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Research Paper

On Integrability up to the Boundary of the Weak Solutions to a Class of non-Newtonian Compressible Fluids with Vacuum

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Abstract. In this paper, we study the integrability up to the boundary of the weak solutions of non-Newtonian compressible fluid with a nonlinear constitutive equation in $\mathbb{R}^3$ bounded domain. Galerkin approximation will be used for existence of weak solutions and by applying the bounded linear operator B, introduced by Bogovskii, we prove the square integrability of the density up to the boundary.

Keywords: Compressible non-Newtonian fluid, Weak solutions, Vacuum, Integrability.

1. Introduction

In many physical situations the motion of fluids can be accurately modeled by using the linear constitutive equations known as Newtonian fluids. On the other hand, in some situations, such as the study of dynamics in Earth's mantle, chemical process, industrial processes, biomechanics and many more [1-3], where the linear models fails and scientists use the nonlinear version of the constitutive laws are known as non-Newtonian fluids. Leray's pioneering work goes back to the global existence of weak solutions of fluid dynamic models. Where he presented the idea of weak solution to Navier-Stokes equations, characterizing the motion of incompressible fluids, this work has become the foundation of the underlying mathematical theory up to present day. The existence theory of viscous compressible fluids has been developed by Lions [4], later improved by Feireisl and his collaborators [5] and has been since then a very active field of study. We describe non-Newtonian compressible fluids by the following system of equations:

$$\begin{cases}\partial_t\rho+\mathrm{div}(\rho\mathbf{u})=0,\\ \partial_t(\rho\mathbf{u})+\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla p(\rho)=\mathrm{div}(2\mu_0\,|\,\mathbb{D}^d(\mathbf{u})\,|^{r-2}\,\mathbb{D}^d(\mathbf{u})+\eta(\mathrm{div}\mathbf{u})\mathrm{div}\mathbf{u}\mathbb{I}),\\ |\,\mathrm{div}\mathbf{u}\,|<\dfrac{1}{b},\\ \eta(\mathrm{div}\mathbf{u})=\dfrac{b}{(1-b^a|\mathrm{div}\mathbf{u}|^a)^{\frac{1}{a}}}\end{cases} \tag{1}$$

where the function $p(\rho)=\rho^{\gamma}$ $(\gamma>3/2)$, the coefficient of viscosity $\mu_0>0$, $t\in(0,\infty)$ being the time, $r\in[11/5,+\infty)$, $\mathbf{u}$ and ρ represent the fluid velocity and density respectively. In addition, the operators div and ∇ act with respect to $x\in\mathbb{R}^n$. Moreover, the symmetric part of $\nabla\mathbf{u}$ is prescribed by $\mathbb{D}\mathbf{u}=(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)/2$. Where its deviatoric part is given by $\mathbb{D}^d(\mathbf{u})=\mathbb{D}\mathbf{u}-1/3\times(\mathrm{div}\mathbf{u})\mathbb{I}$. More generally for any tensor we have $\mathbb{A}^d=\mathbb{A}-1/3\times(tr\mathbb{A})\mathbb{I}$. For continuum mechanical related background, one can refer to [6, 7].

The global existence of one-dimensional compressible fluids has been studied by Kazhikhov [8] with discontinuous initial data. Similarly, one-dimensional compressible fluids, when the data is uniformly away from the vacuum proved in [9, 10] and the citation therein. For the case of multi-dimensions, the uniqueness and local existence of classical solutions without vacuum were studied in [11, 12]. Matsumura [13-15] studied the existence of smooth solutions globally for data close to equilibrium without vacuum. An important result were obtained by Danchin in [16, 17], by proving in critical spaces the existence weak solutions of compressible fluids. The existence of compressible Navier-Stokes equations were developed by Lions [4] and Feireisl [5], where they obtained global existence to weak solutions of suitably large exponent γ, with finite initial energy. However, the uniqueness, dynamical behavior and regularity of weak solutions were studied in [18-20] for compressible fluids with constant coefficients and arbitrary initial data.

For non-Newtonian incompressible fluids, Ladyzhenskaya [21] and Lions [22] investigated the result of uniqueness and existence of weak solutions for $r\geq\frac{3n+2}{n+2}$ by using monotone operator and compactness arguments. Similarly, Mamontov [23] obtained the global existence problem of compressible non-Newtonian fluid of multi-dimensional equations. Where they used

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the technique as in [9], for estimating the global weak solutions of two-dimensional compressible viscous fluid. Feireisl in [24] at first time studied compressible fluids in multi dimensions $\mathbb{R}^n$ $(n \geq 2)$, where the equation of temperature is satisfied as an inequality. Moreover, the existence of weak solutions for non-Newtonian fluids in cylindrical domain was proved by Wolf [25], where the shear stress depends on viscosity. Later on, Bresch and Desjardins in [26] used some different hypothesis from [24] and analyzed in $\mathbb{R}^3$ the global existence of weak solutions with large data, such that equations of temperature are satisfied as equalities in distributional sense. The global and local existence of solutions to one dimensional problem was achieved by Yuan [27, 28] and his cooperators. Recently, the unsteady compressible non-Newtonian fluids properties were studied by Feireisl [29] for existence and large-time result with positive density. Later, in [30] Fang and Guo investigated the blow up criteria of the non-Newtonian fluids. Moreover, they proved analytical solutions of compressible non-Newtonian fluids in [31] and local strong solutions with an assumption of density-dependent viscosity in [32]. By considering the external force and vacuum Yuan and Yang [33] proved the unique local strong solutions to non-Newtonian compressible fluids under compatibility conditions. On the same way the analytical solutions of non-Newtonian compressible fluids by using different methods are studied in [34-37].

Here, our main aim is to consider the compressible non-Newtonian fluids in $\mathbb{R}^3$ and study the square integrability of density up to the boundary by using the linear operator B, as introduced by Bogovskii [38].

In particular, the initial data to the problem (1) is as follows

$$\rho(0,\cdot) = \rho_0 \text{ and } (\rho\mathbf{u})(0,\cdot) = \mathbf{m}_0 \text{ in } \Omega \tag{2}$$

with $\mathbf{m}_0$, $\rho_0 \geq 0$ being initial momenta and density. The boundary conditions of $\mathbf{u}$ are described by

$$\mathbf{u}\,|_{(0,T)\times\partial\Omega} = 0. \tag{3}$$

In [39], the existence of global weak solutions for the problem (1)-(3) is proved. Motivated by the work of Feireisl [40], we investigate the integrability up to boundary of the problem (1)-(3) with large data.

Notations. Let Ω be a domain in $\mathbb{R}^3$, for $p \geq 1$, $L^p = L^p(\Omega)$ being the L^p space whose norm is denoted by $\|\bullet\|_{L^p}$. The space $W^{k,p} = W^{k,p}(\Omega)$ represents the usual Sobolev space, $\|\bullet\|_{W^{k,p}}$ being the norm and $H^k = W^{k,2}(\Omega)$, with $k,p \geq 1$. In addition, C being a general constant that can vary according to various estimates. If it is necessary to specifically point out the dependency, then $C(a,b,\ldots)$ will be used. For η as the standard mollifier in $\mathbb{R}^3$ and $f \in L^1_{loc}(\mathbb{R}^3)$, we set

$$\eta_\varepsilon(\cdot) := \frac{1}{\varepsilon^3}\eta\left(\frac{\cdot}{\varepsilon}\right), \quad [f]_\varepsilon := \eta_\varepsilon * f.$$

also,

$$\nabla = (\partial_1, \partial_2, \partial_3), \quad \partial_i = \partial_{x_i}, i = 1,2,3.$$

The remaining paper is structured as: Sect.2, is concerned to the definition of weak solution and the statement of main result. In Sect.3, some preliminary lemmas are given for later use in the proof of main result. Sect.4, is dedicated to the proof of Theorem 2.2.

2. Main Results

The compressible fluid is defined by the constitutive equation

$$\mathbb{T} = -\rho^\gamma \mathbb{I} + 2\mu \mathbb{D}^d(\mathbf{u}) + \eta \operatorname{div}\mathbf{u}\mathbb{I}. \tag{4}$$

On the same way, as in Feireisl [29], the stress tensor $\mathbb{T}$ in (4), can be generalized as

$$\mathbb{T} = -\rho^\gamma \mathbb{I} + \mathbb{S}(\mathbf{u}) + \eta(\operatorname{div}\mathbf{u})\operatorname{div}\mathbf{u}\mathbb{I}. \tag{5}$$

Here, the symbol $\mathbb{S}$ is used for deviatoric part of stress tensor, described by

$$\mathbb{S}(\mathbf{u}) := 2\mu_0 \,|\, \mathbb{D}^d(\mathbf{u})\,|^{r-2}\, \mathbb{D}^d(\mathbf{u}), \quad \mu_0 > 0 \text{ and } r \in [\tfrac{11}{5}, +\infty). \tag{6}$$

In addition, the coefficient of bulk viscosity η is a continuous function of $\operatorname{div}\mathbf{u}$ and $\eta(\operatorname{div}\mathbf{u}){:}(-\frac{1}{b},\frac{1}{b}) \to [0,\infty)$, such that $\Lambda{:}\,\mathbb{R} \to [0,\infty]$ satisfy

$$\begin{cases} \Lambda(0) = 0, \\ \Lambda'(z) = z\eta(z), \\ \Lambda(z) \to \infty & \text{if } z \to \pm\frac{1}{b}, \\ \Lambda(z) = \infty & \text{if } |z| \geq \frac{1}{b}. \end{cases} \tag{7}$$

Remark 2.1. The convex potential Λ for $\eta(z) = b(1 - b^a z^a)^{\frac{-1}{a}}$ with $z > 0$, can be sketched in Figure 1.

Furthermore, the constitutive relation (5) guarantees the boundedness of $\operatorname{div}\mathbf{u}$, which means that the fluid velocity $|\operatorname{div}\mathbf{u}| < 1/b$, such that the density in the occupied domain of the fluid remains strictly positive.

We define the weak solution $\rho, \mathbf{u}$ of the problem (1)-(3), before stating the main theorem of existence along with the necessary regularity needed for later use.

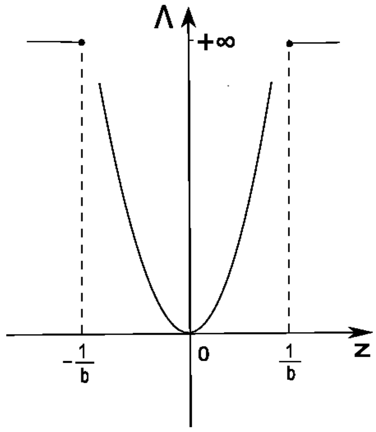

Fig. 1. The convex potential Λ

Definition 2.1. The functions $(\rho,\mathbf{u})$ are known as weak solutions for $T>0$, of the problem (1)-(3), on $(0,T)$, satisfying the conditions

$$\begin{aligned}&\rho\in C([0,T];L^1(\Omega))\cap L^\infty(0,T;L^\gamma(\Omega)),\sqrt{\rho}\mathbf{u}\in L^\infty(0,T;L^2(\Omega)),\\&\mathbf{u}\in L^r(0,T;W_0^{1,r}(\Omega)),\ \eta(\mathrm{div}\mathbf{u})\,|\,\mathrm{div}\mathbf{u}\,|^2\in L^1((0,T)\times\Omega);\end{aligned}\tag{8}$$

- Continuity equation $(1)_1$ holds

$$\int_0^\tau\int_{\mathbb{R}^3}(\rho\partial_t\phi+\rho\mathbf{u}\cdot\nabla\phi)dxdt=\left[\int_{\mathbb{R}^3}\rho\phi dx\right]_0^\tau\tag{9}$$

for $\varphi\in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$ and $\tau\in[0,T]$, with $\varphi(x,0)=\varphi(x,T)=0$ and $x\in\Omega$;

- Weak formulation of Eq. $(1)_2$ holds

$$\begin{aligned}&\left[\frac{1}{2}\int_\Omega\rho\,|\,\mathbf{u}\,|^2\,dx\right]_0^\tau-\left[\int_\Omega\rho\mathbf{u}\cdot\phi dx\right]_0^\tau+\int_0^\tau\int_\Omega[\rho\mathbf{u}\cdot\partial_t\phi+\rho\mathbf{u}\otimes\mathbf{u}\ :\nabla\phi]dxdt\\&+\int_0^\tau\int_\Omega\mathbb{S}(\mathbf{u}):\ \mathbb{D}^d(\mathbf{u}-\phi)dxdt+\int_0^\tau\int_\Omega\rho^\gamma\mathrm{div}(\phi-\mathbf{u})dxdt\\&\le\int_0^\tau\int_\Omega\Lambda(\mathrm{div}\phi)-\Lambda(\mathrm{div}\mathbf{u})dxdt\end{aligned}\tag{10}$$

for $\varphi\in C_c^\infty(((0,T)\times\Omega;\mathbb{R}^3)$ and $\tau\in[0,T]$;

- Equation of continuity $(1)_1$ holds in $D'(0,\infty;\Omega)$, in renormalized sense of solutions, in particular

$$h(\rho)_t+\mathrm{div}(h(\rho)\mathbf{u})+\left(h'(\rho)\rho-h(\rho)\right)\mathrm{div}\,\mathbf{u}=0\tag{11}$$

for each $h\in C^1(\mathbb{R})$ with $h'(z)=0$, $\forall z\in\mathbb{R}$, such that, $|\,z\,|\ge M$, where M will differ for various functions h;

- Formally, the energy inequality is described by

$$\frac{d}{dt}E(t)+\int_\Omega\left(2\mu_0\,|\,\mathbb{D}^d(\mathbf{u})\,|^{r-2}|\,\mathbb{D}^d(\mathbf{u})\,|^2+\eta(\mathrm{div}\mathbf{u})\,|\,\mathrm{div}\mathbf{u}\,|^2\right)dx\le 0\tag{12}$$

with

$$E(t)=\int_\Omega\left(\frac{1}{2}\rho(t)\,|\,\mathbf{u}(t)\,|^2+\frac{1}{\gamma-1}\rho^\gamma(t)\right)dx.$$

The main result related to the problem (1)-(3) is described as follows.

Theorem 2.1. Let $\Omega\subset\mathbb{R}^3$, $p(\rho)=\rho^\gamma$ with $\gamma>\frac{3}{2}$, and suppose that the initial data satisfy

$$\begin{cases}\rho_0\in L^\gamma(\Omega),\rho_0\ge 0\ \text{ on }\ \Omega,\\ \frac{|\mathbf{m}_0|^2}{\rho_0}\in L^1(\Omega).\end{cases}$$

Then for a fixed $T>0$, the weak solution $(\rho,\mathbf{u})$ on $((0,T)\times\Omega)$, is in the class

$$\begin{cases} \rho \in L^{\infty}(0,T;L^{\gamma}(\Omega)) \cap C([0,T];L^{1}(\Omega)), \\ \mathbf{u} \in L^{r}(0,T;W_0^{1,r}(\Omega)),\ \rho\mathbf{u} \in C([0,T];L_{weak}^{\frac{\gamma r}{\gamma+r}}(\Omega)), \\ \eta(|\operatorname{div}\mathbf{u}|)|\operatorname{div}\mathbf{u}|^2 \in L^1((0,T)\times\Omega) \end{cases}$$

and

$$\rho \in L^2((0,T)\times\Omega). \tag{13}$$

Remark 2.2. The solution $(\rho,\mathbf{u})$ satisfies continuity equation in the sense of renormalized solutions, such that

$$\int_0^T\int_\Omega [h(\rho)\partial_t\varphi + h(\rho)\mathbf{u}\cdot\nabla\varphi + (h(\rho)-h'(\rho)\rho\operatorname{div}\mathbf{u}\varphi]dxdt = 0$$

holds for any $h \in C^1[0,\infty)$, $|h'(z)z| \leq cz^{\frac{\gamma}{2}}$ for $z > z_0$, and $\varphi \in C^{\infty}([0,T]\times\overline{\Omega})$ with $\varphi(x,0) = \varphi(x,T) = 0$ for $x \in \Omega$.

Remark 2.3. The pair $(\rho,\mathbf{u})$ of functions prescribed in Theorem 2.1 exists. In fact, in [39], we proved the global existence of weak solution to the problem (1)-(3).

Remark 2.4. It should be mentioned here that one can use the Lebesgue dominated convergence theorem to ensure that Eq. (11) holds for some $h \in C^1(0,\infty)\cap C[0,\infty)$, such that

$$|h'(z)z| \leq c\left(z^{\theta} + z^{\frac{\gamma}{2}}\right), \text{ for all } z > 0 \text{ and } \theta \in \left(0,\frac{\gamma}{2}\right) \tag{14}$$

with weak solution $(\rho,\mathbf{u})$ in the context of Definition 2.1.

Just like, as in the work of Feireisl [40, 41], we establish an approximate problem of the regularized equations as follows:

$$\begin{cases} \partial_t\rho_n + \operatorname{div}(\rho_n\mathbf{u}_n) = 0, \\ \partial_t(\rho_n\mathbf{u}_n) + \operatorname{div}(\rho_n\mathbf{u}_n\otimes\mathbf{u}_n) + \nabla(\rho_n^{\gamma} + \delta\rho_n^{\beta}) = \operatorname{div}(\mathbb{E}(\mathbf{u}_n) + \eta(\operatorname{div}\mathbf{u}_n)\operatorname{div}\mathbf{u}_n\mathbb{I}), \\ \mathbb{E}(\mathbf{u}_n) = 2\mu_0(\delta + |\mathbb{D}^d(\mathbf{u}_n)|^2)^{\frac{r-2}{2}}\mathbb{D}^d(\mathbf{u}_n), \end{cases} \tag{15}$$

where δ is positive and $\beta > \max\{9,\gamma\}$ is a fixed constant. Without loss of generality, let $2\mu_0 = 1$. First, we constructed a modified system (15), for system (1) and will solve this modified system. Finally, we recover the original system from the modified system by taking the limit of the sequences of solutions and show that the obtained weak solutions satisfy our problem (for more detail one can refer to [39]). In addition, the following conditions hold for Eq. (15)

$$\begin{cases} \nabla\rho\cdot\mathbf{n}|_{\partial\Omega} = 0, & \rho|_{t=0} = \rho_{0,n}, \\ \mathbf{u}|_{\partial\Omega} = 0, & \rho\mathbf{u}|_{t=0} = \mathbf{m}_{0,n}, \end{cases} \tag{16}$$

here $\mathbf{n}$ is normal unit vector pointing outward along with $\partial\Omega$. We choose the initial data to satisfy

$$\begin{cases} \rho_{0,n} \in C^3(\overline{\Omega}), & 0 < n \leq \rho_{0,n} \leq n^{-\frac{1}{2\beta}}, \\ \rho_{0,n} \to \rho_0 \text{ in } L^{\gamma}(\Omega), & \text{measure } \{x\in\Omega : \rho_{0,n} < \rho_0\} \to 0 \text{ as } n \to 0, \\ \delta\int_\Omega \rho_{0,n}^{\beta}dx \to 0 \text{ as } \delta \to 0; \\ \mathbf{m}_{0,n} = \begin{cases} \mathbf{m}_0, & \text{if } \rho_{0,n} \geq \rho_0 \\ 0, & \text{if } \rho_{0,n} < \rho_0. \end{cases} \end{cases} \tag{17}$$

Inspired by Feireisl [29], for the function Λ in Eq. (7), we define a regularization Λ_ν as follows

$$\Lambda_\nu(z) = \begin{cases} \Lambda(z) & |z| \leq \frac{1}{b} - \nu, \\ \Lambda(\frac{1}{b}-\nu) + \Lambda'(\frac{1}{b}-\nu)(z-(\frac{1}{b}-\nu)) & z \geq \frac{1}{b} - \nu, \end{cases} \tag{18}$$

$$\Lambda_\nu(z) = \Lambda_\nu(-z) \qquad z \leq -\frac{1}{b} + \nu. \tag{19}$$

Next, fix $\nu > 0$ and let a finite-dimensional space $X_m = \operatorname{span}\{w_m\}_{m=1}^{\infty} \subset C_0^{\infty}(\Omega)^3$ as an orthonormal basis in $\langle\cdot;\cdot\rangle_{L^2(\Omega)^3}$ inner product space. By Schauder's fixed point theorem approximate solutions can be obtained. So, we define $\Theta_n : C([0,T]: X_n) \to C([0,T] : X_n)$, denoting the input of Θ_n by $\bar{\mathbf{u}}_n$ and the output of $\Theta_n\bar{\mathbf{u}}$ by $\mathbf{u}_n$.

By using the input $\bar{\mathbf{u}}_n \in C([0,T]: X_n)$, we have

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\bar{\mathbf{u}}_n) = 0, \\ \rho|_{t=0} = \rho_{0,n},\ \bar{\mathbf{u}}_n|_{t=0} = \mathbf{u}_0,\ \nabla\rho\cdot\mathbf{n}|_{\partial\Omega} = 0. \end{cases} \tag{20}$$

Furthermore, with the help of classical result of Lunardi [42], (Theorem 5.1.21), one can obtain $\rho_n(t,x)$. to the system (20), as a classical solution. Next, comparison principle guarantees that

$$\delta\exp\left(-\int_0^t\|\mathrm{div}\,\mathbf{u}_n(s)\|_{L^\infty(\Omega)}ds\right)\le\rho_n(t,x)\le^{-\frac{1}{2\beta}}\exp\left(\int_0^t\|\mathrm{div}\,\mathbf{u}_n(s)\|_{L^\infty(\Omega)}ds\right)$$

for $x\in\Omega$ and $t\in[0,T]$.

Furthermore, the output $\mathbf{u}_n\in C([0,T];X_n)$ will satisfy $\partial_t\mathbf{u}_n\in L^1(0,T;X_n)$ and the integral equation holds

$$\left[\int_\Omega\rho\mathbf{u}_n\cdot\varphi\mathrm{d}x\right]_0^\tau+\int_0^\tau\int_\Omega[-\rho\mathbf{u}_n\otimes\mathbf{u}_n:\nabla\varphi+\mathbb{E}(\mathbf{u}_n):\mathbb{D}^d\varphi+\Lambda_\nu'(\mathrm{div}\mathbf{u}_n)\mathrm{div}\varphi-(\rho^\gamma+\delta\rho^\beta)\mathrm{div}\varphi]\mathrm{d}x\mathrm{d}t=0 \tag{21}$$

for $\varphi\in X_n$ and $\rho=\rho_n=\rho[\mathbf{u}_n]$.

The energy inequality may be prescribed by

$$\left[\int_\Omega\frac{1}{2}\rho_n|\mathbf{u}_n|^2+\frac{\rho_n^\gamma}{\gamma-1}+\frac{\delta}{\beta-1}\rho_n^\beta dx\right]_0^\tau+\int_0^t\int_\Omega\left[|\mathbb{D}^d(\mathbf{u}_n)|^r+\Lambda_\nu'(\mathrm{div}\mathbf{u}_n)\mathrm{div}\mathbf{u}_n\right]dxdt\le 0. \tag{22}$$

Moreover, (22) ensures the estimates

$$\|\rho_n\|_{L^\infty(0,T;L^\gamma(\Omega))}\le C(\varepsilon,\delta,T,\mathbf{m}_0,\rho_0), \tag{23}$$

$$\|\rho_n\|_{L^\infty(0,T;L^\gamma(\Omega))}\le C(\varepsilon,\delta,T,\mathbf{m}_0,\rho_0), \tag{24}$$

$$\left\|\sqrt{\rho_n}\mathbf{u}_n\right\|_{L^\infty(0,T;L^2(\Omega))}\le C(\varepsilon,\delta,T,\mathbf{m}_0,\rho_0), \tag{25}$$

$$\left\|\eta(\mathrm{div}\mathbf{u}_n)|\mathrm{div}\mathbf{u}_n|^2\right\|_{L^1((0,T)\times\Omega)}\le C(\varepsilon,\delta,T,\mathbf{m}_0,\rho_0), \tag{26}$$

$$|\mathrm{div}\mathbf{u}_n|<\frac{1}{b} \tag{27}$$

also, with the help of Poincare` inequality [43], we have

$$\int_0^T\|\mathbf{u}_n(t)\|_{W_0^{1,r}(\Omega)}^r dt\le C(\varepsilon,\delta,T,\mathbf{m}_0,\rho_0). \tag{28}$$

Here, C is independent of n and only dependent on initial data. Next, the main result is prescribed as follows.

Theorem 2.2. Let the functions $(\rho_n,\mathbf{u}_n)$ are known to be the approximate solution of the modified system (15), under the assumption of estimates (23)-(28). Then,

$$\int_0^T\int_\Omega\rho_n^{\gamma+\frac{1}{5}}+\delta_n\rho_n^{\beta+\frac{1}{5}}dxdt\le K \tag{29}$$

here, the constant K is independent of n.

Similarly, as shown in [29, 39], the limit of the approximate solution $(\rho_n,\mathbf{u}_n)$ is weak solution to the problem (1)-(3). Therefore, taking limit in (29) will lead to the conclusion of Theorem 2.1. Moreover, it remains to prove Theorem 2.2.

3. A Bounded Linear Operator B

The key point of (29) may be obtained on the basis of operators introduced by Bogovskii [38], given as follows. These operators are usually called the solutions of the problem

$$\begin{cases}\mathrm{div}\,\mathbf{v}=f\\ \mathbf{v}|_{\partial\Omega}=0,\quad f\in L^p(\Omega).\end{cases} \tag{30}$$

Lemma 3.1. [38, 44] For the problem (18), the linear operator B satisfy the following properties:

$$\mathbf{B}=[\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3]:\left\{f\in L^p(\Omega)\,\Big|\,\int_\Omega fdx=0\right\}\to W_0^{1,p}(\Omega)$$

here, the operator B,

$$\|\mathbf{B}\{f\}\|_{W_0^{1,p}(\Omega)}\le C(p,\Omega)\|f\|_{L^p(\Omega)}$$

for any $p\in(1,\infty)$.

- $\mathbf{v}=\mathbf{B}\{f\}$ will solve the problem

$$\operatorname{div}\mathbf{v} = f \quad \text{a.e. in} \quad \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0.$$

- Moreover, if $f = \operatorname{div}\mathbf{g}$, then

$$\mathbf{g} \in L^r(\Omega), \quad \mathbf{g}\cdot n|_{\partial\Omega} = 0$$

with n being a unit normal vector pointing outward along the $\partial\Omega$, such that

$$\|\mathbf{B}\{f\}\|_{L^r(\Omega)} \le C(p,r,\Omega)\|\mathbf{g}\|_{L^r(\Omega)}$$

for an arbitrary $r \in (1,\infty)$.

Proof. Complete proof can be found in Galdi [44] or Bochers and Sohr [45].

Lemma 3.2. [46] Let a function f defined on $\mathbb{R}^3$ and identically vanishes outside the domain $\Omega \subset \mathbb{R}^3$.

$$\begin{aligned} &\text{(i)} \quad \text{If } f \in L^1_{\text{loc}}(\mathbb{R}^3), \text{ then } [f]_n \in C^\infty(\mathbb{R}^3). \\ &\text{(ii)} \quad \text{If } f \in L^p(\Omega) \text{ with } 1 \le p < \infty, \text{ then } [f]_n \in L^p(\Omega). \text{ Also } \|[f]_n\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)}, \ \lim_{n\to 0^+}\|[f]_n - f\|_{L^p(\Omega)} = 0. \end{aligned} \tag{31}$$

Next, the commutator estimates be stated as follows that play an important role in further analysis.

Lemma 3.3. [5, 47] Let $\Omega \subset \mathbb{R}^3$ and $\rho \in L^p(\Omega), \mathbf{u} \in [W^{1,q}(\Omega)]^3$ with $p,q \in (1,\infty)$ and $\frac{1}{p}+\frac{1}{q} \le 1$. Then $K \subset \Omega$, (K is compact set)

$$\begin{cases} \|[\operatorname{div}(\rho\mathbf{u})]_n - \operatorname{div}([\rho]_n\mathbf{u})\|_{L^r(K)} \le C(K)\|\rho\|_{L^p(\Omega)}\|\mathbf{u}\|_{W^{1,q}(\Omega)} \\ \|[\operatorname{div}(\rho\mathbf{u})]_n - \operatorname{div}([\rho]_n\mathbf{u})\|_{L^r(K)} \to 0, \text{ as } n \to 0 \end{cases}$$

provided $n > 0$ is small enough and $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$. Moreover, if $\Omega = \mathbb{R}^n$, then K can be substituted by $\mathbb{R}^n$.

Lemma 3.4. Let equation (1) holds in $D'(0,\infty;\Omega)$ and $(1)_1$ holds in $D'(0,\infty;\mathbb{R}^3)$ given that $\rho,\mathbf{u}$ be zero on $\mathbb{R}^3\setminus\Omega$, then

$$\partial_t\rho_n + \operatorname{div}(\rho_n\mathbf{u}_n) = 0 \text{ in } D'((0,T)\times\mathbb{R}^3). \tag{32}$$

Next, denoting $\tilde{\rho}_n = \vartheta_\varepsilon * \rho_n$ with $\vartheta_\varepsilon = \vartheta_\varepsilon(x)$ is a regularizing sequence, implies that

$$\partial_t\tilde{\rho}_n + \operatorname{div}(\tilde{\rho}_n\mathbf{u}_n) = r^n_\varepsilon \text{ on } (0,T)\times\mathbb{R}^3 \tag{33}$$

where

$$r^n_\varepsilon \to 0 \text{ in } L^2(0,T;L^\alpha(\Omega)) \text{ as } \varepsilon \to 0 \text{ for fixed } n \tag{34}$$

with

$$\alpha = \frac{2\beta}{\beta+2}. \tag{35}$$

Proof. Please refer to ([48], Corollary 2.4) for the proof.

4. Proof of Theorem 2.2

This section deals with the proof of Theorem 2.2, in order to prove the required result, we consider the quantities

$$\psi(t)\mathbf{B}\left[h(\tilde{\rho}_n) - \int_\Omega h(\tilde{\rho}_n)dx\right], \ \psi \in (0,T), \ 0 \le \psi \le 1 \tag{36}$$

as a test functions for $(15)_2$, where $h \in C^1(\mathbb{R}), h(z) = z^{\frac{1}{5}}$, for $z \ge 1$. By using the quantities defined in (36), from (15), after a straightforward calculation, one can obtain:

$$\begin{aligned} &\int_0^T\int_\Omega \psi(\rho_n^\gamma + \delta_n\rho_n^\beta)h(\tilde{\rho}_n)dxdt \\ &= \int_0^T\psi\left(\int_\Omega \rho_n^\gamma + \delta_n\rho_n^\beta dx\right)\left(\int_\Omega h(\tilde{\rho}_n)dx\right)dt + \int_0^T\psi\int_\Omega \eta(\operatorname{div}\mathbf{u}_n)\operatorname{div}\mathbf{u}_n h(\tilde{\rho}_n)dxdt \\ &-\int_0^T\psi_t\int_\Omega \rho_n\mathbf{u}_n\mathbf{B}\left[h(\tilde{\rho}_n) - \int_\Omega h(\tilde{\rho}_n)dx\right]dxdt \\ &+\int_0^\infty\psi\int_\Omega |\mathbb{D}^d(\mathbf{u}_n)|^{r-2}\,\mathbb{D}^d(\mathbf{u}_n) : \nabla\mathbf{B}\left[h(\tilde{\rho}_n) - \int_\Omega h(\tilde{\rho}_n)dx\right]dxdt \\ &-\int_0^T\psi\int_\Omega \rho_n\mathbf{u}_n\otimes\mathbf{u}_n : \nabla\mathbf{B}\left[h(\tilde{\rho}_n) - \int_\Omega h(\tilde{\rho}_n)dx\right]dxdt \end{aligned} \tag{37}$$

$$+\int_0^T\psi\int_\Omega\rho_n\mathbf{u}_n\mathbf{B}\Big[\big(h(\tilde\rho_n)-h'(\tilde\rho_n)\tilde\rho_n\big)\operatorname{div}\mathbf{u}_n-\int_\Omega\big(h(\tilde\rho_n)-h'(\tilde\rho_n)\tilde\rho_n\big)\operatorname{div}\mathbf{u}_n dx\Big]dxdt$$
$$+\int_0^T\psi\int_\Omega\rho_n\mathbf{u}_n\mathbf{B}\Big[r_\varepsilon h'(\tilde\rho_n)-\int_\Omega r_\varepsilon h'(\tilde\rho_n)dx\Big]dxdt$$
$$-\int_0^T\psi\int_\Omega\rho_n\mathbf{u}_n\mathbf{B}\big[\operatorname{div}\big(h(\tilde\rho_n)\mathbf{u}_n\big)\big]dxdt=\sum_{j=1}^{8}I_j.$$

To simplify the notation, summation convention is used here and in what follows. In the following, we calculate the integrals one by one used in summation:

(I) The first integral I_1, with help of (22) and (23) may be estimated as

$$|I_1|=\left|\int_0^T\psi\Big(\int_\Omega a\rho_n^\gamma+\delta_n\rho_n^\beta dx\Big)\Big(\int_\Omega h(\tilde\rho_n)dx\Big)dt\right|\le C(\rho_0,\mathbf{m}_0,T)$$

where C is independent of n and ε.

(II) For the second integral I_2, we have

$$|I_2|=\left|\int_0^T\psi\int_\Omega\eta(\operatorname{div}\mathbf{u}_n)\operatorname{div}\mathbf{u}_n h(\tilde\rho_n)dxdt\right|\le C(\rho_0,\mathbf{m}_0,T)$$

I_2 may be computed as follow, for $\eta(\operatorname{div}\mathbf{u}_n)\operatorname{div}\mathbf{u}_n$, since $|\operatorname{div}\mathbf{u}_n|<\frac{1}{b}$ a.e., on $(0,T)\times\Omega$. Let $C_0\in\mathbb{R}$ be a constant such that $0<C_0<\frac{1}{b}$

$$\int_0^T\int_\Omega\eta(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|dxdt=\int_{\Omega\times(0,T)\cap\{(x,t)||\operatorname{div}\mathbf{u}_n|<C_0\}}\eta(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|dxdt$$
$$+\int_{\Omega\times(0,T)\cap\{(x,t)|C_0<|\operatorname{div}\mathbf{u}_n|<\frac{1}{b}\}}\eta(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|dxdt. \tag{38}$$

Furthermore, from Eq. (38), we have

$$\int_{\Omega\times(0,T)\cap\{(x,t)||\operatorname{div}\mathbf{u}_n|<C_0\}}\eta(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|dxdt\le\eta^{\frac12}(C_0)\int_{\Omega\times(0,T)}\eta^{\frac12}(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|dxdt$$
$$\le\eta^{\frac12}(C_0)\Big(\int_{\Omega\times(0,T)}\eta^{\frac12}(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|^2dxdt\Big)^{\frac12}\times C(\Omega). \tag{39}$$

Similarly, on the right-hand side of Eq. (38), the second term may be computed as follows

$$\int_{\Omega\times(0,T)\cap\{(x,t)|C_0<|\operatorname{div}\mathbf{u}_n|<\frac{1}{b}\}}\eta(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|dxdt=\int_{\Omega\times(0,T)\cap\{(x,t)|C_0<|\operatorname{div}\mathbf{u}_n|<\frac{1}{b}\}}\frac{1}{|\operatorname{div}\mathbf{u}_n|}\eta(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|^2dxdt$$
$$\le\frac{1}{C_0}\int_{\Omega\times((0,T)}\eta(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|^2dxdt.$$

Hence, it yields

$$\int_0^T\int_\Omega\eta(\operatorname{div}\mathbf{u}_n)\operatorname{div}\mathbf{u}_n dxdt$$
$$\le\left(\eta^{\frac12}(C_0)\Big(\int_{\Omega\times(0,T)}\eta^{\frac12}(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|^2dxdt\Big)^{\frac12}\times C(\Omega)+\frac{1}{C_0}\int_{\Omega\times(0,T)}\eta(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|^2dxdt\right).$$

As $\eta(\operatorname{div}\mathbf{u}_n)|\operatorname{div}\mathbf{u}_n|^2\in L^1((0,T)\times\Omega)$. So, we get the required result.

(III) The integral I_3 may be computed with the help of (23), (25) and Holder inequality, as

$$|I_3|=\left|\int_0^T\psi_t\int_\Omega\rho_n\mathbf{u}_n\mathbf{B}\Big[h(\tilde\rho_n)-\int_\Omega h(\tilde\rho_n)dx\Big]dxdt\right|$$
$$\le C\int_0^T|\psi_t|\big\|\sqrt{\rho_n}\big\|_{L^{2\gamma}(\Omega)}\big\|\sqrt{\rho_n}\mathbf{u}_n\big\|_{L^2(\Omega)}\Big\|\mathbf{B}\Big[h(\tilde\rho_n)-\int_\Omega h(\tilde\rho_n)dx\Big]\Big\|_{L^\infty(\Omega)}dt$$
$$\le C\int_0^T|\psi_t|\|\rho_n\|_{L^\gamma(\Omega)}^{\frac12}\big\|\sqrt{\rho_n}\mathbf{u}_n\big\|_{L^2(\Omega)}\|\tilde\rho_n\|_{L^\gamma(\Omega)}^{\frac15}dt\le C(\rho_0,\mathbf{m}_0,T)\int_0^T|\psi_t|dt.$$

(IV) Moreover, the integral I_4 may be computed as follows

$$|I_4|=\left|\int_0^T\psi\int_\Omega\big|\mathbb{D}^d(\mathbf{u}_n)\big|^{r-2}\mathbb{D}^d(\mathbf{u}_n):\nabla\mathbf{B}\Big[h(\tilde\rho_n)-\int_\Omega h(\tilde\rho_n)dx\Big]dxdt\right|$$
$$\le C\int_0^T\int_\Omega\big|\mathbb{D}^d(\mathbf{u}_n)\big|^{r-1}\Big|\nabla\mathbf{B}\Big[h(\tilde\rho_n)-\int_\Omega h(\tilde\rho_n)dx\Big]\Big|dxdt$$
$$\le C\int_0^T\big\|\mathbb{D}^d(\mathbf{u}_n)\big\|_{L^r(\Omega)}^{r-1}\Big\|h(\tilde\rho_n)-\int_\Omega h(\tilde\rho_n)dx\Big\|_{L^r(\Omega)}dxdt\le C(\rho_0,\mathbf{m}_0,T).$$

(V) Furthermore, for I_5, we have

$$\begin{aligned}|I_5| &= \left|\int_0^T \psi \int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{B}\left[h(\tilde{\rho}_n) - \int_\Omega h(\tilde{\rho}_n)\,dx\right] dxdt\right| \\ &\le C\int_0^T \|\rho n\|_{L^\gamma(\Omega)} \|\mathbf{u} n\|^2_{L^{\frac{3r}{3-r}}(\Omega)} \left\|\mathbf{B}\left[h(\tilde{\rho}_n) - \int_\Omega h(\tilde{\rho}_n)\,dx\right]\right\|_{W^{1,p_1}(\Omega)} dt\end{aligned}$$

where $p_1 = \frac{3r\gamma}{5r\gamma - 3r - 6\gamma}$. Furthermore, with the help of (23), (25) and (28), we get that the above inequality is bounded and C does not depend on n and ε.

(VI) Next, I_6 may be computed as follows

$$\begin{aligned}|I_6| &= \left|\int_0^T \psi \int_\Omega \rho_n \mathbf{u}_n \mathbf{B}\left[(h(\tilde{\rho}_n) - h'(\tilde{\rho}_n)\tilde{\rho}_n)\operatorname{div}\mathbf{u}_n - \int_\Omega (h(\tilde{\rho}_n) - h'(\tilde{\rho}_n)\tilde{\rho}_n)\operatorname{div}\mathbf{u}_n dx\right] dxdt\right| \\ &\le C\int_0^T \|\rho_n\|_{L^\gamma(\Omega)} \|\mathbf{u}_n\|_{L^{\frac{3r}{3-r}}(\Omega)} \|\mathbf{B}[\ldots]\|_{L^{p_2}(\Omega)} \\ &\le C \sup_{t\in[0,T]} \|\rho_n(t)\|_{L^\gamma(\Omega)} \int_0^T \|\mathbf{u}_n\|^r_{W_0^{1,r}(\Omega)} dt \le C(\rho_0, \mathbf{m}_0, T)\end{aligned}$$

with

$$\begin{aligned}&\left\|\mathbf{B}\left[(h(\tilde{\rho}_n) - h'(\tilde{\rho}_n)\tilde{\rho}_n)\operatorname{div}\mathbf{u}_n - \int_\Omega (h(\tilde{\rho}_n) - h'(\tilde{\rho}_n)\tilde{\rho}_n)\operatorname{div}\mathbf{u}_n dx\right]\right\|_{L^{p_2}(\Omega)} \\ &\le \|h(\rho_n)\operatorname{div}\mathbf{u}_n\|_{L^{p_3}(\Omega)} \le \|h(\rho_n)\|_{L^{p_1}(\Omega)} \|\mathbf{u}_n\|_{W_0^{1,r}(\Omega)}\end{aligned}$$

where

$$p_2 = \frac{3r\gamma}{4r\gamma - 3(r+\gamma)} \quad \text{and} \quad p_3 = \frac{3r\gamma}{4r\gamma + 3(r+\gamma)}.$$

(VII) Moreover, for integral I_7, we have

$$\begin{aligned}|I_7| &= \left|\int_0^T \psi \int_\Omega \rho_n \mathbf{u}_n \mathbf{B}\left[r_\varepsilon h'(\tilde{\rho}_n) - \int_\Omega r_\varepsilon h'(\tilde{\rho}_n)\,dx\right] dxdt\right| \\ &\le C\int_0^T \|\rho_n\|_{L^\gamma(\Omega)} \|\mathbf{u}_n\|_{L^{\frac{3r}{3-r}}(\Omega)} \|r_\varepsilon\|_{L^{p_3}(\Omega)} dt \le C\|r_\varepsilon\|_{L^2(0,T;L^\alpha(\Omega))}\end{aligned}$$

where, the last inequality may be obtained with the help of Lemma 3.4.

(VIII) For integral I_8, we conclude

$$\begin{aligned}|I_8| &= \left|\int_0^T \psi \int_\Omega \rho_n \mathbf{u}_n \mathbf{B}\left[\operatorname{div}(h(\tilde{\rho}_n)\mathbf{u}_n)\right] dxdt\right| \le C\int_0^T \|\rho_n\|_{L^\gamma(\Omega)} \|\mathbf{u}_n\|_{L^{\frac{3r}{3-r}}(\Omega)} \|h(\rho_n)\mathbf{u}_n\|_{L^{p_2}(\Omega)} dt \\ &\le C\int_0^T \|\rho_n\|_{L^\gamma(\Omega)} \|\mathbf{u}_n\|^2_{L^{(\Omega)}} \|h(\tilde{\rho}_n)\|_{L^{p_1}(\Omega)} \le C\int_0^T \|\rho_n\|_{L^\gamma(\Omega)} \|\mathbf{u}_n\|^2_{L^{\frac{3r}{3-r}}(\Omega)} \|\tilde{\rho}_n\|^{\frac{1}{5}}_{L^\gamma(\Omega)} dt\end{aligned}$$

the above inequality is bounded and is independent of n and ε.

Moreover, combining the estimates (I)-(VIII), implies that

$$\int_0^T \psi \int_\Omega (\rho_n^\gamma + \delta_n \rho_n^\beta) h(\tilde{\rho}_n)\,dxdt \le C\left(1 + \|r_\varepsilon^n\|_{L^2(0,T;L^\alpha(\Omega))} + \int_0^T |\psi_t|\,dt\right)$$

where C does not depend on n and ε.

Next, taking $\psi \to 1$, ensures that

$$\int_0^T \int_\Omega (\rho_n^\gamma + \delta_n \rho_n^\beta) h(\tilde{\rho}_n)\,dxdt \le C\left(1 + \|r_\varepsilon^n\|_{L^2(0,T;L^\alpha(\Omega))}\right).$$

Next, by using (28), yields

$$\int_0^T \int_\Omega (\rho_n^\gamma + \delta_n \rho_n^\beta) h(\tilde{\rho}_n)\,dxdt \le C(\rho_0, \mathbf{m}_0, T).$$

Hence, (29) is followed and the proof is completed.

5. Conclusion

In this paper, we studied the integrability up to boundary of weak solutions of a compressible non-Newtonian fluid, such that the initial density $\rho_0 \ge 0$, where the global weak solutions of this model is studied in [39]. It is investigated that the density is square integrable up to the boundary by using the bounded linear operator B. Our approach for this study is on the base of the work of Feireisl [40], where they studied the integrability up to the boundary of weak solutions of compressible fluid. Moreover, we have a plan to study this model with numerical methods in future.

Author Contributions

The authors have equally contributed about publishing this research paper.

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Conflict of Interest

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