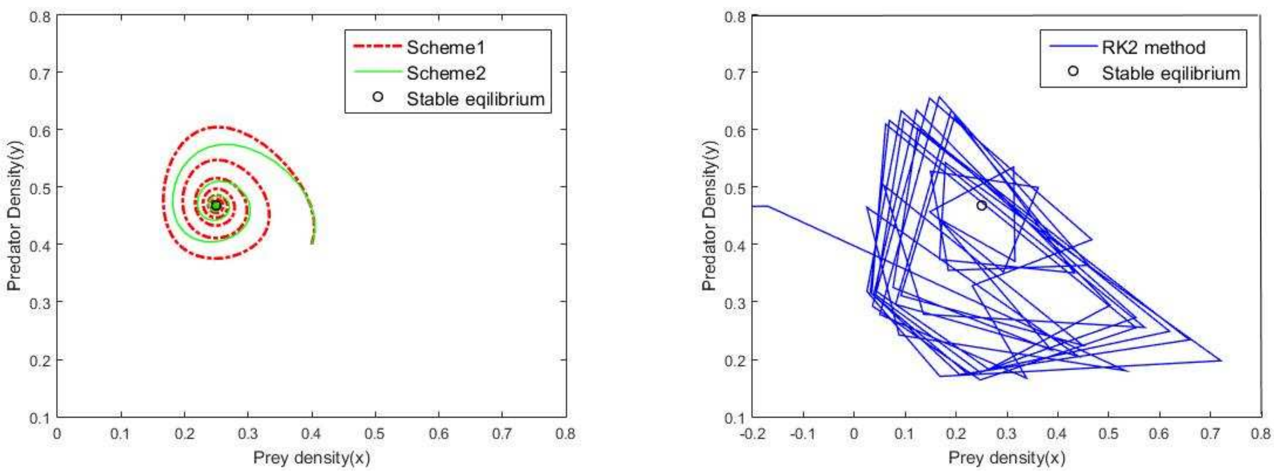
(a) $h = 0.2, x_0 = 1, y_0 = 6.5$ (b) $h = 0.2, x_0 = 0.3, y_0 = 7.5$

Fig. 2. Numerical results with $a = 2$, $b = 1$, $r = 0.5$, $c = 6$ and $q = 3.1$.



(a) $h = 2.1, x_0 = 0.1, y_0 = 0.2$ (b) $h = 1.3, x_0 = .4, y_0 = .4$

Fig. 3. Numerical results with $a = 2$, $b = 1$, $r = 1$, $c = 0.2$ and $q = 1.2$.



(a) $h = 4.6, x_0 = 0.4, y_0 = 0.4$ (b) $h = 4.6, x_0 = 0.4, y_0 = 0.4$

Fig. 4. Numerical results with $a = 2$, $b = 1$, $r = 1$, $c = 0.2$ and $q = 1.2$.

Table 1. Qualitative behavior with respect to E_0 of the schemes considered on the problem (4) with $a = 2, b = 1, r = 0.5, c = 6, q = 3.1, x_0 = 1$, and $y_0 = 6.5$.

h	$Euler$	$RK2$	$RK4$	$Scheme1$	$Scheme2$
0.01	Convergence	Convergence	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Convergence	Convergence	Convergence
0.2	Divergence	Convergence	Convergence	Convergence	Convergence
0.3	Divergence	Divergence	Convergence	Convergence	Convergence
1	Divergence	Divergence	Divergence	Convergence	Convergence
2	Divergence	Divergence	Divergence	Convergence	Convergence
10	Divergence	Divergence	Divergence	Convergence	Convergence

Table 2. The computational time used to obtain Fig. 2-5.

Figure	(a)	(b)
Fig. 2	3.756s	2.415s
Fig. 3	3.295s	3.297s
Fig. 4	3.296s	2.246s
Fig. 5	3.416s	2.227s

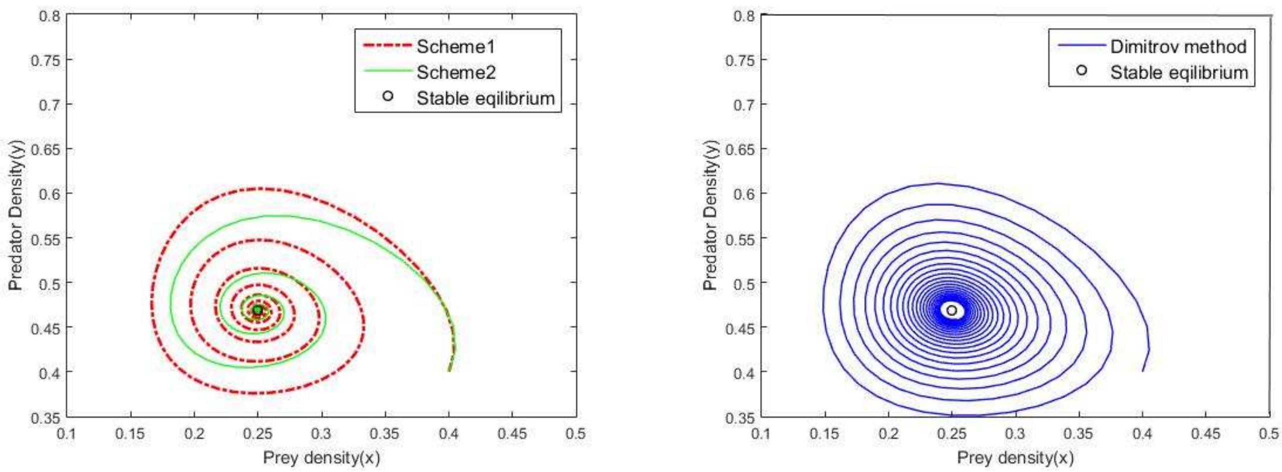
7. Numerical Results

Here, we give numerical results to check the accuracy of theoretical results and the efficiency of the new PESN schemes. Comparisons the numerical results of the new schemes and the scheme presented in [1] show the better performance of the new schemes with a large step size. Here, $g(x) = \frac{1}{r+x}$ and all the parameters used in drawing figures are selected from [1].

As is evident in Fig. 2(a), the proposed PESN schemes maintain the stability of the equilibrium E_1, while the standard explicit Euler method is not able to produce nonnegative solution and it diverges. The behavior of RK2 method is seen to be similar, (see Fig. 2 (b)). In Table 1, we see that the Euler and RK2 methods diverge with increasing the step-size, but the new schemes are convergent for all h. Also, the numerical solution for the new methods for $a = 2$, $b = 1$, $r = 1$, $c = 0.2$ and $q = 1.2$, leads to $E^* = (\frac{1}{4}, \frac{15}{32})$ are shown with different initial values and step sizes Fig. 3. Fig. 3(a), 3(b) and 4(a) show that our proposed schemes produce smooth and nonoscillatory solution and preserves the stability of the equilibrium E^*, but the results obtained by the RK2 method quickly deteriorates (see Fig. 4(b)). The $X - Y$ phase portraits obtained using the new PESN methods and the NSFD method presented in [1] with $h = 4$ for system (4) are shown in Fig. 5. We can see that the proposed schemes preserve the stability of the equilibrium and perform very well. In Table 2, computational time used to obtain Fig. 2-5 presented. In Fig. 6-7 we present the numerical results obtained from the new PESN methods, [1] and the RK4 with the step sizes $h = 0.4$ and $h = 4$ for the system (4). It was found that solutions of the PESN methods agree very well with the RK4 solutions. In this case, we see that the new PESN methods have an advantage over scheme [1] because of achieving good accuracy with different large time steps. It can also be seen that scheme 2 is better than scheme 1. Numerical results are coded in MATLAB software with the feature of computer RAM 4GB and CPU 2.40GHz.

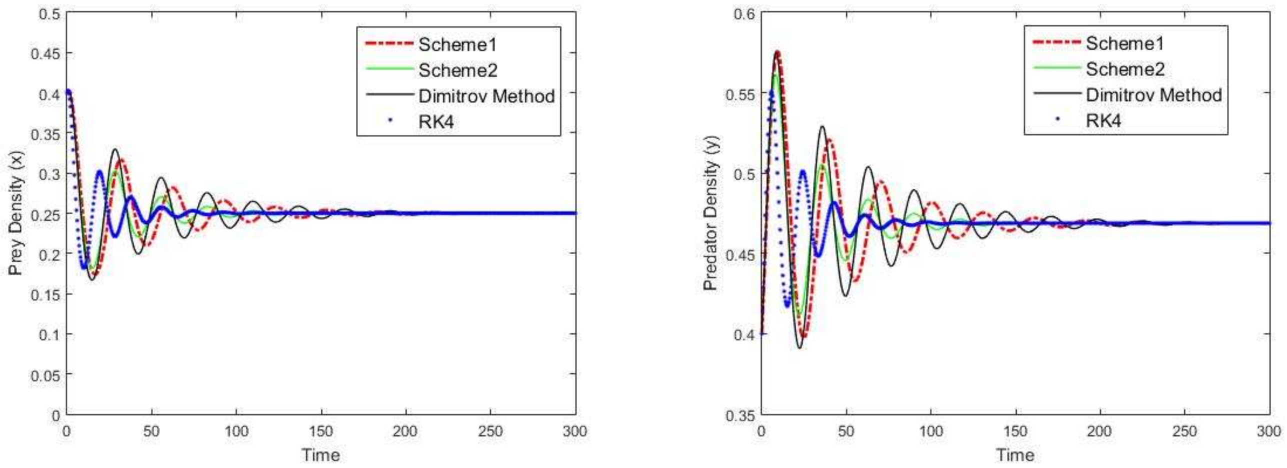
8. Conclusions and Discussion

In this paper, we have developed two new nonstandard finite difference method based on Mickens' rules to solve predator-prey model. We have shown that the proposed methods preserve positivity property and stability. By numerical comparisons we have shown that the new schemes converge with any step size and perform better than the NSFD method presented in [1], the RK2 method and the Euler method.



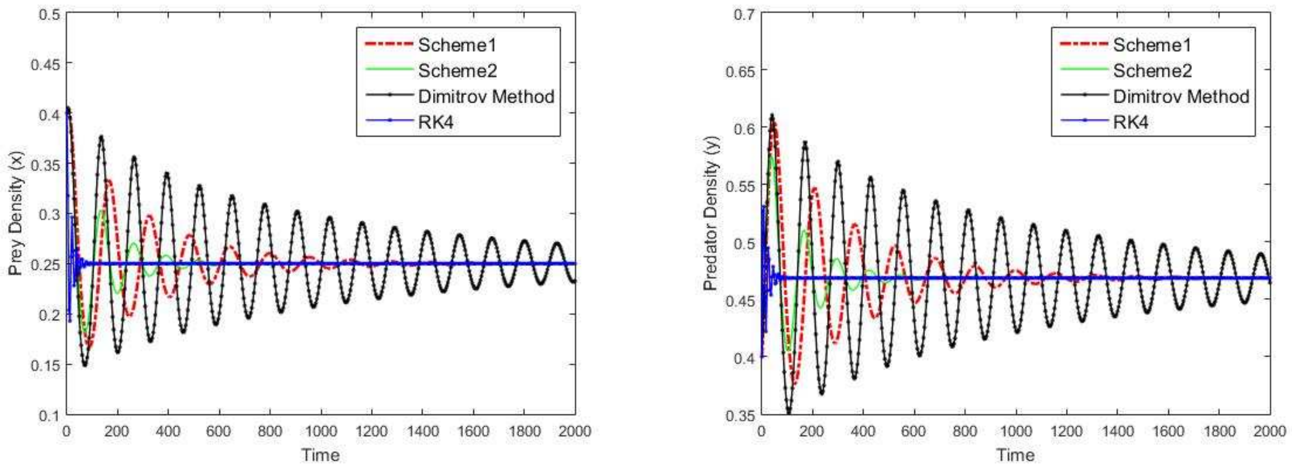
(a) $h = 4, x_0 = 0.4, y_0 = 0.4$ (b) $h = 4, x_0 = 0.4, y_0 = 0.4$

Fig. 5. Numerical results with $a = 2$, $b = 1$, $r = 1, c = 0.2$ and $q = 1.2$.



(a) Prey density (b) Predator density

Fig. 6. Numerical results for proposed schemes, Dimitrov method [6] and RK4 with $\boldsymbol{a = 2, b = 1, r = 1, c = 0.2\ q = 1.2, x_0 = 0.4, y_0 = 0.4}$ and $\boldsymbol{h = 0.4}$.



(a) Prey density (b) Predator density

Fig. 7. Numerical results for the proposed schemes, Dimitrov method [6] and RK4 with $\boldsymbol{a = 2, b = 1, r = 1, c = 0.2\ q = 1.2, x_0 = 0.4, y_0 = 0.4}$ and $\boldsymbol{h = 4}$.

Author Contributions

All authors planned the scheme, developed the mathematical modeling and examined the theory validation. The manuscript was written through the contribution by all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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References

[1] Dimitrov, D.T., Kojouharov, H.V., Analysis and numerical simulation of phytoplankton-nutrient systems with nutrient loss, *Mathematics and Computers in Simulation*, 70(1), 2005, 33-43.
[2] Dimitrov, D.T., Kojouharov, H.V., Positive and elementary stable nonstandard numerical methods with applications to predator-prey models, *Journal of Computational and Applied Mathematics*, 189(1-2), 2006, 98-108.
[3] Dimitrov, D.T., Kojouharov, H.V., Nonstandard numerical methods for a class of predator-prey models with predator interference, *Electronic Journal of Differential Equations (EJDE)*, 2007, 2007, 67-75.

[4] Dumont, Y., Chiroleu, F., Domerg, C., On a temporal model for the Chikungunya disease:modeling theory and numerics, *Mathematical Biosciences,* 213(1), 2008, 80-91.
[5] Mehdizadeh Khalsaraei, M., Positivity of an explicit Runge-Kutta method, *Ain Shams Engineering Journal,* 6(4), 2015, 1217-1223.
[6] Mehdizadeh Khalsaraei, M., Khodadoosti, F., Qualitatively stability of nonstandard 2-stage explicit Runge-Kutta methods of order two, *Computational Mathematics and Mathematical Physics,* 56(2), 2016, 235-242.
[7] Mickens, R.E., A nonstandard finite-difference scheme for the Lotka-Volterra system, *Applied Numerical Mathematics,* 45(2-3), 2003, 309-314.
[8] Mickens, R.E., Jordan, P.M., A positivity-preserving nonstandard finite difference scheme for the damped wave equation, *Numerical Methods for Partial Differential Equations,* 20(5), 2004, 639-649.
[9] Anguelov, R., Dumontb, Y., Lubumaa, J.M.S., Shillor, M., Dynamically consistent nonstandard finite difference schemes for epidemiological models, *Journal of Computational and Applied Mathematics,* 255, 2014, 161-182.
[10] Ashyralyev, A., Agirseven, D., Agarwal, R.P., Stability estimates for delay parabolic differential and difference equations, *Applied and Computational Mathematics,* 19(2), 2020, 175-204.
[11] Gonzalez-Parra, G., Arenas, A.J., Cogollo, M.R., Numerical-analytical solutions of predator-prey models, *WSEAS Transactions on Biology and Biomedicine,* 10(2), 2013, 79-87.
[12] Gurski, K.F., A simple construction of nonstandard finite-difference schemes for small nonlinear systems applied to SIR models, *Computers & Mathematics with Applications*, 66(11), 2013, 2165-2177.
[13] Mehdizadeh Khalsaraei, M., An improvement on the positivity results for 2-stage explicit Runge-Kutta methods*, Journal of Computational and Applied Mathematics,* 235(1), 2010, 137-143.
[14] Mehdizadeh Khalsaraei, M., Khodadoosti, F., A new total variation diminishing implicit nonstandard finite difference scheme for conservation laws, *Computational Methods for Differential Equations*, 2(2), 2014, 85-92.
[15] Mehdizadeh Khalsaraei, M., Shokri Jahandizi, R., Positivity-preserving nonstandard finite difference schemes for simulation of advection-diffusion reaction equations, *Computational Methods for Differential Equations,* 2(4), 2014, 256-267.
[16] Mehdizadeh Khalsaraei, M., Shokri, A., Ramos, H., Heydari, S., A positive and elementary stable nonstandard explicit scheme for a mathematical model of the influenza disease, *Mathematics and Computers in Simulation*, 182, 2021, 397-410.
[17] Bacaer, N., Ouifki, R., Pretorius, C., Wood, R., Williams, B., Modeling the joint epidemics of TB and HIV in a south African township, *Journal of Mathematical Biology,* 57(4), 2008, 557-593.
[18] Brauer, F., Castillo-Chavez, C., *Mathematical Models in Population Biology and Epidemiology,* Springer, New York, 2001.
[19] Golbabai, A., Nikan O., Molavi-Arabshahi, M., Numerical approximation of time fractional advection-dispersion model arising from solute transport in rivers, *TWMS Journal of Pure and Applied Mathematics*, 10(1), 2019, 117-131.
[20] Moshtaghi, N., Saadatmandi, A., Numerical solution for diffusion equations with distributed-order in time based on sinc-Legendre collocation method, *Applied and Computational Mathematics*, 19(3), 2020, 317-355.
[21] Murry, J.D., *Mathematical Biology*, Springer-Verlag, 1989.
[22] Qalandarov, A.A., Khaldjigitov, A.A., Mathematical and numerical modeling of the coupled dynamic thermoelastic problems for isotropic bodies, *TWMS Journal of Pure and Applied Mathematics*, 11(1), 2020, 119-126.
[23] Anguelov, R., Lubuma, J.M.S., Contributions to the mathematics of the nonstandard finite difference method and applications, *Numerical Methods for Partial Differential Equations*, 17(5), 2001, 518-543.
[24] Dimitrov, D.T., Kojouharov, H.V., Stability-Preserving finite-difference methods for general multi-dimensional autonomous dynamical system, *International Journal of Numerical Analysis and Modeling*, 4(2), 2007, 282-292.
[25] Mickens, R.E., *Nonstandard Finite Difference Models of Differential Equations,* World Scientific, Singapore, 1994.
[26] Portalier, S.M., Fussmann, G.F., Loreau, M., Cherif, M., The mechanics of predator–prey interactions: First principles of physics predict predator-prey size ratios, *Functional Ecology*, 33(2), 2019, 323-334.

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