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Research Paper

Variational Principles and Solitary Wave Solutions of Generalized Nonlinear Schrödinger Equation in the Ocean

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Abstract. Internal solitary waves are very common physical phenomena in the ocean, which play an important role in the transport of marine matter, momentum and energy. Because the generalized nonlinear Schrödinger equation can well explain the effects of nonlinearity and dispersion in the ocean, it is more suitable for describing the deep-sea internal wave propagation and evolution than other mathematical models. At first, by designing skillfully the trial-Lagrange functional, different kinds of variational principles are successfully established for a generalized nonlinear Schrödinger equation by the semi-inverse method. Then, the constructed variational principles are proved correct by minimizing the functionals with the calculus of variations. Furthermore, some kinds of internal solitary wave solutions are obtained and demonstrated by semi-inverse variational principle for the generalized nonlinear Schrödinger equation.

Keywords: Generalized nonlinear Schrödinger equation; semi-inverse method; variational principle; internal solitary waves.

1. Introduction

Internal solitary waves [1-3] are a kind of physical motions that occur frequently in the interior of fluid, and they happen almost everywhere in the world ocean. The study of internal waves in the ocean is of great significance to the theoretical research of ocean science, utilization of marine resources, avoiding marine disasters, as well as marine military and engineering. Internal solitary waves play an important role in ocean dynamics, which affect the transport of marine matter, momentum, and energy. At present, the well-known KdV equation is only suitable for describing the propagation of small amplitude internal waves in shallow water, [4-8] but there will be intolerable errors when it is used to model large-amplitude internal waves in the deep sea. For deep-sea internal waves, the Benjamin-Ono equation is constructed by Benjamin [9] and Ono [10], while the intermediate long wave (ILW) equation is obtained by Kubota [11] et al. Chio and Camass [12] obtained the fully nonlinear evolution equation of the internal wave at the two-layer interface. The derived equation can be reduced to the ILW equation when it is weakly nonlinear and propagates along one direction, and can be reduced to the Benjamin-Ono equation in infinite water depth. Song et al. [13] established the nonlinear Schrödinger (NLS) equation under two-layer stratification, trying to develop a more accurate equation of the ocean internal wave characteristics in a specific environment. Solving the nonlinear partial differential equations (PDEs) with integer or fractional orders is always an attractive and hot topic for many researchers in different scientific fields, because of their excellent ability for modeling nonlinear phenomena [14-18]. Numerous mathematical techniques have been developed to explore the approximate and exact solutions, of which variational-based methods have been very effective and successful, such as the Ritz technique [19-20], variational iteration method [21-24], and variational approximation method [25-28] et al. When contrasted with other methods, variational ones show some outstanding advantages. In this paper, a generalized nonlinear Schrödinger (GNLS) equation for modelling ocean internal waves is studied by the semi-inverse method, which was first proposed in 1997 by Dr. Ji-Huan He[29], who is a famous Chinese mathematician. At first, by designing skillfully the trial-Lagrange functional, different forms of variational principles are successfully established for the generalized nonlinear Schrödinger equation based on the semi-inverse method and variational theory. Then, different kinds of internal solitary wave solutions are obtained by semi-inverse variational principle for the GNLS equation. Furthermore, some different solutions of the solitary wave with the same trial-Lagrange functional form for the GNLS equation are demonstrated.

2. Variational principles for a GNLS equation

For inviscid fluids, ignoring the influence of Coriolis force, if the fluid is selected as a two-layer structure, a generalized nonlinear Schrödinger equation for deep-sea internal waves can be derived from the continuity equation and Bernoulli equation. It can be used to describe the propagation of internal solitary waves in the ocean:

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$$-iA_t + \alpha A_{xx} + \alpha_1 iA_{xxx} + \beta|A|^2 A = 0 \tag{1}$$

where x and t represents the spatial and temporal variables respectively. In eq.(1), A represents complex amplitude fields of the internal solitary wave and $i=\sqrt{-1}$. α and α_1 is the dispersion and the high-order dispersion coefficient, respectively, and β is the nonlinear coefficient. All these coefficients are related to the local ocean depth, layer structure and the density of the seawater et al, which are physical parameters impacting the amplitude of the internal solitary waves. In eq.(1), the first term is the evolution term, and the second one is the group velocity dispersion term. The third term is the high-order dispersion term, and the fourth one is the nonlinear term in the equation. After substituting $A(x,t) = q_1(x,t) + iq_2(x,t)$ and $|A|^2 = q_1^2 + q_2^2$ into eq. (1), where q_1 and q_2 are the real-valued functions of t and x, we obtain the following coupled partial differential equations for q_1 and q_2 in real space

$$-\frac{\partial q_1}{\partial t} + \alpha\frac{\partial^2 q_2}{\partial x^2} + \alpha_1\frac{\partial^3 q_1}{\partial x^3} + \beta(q_1^2+q_2^2)q_2 = 0 \tag{2}$$

$$\frac{\partial q_2}{\partial t} + \alpha\frac{\partial^2 q_1}{\partial x^2} - \alpha_1\frac{\partial^3 q_2}{\partial x^3} + \beta(q_1^2+q_2^2)q_1 = 0 \tag{3}$$

The target is searching for variational formulations whose stationary conditions satisfy eq. (2) and eq. (3) simultaneously. With the help of He's semi-inverse method [30-31], a trial-functional is constructed in the following form

$$J(q_1,q_2) = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} L dx = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \{q_1\frac{\partial q_2}{\partial t} - \frac{\alpha}{2}[(\frac{\partial q_1}{\partial x})^2 + (\frac{\partial q_2}{\partial x})^2] - \alpha_1 q_1\frac{\partial^3 q_2}{\partial x^3} + F(q_1,q_2)\}dx \tag{4}$$

where F is an unknown function of q_1, q_2 and their derivatives. There are various alternative approaches to the construction of trial-functional, illustrating examples can be found in Refs. [38-40], and detailed discussion about how to construct a suitable trial-functional is given in Ref. [32]. The main merit of the above trial-functional lies on the fact that the stationary condition with respect to q_1, q_2 results in eq. (2) and eq. (3), respectively.

Now calculating the variational derivative of the functional, in eq. (4), with respect to q_1, q_2, we obtain the following Euler equations:

$$-\frac{\partial q_1}{\partial t} + \alpha\frac{\partial^2 q_2}{\partial x^2} + \alpha_1\frac{\partial^3 q_1}{\partial x^3} + \frac{\delta F}{\delta q_2} = 0 \tag{5}$$

$$\frac{\partial q_2}{\partial t} + \alpha\frac{\partial^2 q_1}{\partial x^2} - \alpha_1\frac{\partial^3 q_2}{\partial x^3} + \frac{\delta F}{\delta q_1} = 0 \tag{6}$$

where $\delta F/\delta q_i$ is called He's variational derivative with respect to q_i, defined as [32]

$$\frac{\delta F}{\delta q_i} = \frac{\partial F}{\partial q_i} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial q_{ix}}) - \frac{\partial}{\partial t}(\frac{\partial F}{\partial q_{it}}) + \frac{\partial^2}{\partial x^2}(\frac{\partial F}{\partial q_{ixx}}) + \cdots$$

We search for such an F so that eq. (5) turns into eq. (2), and eq. (6) becomes eq. (3) separately. Accordingly, we set

$$\frac{\delta F}{\delta q_1} = \beta(q_1^2+q_2^2)q_1 \tag{7}$$

$$\frac{\delta F}{\delta q_2} = \beta(q_1^2+q_2^2)q_2 \tag{8}$$

from which the unknown F can be determined as follows

$$F = \frac{\beta}{4}(q_1^2+q_2^2)^2 \tag{9}$$

After embedding eq. (9) into eq. (4), the variational principle in real space is established for the generalized nonlinear Schrödinger equations (1), as following

$$J(q_1,q_2) = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \{q_1\frac{\partial q_2}{\partial t} - \frac{\alpha}{2}[(\frac{\partial q_1}{\partial x})^2 + (\frac{\partial q_2}{\partial x})^2] - \alpha_1 q_1\frac{\partial^3 q_2}{\partial x^3} + \frac{\beta}{4}(q_1^2+q_2^2)^2\}dx \tag{10}$$

Similarly, another variational principle can be obtained as

$$J(q_1,q_2) = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} [q_1\frac{\partial q_2}{\partial t} + \frac{\alpha}{2}(q_1\frac{\partial^2 q_1}{\partial x^2} + q_2\frac{\partial^2 q_2}{\partial x^2}) - \alpha_1 q_1\frac{\partial^3 q_2}{\partial x^3} + \frac{\beta}{4}(q_1^2+q_2^2)^2]dx \tag{11}$$

If the trial-Lagrange functional is preset to be in the following form

$$J(q_1,q_2) = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} L dx = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \{-q_2\frac{\partial q_1}{\partial t} - \frac{\alpha}{2}[(\frac{\partial q_1}{\partial x})^2 + (\frac{\partial q_2}{\partial x})^2] + \alpha_1 q_2\frac{\partial^3 q_1}{\partial x^3} + F(q_1,q_2)\}dx \tag{12}$$

using the variational theories, two diverse variational principles in different formulations can be constructed as

$$J(q_1,q_2)=\int_{t_1}^{t_2}dt\int_{x_1}^{x_2}\{-q_2\frac{\partial q_1}{\partial t}-\frac{\alpha}{2}[(\frac{\partial q_1}{\partial x})^2+(\frac{\partial q_2}{\partial x})^2]+\alpha_1 q_2\frac{\partial^3 q_1}{\partial x^3}+\frac{\beta}{4}(q_1^2+q_2^2)^2\}dx \tag{13}$$

and

$$J(q_1,q_2)=\int_{t_1}^{t_2}dt\int_{x_1}^{x_2}[-q_2\frac{\partial q_1}{\partial t}+\frac{\alpha}{2}(q_1\frac{\partial^2 q_1}{\partial x^2}+q_2\frac{\partial^2 q_2}{\partial x^2})+\alpha_1 q_2\frac{\partial^3 q_1}{\partial x^3}+\frac{\beta}{4}(q_1^2+q_2^2)^2]dx \tag{14}$$

Proof. Making any one of the above functionals of variational principles eq. (10), eq. (11), eq. (13), and eq. (14) stationary with respect to all independent functions q_1 and q_2 severally, the following Euler-Lagrange equations can be obtained:

$$\delta q_1:\quad \frac{\partial q_2}{\partial t}+\alpha\frac{\partial^2 q_1}{\partial x^2}-\alpha_1\frac{\partial^3 q_2}{\partial x^3}+\beta(q_1^2+q_2^2)q_1=0 \tag{15}$$

$$\delta q_2:\quad -\frac{\partial q_1}{\partial t}+\alpha\frac{\partial^2 q_2}{\partial x^2}+\alpha_1\frac{\partial^3 q_1}{\partial x^3}+\beta(q_1^2+q_2^2)q_2=0 \tag{16}$$

in which δq_1 and δq_2 are the first-order variations for q_1 and q_2. Obviously, the equations (15)-(16) are totally equivalent to the field equations eq. (3), eq. (2) in turn. So, we successfully proved the obtained four different variational principles eq. (10)-(11), eq. (13)-(14) correct.

3. Solitary wave solutions for the GNLS equation

There are various techniques of integration that have been recently developed to integrate the nonlinear PDEs. They are the Lie symmetry approach, variational iteration method, homotopy analysis method, ansatz method, exponential function method and many others. These are besides the well-known and powerful technique of integration that was known for a fairly long time. In this article, one such modern method of integrability will be employed to integrate the GNLS equation (1) in the ocean. This is the He's semi-inverse variational principle (HVP) that has become very popular since its first appearance in 1997 [25]. In this method, the given PDEs are transformed into ODEs based on the traveling wave function transformation, and the variational formulas corresponding to ordinary differential equations are established in the framework of the variational method with the help of semi-inverse technique[33-36]. The solitary wave solution of the given equation is constructed by substituting the assumed solution into the variational formula and finding its stationary point. The fractal variational principle is the last development of the semi-inverse method[36-40],which can greatly widen our sight and richen our knowledge on solitary wave theory and nonlinear vibration theory.

The solitary wave solution of the given equation is constructed by substituting the assumed solution into the variational formula and finding its stationary point. Subsequently, it will be applied to carry out the integration of the generalized nonlinear Schrödinger equation eq. (1).

The starting point is the solitary wave ansatz that is given by

$$A(x,t)=f(\xi)e^{i(mx-nt)} \tag{17}$$

where the travelling wave transform is:

$$\xi=x-Et \tag{18}$$

and both m and n are constants. f is an undetermined real function, and E is the wave velocity. Substituting the solitary wave ansatz eq. (17) into eq. (1) and decomposing into real and imaginary parts yields the following pair of relations, respectively

$$\alpha_1 f'''+\left(E+2\alpha m-3\alpha_1 m^2\right)f'=0 \tag{19}$$

$$(\alpha-3\alpha_1 m)f''-\left(n+\alpha m^2-\alpha_1 m^3\right)f+\beta f^3=0 \tag{20}$$

In the above two equations, $f'=df/d\xi$, $f''=d^2f/d\xi^2$ and $f'''=d^3f/d\xi^3$. By using semi-inverse method [41-62], the variational formulation of eq. (20) can be obtained:

$$J=\int_0^\infty\left[\frac{1}{2}(\alpha-3\alpha_1 m)(f')^2+\frac{1}{2}\left(n+\alpha m^2-\alpha_1 m^3\right)f^2-\frac{1}{4}\beta f^4\right]d\xi \tag{21}$$

Now, f is assumed to have the following form

$$f=p\,\mathrm{sech}(q\xi),\quad \xi=x-Et \tag{22}$$

where p and q are unknown parameters to be determined.

In order to obtain the two parameters function f, eq. (22) is inserted into eq. (21), and after some manipulations, we get:

$$\begin{aligned}J&=\int_0^\infty\frac{1}{2}(\alpha-3\alpha_1 m)p^2q^2\tanh^2(q\xi)\mathrm{sech}^2(q\xi)d\xi+\int_0^\infty\frac{1}{2}\left(n+\alpha m^2-\alpha_1 m^3\right)p^2\mathrm{sech}^2(q\xi)d\xi-\int_0^\infty\frac{1}{4}\beta p^4\mathrm{sech}^4(q\xi)d\xi\\&=\frac{(\alpha-3\alpha_1 m)}{6}p^2q+\frac{\left(n+\alpha m^2-\alpha_1 m^3\right)p^2}{2q}-\frac{\beta p^4}{6q}\end{aligned} \tag{23}$$

In order to get the stagnation point of J on p and q, we minimizing the above functional with respect to two unknown parameters. And the following equations can be obtained:

$$\begin{aligned}\frac{\partial J}{\partial p}&=\frac{\alpha-3\alpha_1 m}{3}pq+\frac{\left(n+\alpha m^2-\alpha_1 m^3\right)}{q}p-\frac{2\beta}{3q}p^3\\ \frac{\partial J}{\partial q}&=\frac{\alpha-3\alpha_1 m}{6}p^2-\frac{\left(n+\alpha m^2-\alpha_1 m^3\right)}{2q^2}p^2+\frac{2\beta}{6q^2}p^4\end{aligned} \tag{24}$$

The above two equations can be transformed into:

$$\begin{aligned}&(\alpha-3\alpha_1 m)q^2+3\left(n+\alpha m^2-\alpha_1 m^3\right)-2\beta p^2=0\\ &(\alpha-3\alpha_1 m)q^2-3\left(n+\alpha m^2-\alpha_1 m^3\right)+\beta p^2=0\end{aligned} \tag{25}$$

After solving the above algebraic equations, we can get:

$$\begin{aligned}p&=\pm\sqrt{\frac{2\left(n+\alpha m^2-\alpha_1 m^3\right)}{\beta}}\\ q&=\pm\sqrt{\frac{\left(n+\alpha m^2-\alpha_1 m^3\right)}{\alpha-3\alpha_1 m}}\end{aligned} \tag{26}$$

provided

$$\left(n+\alpha m^2-\alpha_1 m^3\right)\cdot\beta>0$$

$$\left(n+\alpha m^2-\alpha_1 m^3\right)\cdot(\alpha-3\alpha_1 m)>0$$

Finally, the solitary wave solutions to eq. (1) are obtained:

$$A(x,t)=\pm\sqrt{\frac{2\left(n+\alpha m^2-\alpha_1 m^3\right)}{\beta}}\,\mathrm{sech}\left[\pm\sqrt{\frac{\left(n+\alpha m^2-\alpha_1 m^3\right)}{\alpha-3\alpha_1 m}}\xi\right]e^{i(mx-nt)} \tag{27}$$

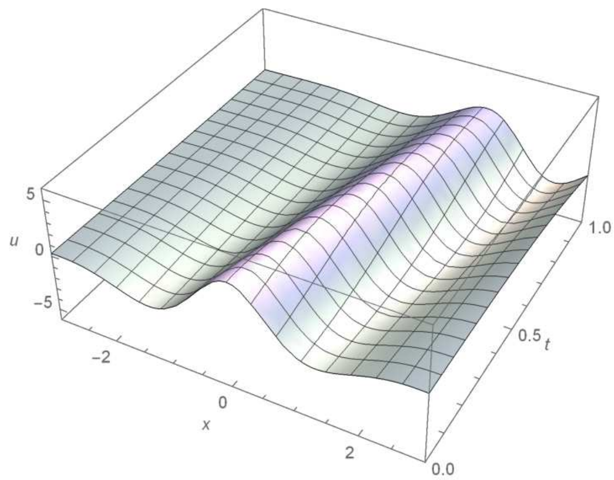

Fig. 1. The shape of the solitary wave solution given by eq. (27)

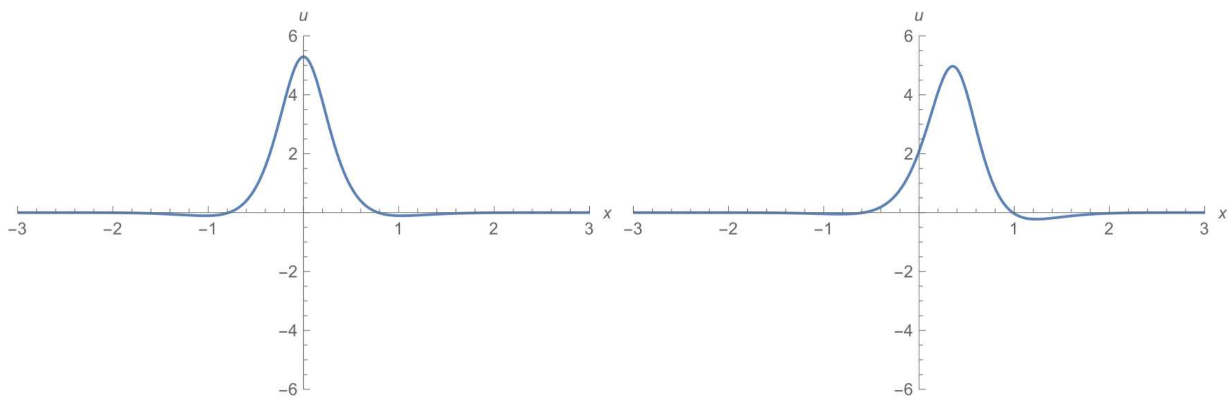

Fig. 2. The shape of the solitary wave solution given by eq. (27) at different time (when $t=0$, $t=0.2$, $t=0.4$, $t=0.6$, $t=0.8$, $t=1$)

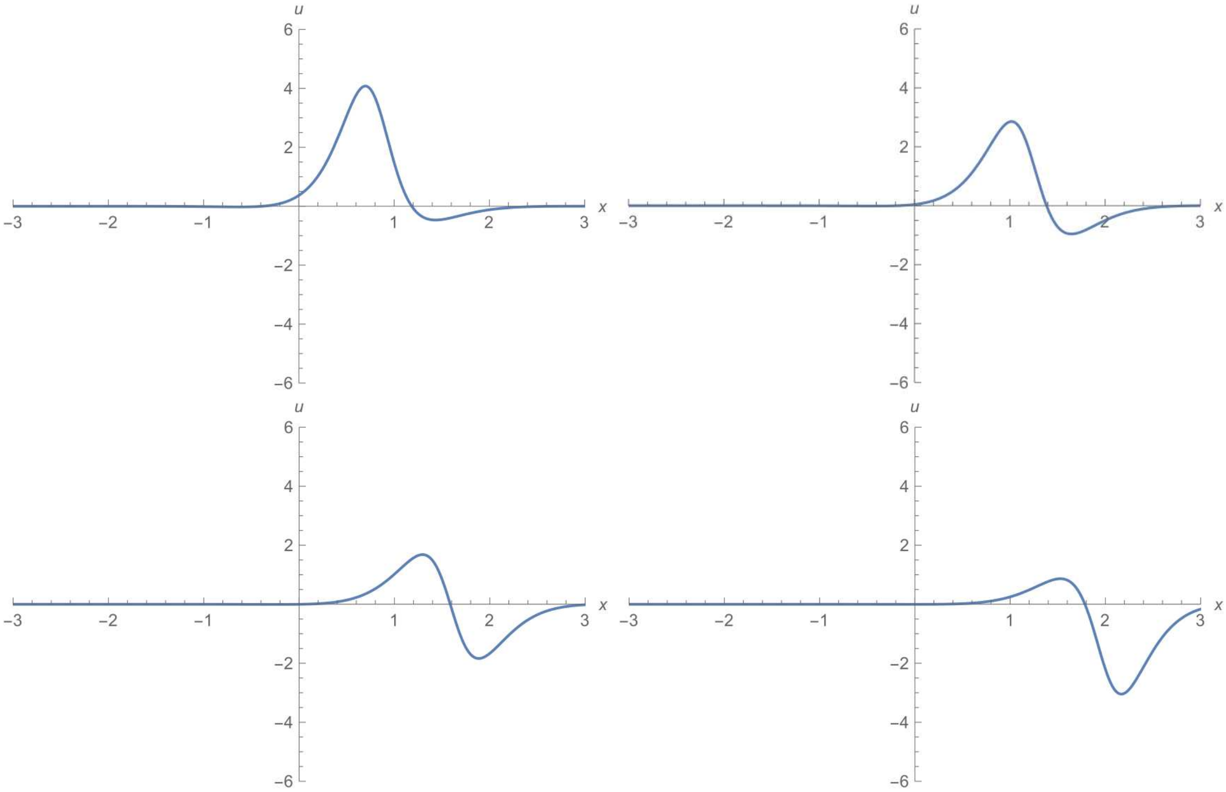

Fig. 2. Continued.

From the exact solution formula eq. (27), it can be concluded that high-order dispersion term α_1 and nonlinear term β both have a great influence on internal waves, which cannot be ignored. Obviously, by giving different values to the parameters for α, β, α_1, m, n and E, we will get different solitary wave solutions. If the parameters are set as $\alpha = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, m = 2, n = 2, and E = 2. The value of x is between -3 and 3, and the value of t is between 0 and 1. We can plot the solitary wave solution as figure 1. From figure 1 and figure 2, it is easy to show that the amplitude of wave solution is very local in space and has characteristics of soliton.

Similarly, in order to get new solutions, we can choose a different form of solution functional as

$$f = p\,\mathrm{sech}^2(q\xi), \quad \xi = x - Et \tag{28}$$

The calculation procedure is similar to above, and the letters p, q are undetermined parameters. In order to obtain the following two-parameter function, we insert eq.(28) into eq.(21):

$$\begin{aligned} J &= \int_0^{\infty} 2(\alpha - 3\alpha_1 m)p^2q^2\tanh^2(q\xi)\mathrm{sech}^4(q\xi)d\xi + \int_0^{\infty}\frac{1}{2}\left(n + \alpha m^2 - \alpha_1 m^3\right)p^2\mathrm{sech}^4(q\xi)d\xi - \int_0^{\infty}\frac{1}{4}\beta p^4\mathrm{sech}^8(q\xi)d\xi \\ &= \frac{4}{15}(\alpha - 3\alpha_1 m)p^2q + \frac{\left(n + \alpha m^2 - \alpha_1 m^3\right)p^2}{3q} - \frac{4\beta p^4}{35q} \end{aligned} \tag{29}$$

In order to get the stagnation point of J on p and q, we set up the following equations:

$$\begin{aligned} \frac{\partial J}{\partial p} &= \frac{8}{15}(\alpha - 3\alpha_1 m)pq + \frac{2\left(n + \alpha m^2 - \alpha_1 m^3\right)}{3q}p - \frac{16(\beta - \beta_1 m)}{35q}p^3 \\ \frac{\partial J}{\partial q} &= \frac{4}{15}(\alpha - 3\alpha_1 m)p^2 - \frac{\left(n + \alpha m^2 - \alpha_1 m^3\right)}{3q^2}p^2 + \frac{4(\beta - \beta_1 m)}{35q^2}p^4 \end{aligned} \tag{30}$$

Or simplify to get:

$$\begin{aligned} &28(\alpha - 3\alpha_1 m)q^2 - 35\left(n + \alpha m^2 - \alpha_1 m^3\right) + 12\beta p^2 = 0 \\ &28(\alpha - 3\alpha_1 m)q^2 + 35\left(n + \alpha m^2 - \alpha_1 m^3\right) - 24\beta p^2 = 0 \end{aligned} \tag{31}$$

After solving the above algebraic equations, we can get:

$$\begin{aligned} p &= \pm\sqrt{\frac{35\left(n + \alpha m^2 - \alpha_1 m^3\right)}{18\beta}} \\ q &= \pm\sqrt{\frac{5\left(n + \alpha m^2 - \alpha_1 m^3\right)}{12(\alpha - 3\alpha_1 m)}} \end{aligned} \tag{32}$$

provided

$$\left(n+\alpha m^{2}-\alpha_{1}m^{3}\right)\cdot\beta>0$$

$$\left(n+\alpha m^{2}-\alpha_{1}m^{3}\right)\cdot\left(\alpha-3\alpha_{1}m\right)>0$$

and the result is:

$$A(x,t)=\pm\frac{1}{3}\sqrt{\frac{35\left(n+\alpha m^{2}-\alpha_{1}m^{3}\right)}{2\beta}}\operatorname{sech}^{2}\left[\pm\frac{1}{2}\sqrt{\frac{5\left(n+\alpha m^{2}-\alpha_{1}m^{3}\right)}{3\left(\alpha-3\alpha_{1}m\right)}}\xi\right]e^{i(mx-nt)} \tag{33}$$

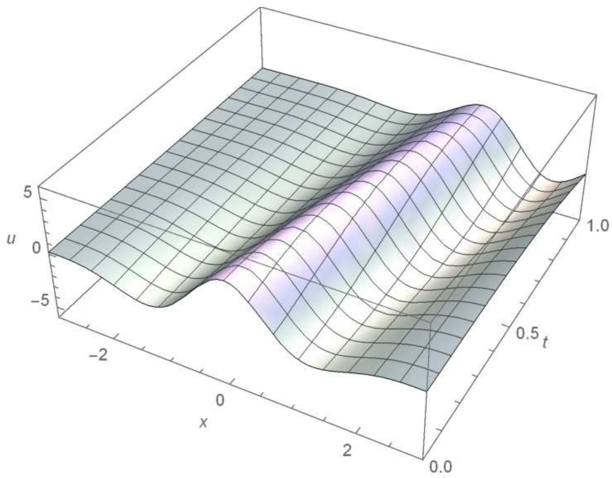

Fig. 3. The shape of the solitary wave solution given by eq. (33)

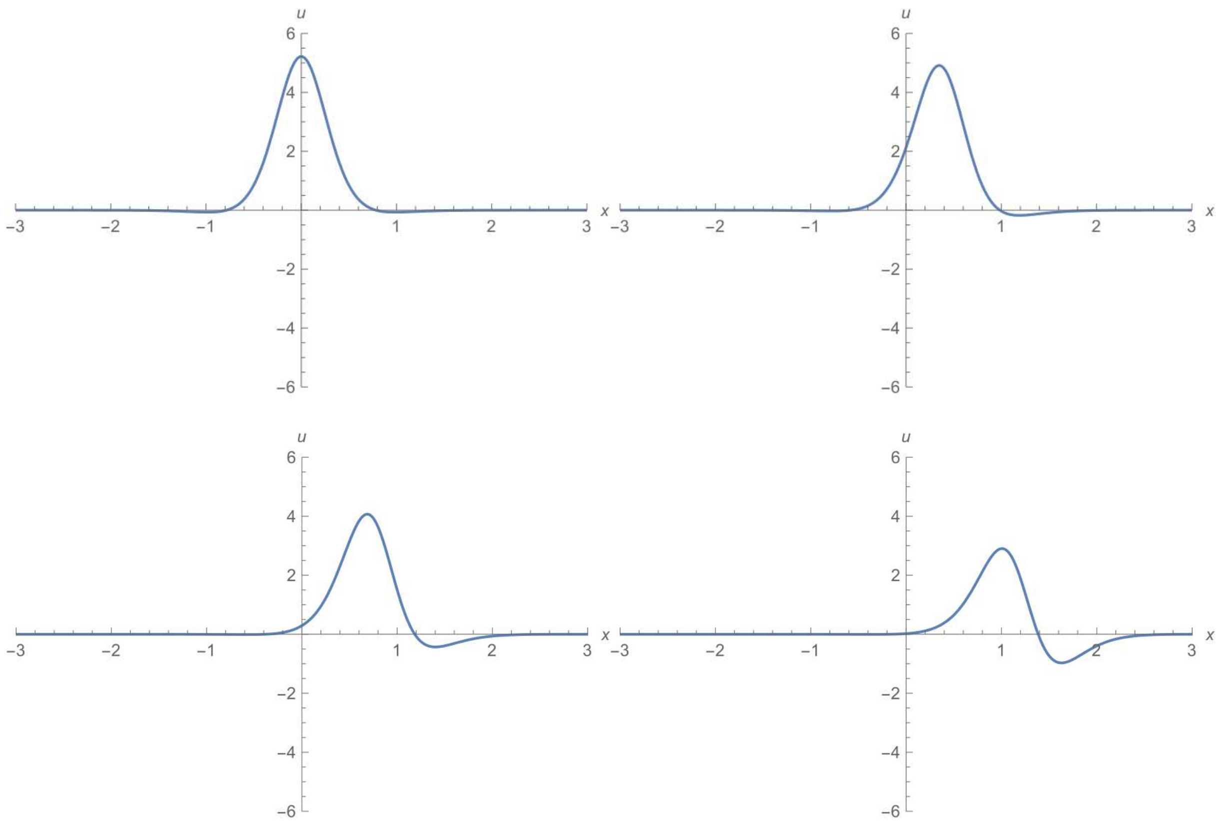

Fig. 4. The shape of the solitary wave solution given by eq. (33) at different time
(when $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$, $t = 1$)

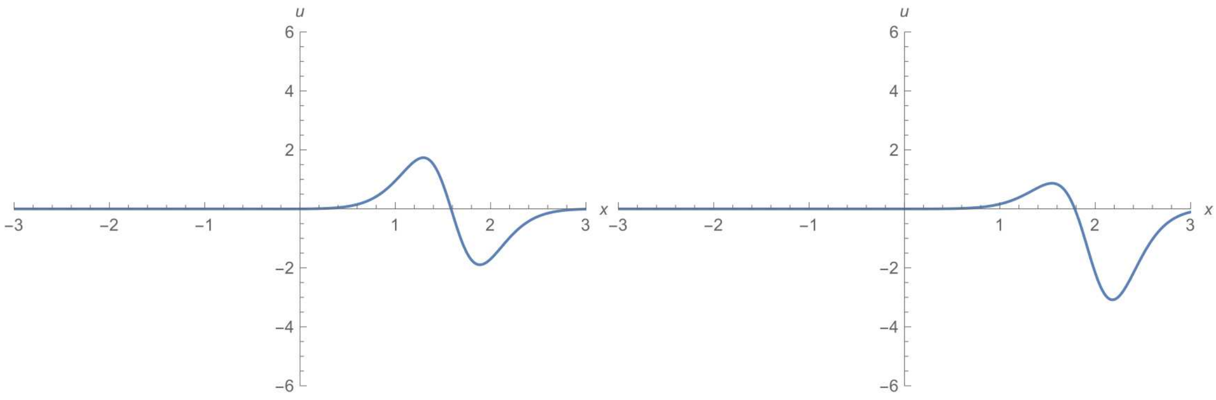

Fig. 4. Continued.

From the exact solution formula eq. (31), it can be concluded that high-order dispersion term α_1 and nonlinear term β both have a great influence on internal waves, which cannot be ignored. Obviously, by giving different values to the parameters for α, β, α_1, m, n and E, we will get different solitary wave solutions. If the parameters are set as $\alpha = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, m = 2, n = 2, and E = 2. The value of x is between -3 and 3, and the value of t is between 0 and 1. We can plot the solitary wave solution as figure 1. From figure 3 and figure 4, it is easy to show that the amplitude of wave solution is very local in space and has characteristics of soliton.

And we can also choose

$$f = p\sqrt{\operatorname{sech}(q\xi)}, \quad \xi = x - Et \tag{34}$$

$$f' = \frac{1}{2} pq \tanh(q\xi)\sqrt{\operatorname{sech}(q\xi)} \tag{35}$$

Inserting eq. (34) and eq. (35) into eq. (19), eq. (19) is:

$$\begin{aligned} J &= \int_0^\infty \frac{1}{8}(\alpha - 3\alpha_1 m)p^2q^2\tanh^2(q\xi)\operatorname{sech}(q\xi) + \frac{1}{2}(n + \alpha m^2 - \alpha m^3)p^2\operatorname{sech}(q\xi) - \frac{1}{4}\beta p^4\operatorname{sech}^2(q\xi)d\xi \\ &= \frac{(\alpha - 3\alpha_1 m)\pi p^2 q}{32} + \frac{(n + \alpha m^2 - \alpha m^3)\pi p^2}{4q} - \frac{\beta p^4}{4q} \end{aligned} \tag{36}$$

In order to get the stagnation point of J on p and q, we set up the following equations:

$$\begin{aligned} \frac{\partial J}{\partial p} &= \frac{(\alpha - 3\alpha_1 m)\pi pq}{16} + \frac{(n + \alpha m^2 - \alpha m^3)\pi p}{2q} - \frac{\beta p^3}{q} \\ \frac{\partial J}{\partial q} &= \frac{(\alpha - 3\alpha_1 m)\pi p^2}{32} - \frac{(n + \alpha m^2 - \alpha m^3)\pi p^2}{4q^2} + \frac{\beta p^4}{4q^2} \end{aligned} \tag{37}$$

Or simplify to get:

$$\begin{aligned} (\alpha - 3\alpha_1 m)\pi q^2 - 8(\alpha_1 m^3 - n - \alpha m^2) - 16\beta p^2 = 0 \\ (\alpha - 3\alpha_1 m)\pi q^2 + 8(\alpha_1 m^3 - n - \alpha m^2) + 8\beta p^2 = 0 \end{aligned} \tag{38}$$

After solving the above algebraic equations, we can get:

$$\begin{aligned} p &= \pm\sqrt{\frac{2(n + \alpha m^2 - \alpha m^3)\pi}{3\beta}} \\ q &= \pm\sqrt{\frac{8(n + \alpha m^2 - \alpha m^3)}{3(\alpha - 3\alpha_1 m)}} \end{aligned} \tag{39}$$

provided

$$(n + \alpha m^2 - \alpha_1 m^3)\cdot\beta > 0$$

$$(n + \alpha m^2 - \alpha_1 m^3)\cdot(\alpha - 3\alpha_1 m) > 0$$

After solving the above algebraic equations, we can get:

$$A(x,t) = \pm\sqrt{\frac{2(n + \alpha m^2 - \alpha m^3)\pi}{3\beta}} \cdot \sqrt{\operatorname{sech}\left(\pm\sqrt{\frac{8(n + \alpha m^2 - \alpha m^3)}{3(\alpha - 3\alpha_1 m)}}\xi\right)} \tag{40}$$

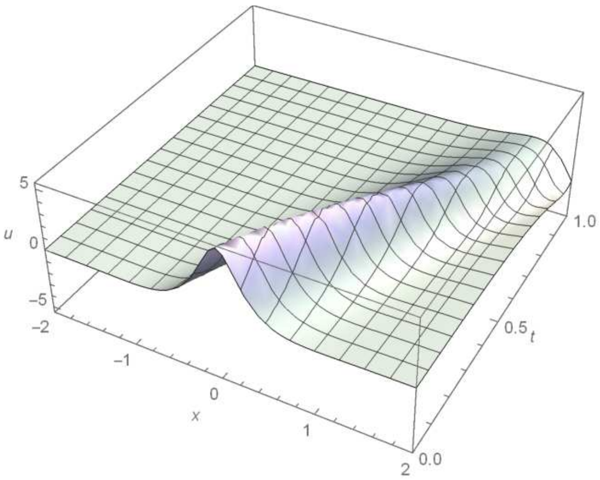

Fig. 5. The shape of the solitary wave solution given by eq. (40)

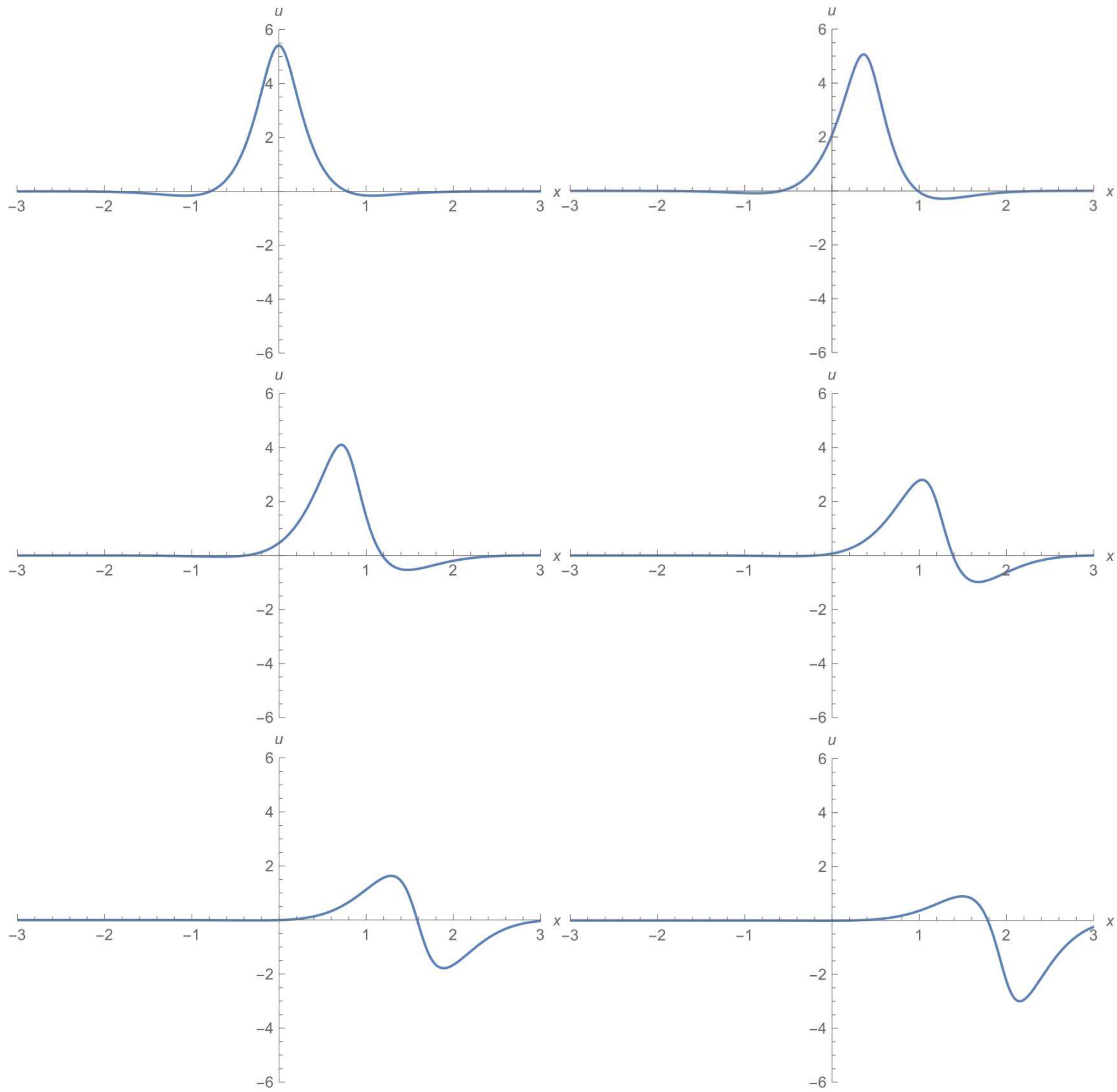

Fig. 6. The shape of the solitary wave solution given by eq.(40) at different time
(when $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$, $t = 1$)

From the exact solution formula eq. (40), it can be concluded that high-order dispersion term α_1 and nonlinear term β both have a great influence on internal waves, which cannot be ignored. Obviously, by giving different values to the parameters for α, β, α_1, m, n and E, we will get different solitary wave solutions. If the parameters are set as $\alpha = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, m = 2, n = 2, and E = 2. The value of x is between -3 and 3, and the value of t is between 0 and 1. We can plot the solitary wave solution as figure 1. From figure 5 and figure 6, it is easy to show that the amplitude of wave solution is very local in space and has characteristics of soliton.

4. Conclusion

The generalized nonlinear Schrödinger equation is widely applied in mathematics and physics. It is closely related to many nonlinear problems in theoretical physics such as nonlinear optics, ion acoustic waves in plasma, etc. Especially, it is very suitable for describing the deep-sea internal wave propagation. In this paper, different kinds of variational principles have been successfully constructed for a generalized nonlinear Schrödinger equation, by the semi-inverse method and designing skillfully trial-Lagrange functionals. Then, the constructed variational principles are proved correct by minimizing the functionals with the calculus of variations. Subsequently, different solution structures for solitary waves are obtained by semi-inverse variational principle for the GNLS equation. From the figures of solutions, it is observed that on one hand the amplitude of solitary wave solution is very local in space, which displays the characteristics of soliton, on the other hand the shape of the solitary wave solution varies greatly over time. From the exact solution formulas, it can be concluded that high-order dispersion term and nonlinear term both have a great influence on internal wave solutions in the GNLS equation, and they cannot be ignored.

Author Contributions

All authors have important contributions in this paper. The details are as following. Conceptualization, M.-Z. L. and X.-Q. C.; methodology, X.-Q. C. and M.-Z. L.; validation, K.-C. P.; writing—original draft preparation, M.-Z. Liu; writing—review and editing, X.-Q. C., X. -Q. Z., and B.-N. L. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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References

[1] Jiang, Z.H., Huang, S.X., You, X. B., Ocean internal waves interpreted as oscillation travelling waves in consideration of ocean dissipation, *Chinese Physics B*, 23, 2014, 52-59.
[2] Lee, C.Y., Beardsley, R.C., The generation of long nonlinear internal waves in a weakly stratified shear flow, *Journal of Geophysical Research*, 79, 1974, 453-462.
[3] Wang, Z., Zhu Y.K., Theory, modelling and computation of nonlinear ocean internal waves, *Chinese Journal of Theoretical and Applied Mechanics*, 51, 2019, 1589-1604.
[4] Karunakar, P., Chakraverty, S., Effect of Coriolis constant on Geophysical Korteweg-de Vries equation, *Journal of Ocean Engineering and Science*, 4, 2019, 113-121.
[5] Kaya, D., Explicit and Numerical Solutions of Some Fifth-order KdV Equation by Decomposition Method, *Applied Mathematics and Computation*, 144, 2003, 353-363.
[6] Wazwaz, A. M., A Study on Compacton-Like Solutions for the Modified KdV and Fifth Order KdV-Like Equations, *Applied Mathematics and Computation*, 147, 2004, 439-447.
[7] Wazwaz, A.M., Helal, M.A., Variants of the Generalized Fifth-Order KdV Equation with Compact and Noncompact Structures, *Chaos Solitons and Fractals*, 21, 2004, 579-589.
[8] Li, J., Gu X.F., Yu T., Sun Y. Simulation investigation on the internal wave via the analytical solution of Korteweg-de Vries equation, *Marine Science Bulletin*, 30, 2011, 23-28.
[9] Benjamin, B.T., Internal waves of permanent form in fluids of great depth, *Journal of Fluid Mechanics*, 29(03), 1967, 559-592.
[10] Hiroaki, O., Algebraic Solitary Waves in Stratified Fluids, *Journal of the Physical Society of Japan*, 39, 1975, 1082-1091.
[11] Kubota, T., Ko, D., Dobbs, L., Propagation of weakly nonlinear internal waves in a stratified fluid of finite depth, *AIAA Journal of Hydronautics*, 12, 1978, 157-165.
[12] Choi, W., Camassa, R., Fully nonlinear internal waves in a two-fluid system, *Journal of Fluid Mechanics*, 396, 1999, 1-36.
[13] Song, S.Y., Wang, J., Meng, J.M., Wang, J.B., Hu, P. X., Nonlinear Schrödinger equation for internal waves in deep sea, *Acta Physica Sinica*, 59(02), 2010, 1123-1129
[14] Liu, S.K., Fu, Z.T., Expansion method about the Jacobi elliptic function and its applications to nonlinear wave equations, *Acta Physica Sinica*, 50, 2001, 2068-2073.
[15] He, J.H., Exp-function method for fractional differential equations, *International Journal Nonlinear Science and Numerical Simulation*, 14, 2013, 363-366.
[16] He, J.H., On the fractal variational principle for the Telegraph equation, *Fractals*, 29, 2021, https://doi.org/10.1142/S0218348X21500225.
[17] Wu, Y., Variational approach to higher-order water-wave equations, *Chaos Solitons and Fractals*, 32, 2007, 195-203.
[18] Gazzola, F., Wang, Y., Pavani, R., Variational formulation of the Melan equation, *Mathematical Methods in the Applied Sciences*, 41, 2018, 943-951.
[19] He, J.H., Liu, F.J., Local Fractional Variational Iteration Method for Fractal Heat Transfer in Silk Cocoon Hierarchy, *Nonlinear Science Letters A*, 2013, 4, 15-20.
[20] He, J.H., Ji, F.Y., Taylor Series Solution for Lane-Emden Equation, *Journal of Mathematical Chemistry*, 57(8), 2019, 1932-1934.
[21] He, C.H., Shen, Y., Ji, F.Y., He, J.H., Taylor series solution for fractal Bratu-type equation arising in electrospinning process, *Fractals*, 28(1), 2020, 2050011.
[22] He, J.H., Taylor series solution for a third order boundary value problem arising in architectural engineering, *Ain Shams Engineering Journal*, 11(4), 2020, 1411-1414.
[23] Kaup, D.J., Variational solutions for the discrete nonlinear Schrödinger equation, *Mathematics and Computers in Simulation*, 69, 2005, 322-333.

[24] Putri, N.Z., Asfa, A.R., Fitri, A., Bakri, I., Syafwan, M., Variational approximations for intersite soliton in a cubic-quintic discrete nonlinear Schrödinger equation, *Journal of Physics, Conference Series*, 2019, 1317, 012-015.
[25] He, J.H., Variational principles for some nonlinear partial differential equations with variable coefficients, *Chaos Solitons and Fractals*, 19, 2004, 847-851.
[26] He, J.H., A modified Li-He's variational principle for plasma, *International Journal Numerical Methods for Heat & Fluid Flow*, 2019, doi,10.1108/HFF-06-2019-0523.
[27] He, J.H., Generalized equilibrium equations for shell derived from a generalized variational principle, *Applied Mathematics Letters*, 64, 2017, 94-100.
[28] He, J.H., Sun, C., A variational principle for a thin film equation, *Journal of Mathematical Chemistry*, 57, 2019, 2075-2081.
[29] He, J.H., Semi-Inverse Method of Establishing Generalized Variational Principles for Fluid Mechanics With Emphasis on Turbomachinery Aerodynamics, *International Journal of Turbo & Jet-Engines*, 14, 1997, 23-28.
[30] He, J.H., Generalized Variational Principles for Buckling Analysis of Circular Cylinders, *Acta Mechanica*, 231, 2020, 899-906.
[31] He, J.H., A fractal variational theory for one-dimensional compressible flow in a microgravity space, *Fractals*, 2019, https://doi.org/10.1142/S0218348X20500243.
[32] Yue, S., He, J.H., Variational principle for a generalized KdV equation in a fractal space, *Fractals*, 28(4), 2020, 2050069.
[33] Biswas, A., Zhou, Q., Ullah, M.Z., Triki, H., Moshokoa, S.P., Belic, M., Optical soliton perturbation with anti-cubic nonlinearity by semi-inverse variational principle, *Optik,* 143, 2017, 131-134.
[34] Cao, X.Q., Guo Y.N., Hou, S.C., Zhang, C.Z., Peng, K.C., Variational Principles for Two Kinds of Coupled Nonlinear Equations in Shallow Water, *Symmetry*, 12, 2020, 850.
[35] Cao, X.Q., Generalized variational principles for Boussinesq equation systems, *Acta Physica Sinica*, 60, 2011, 105-113.
[36] Kohl, R.W., Biswas, A., Zhou, Q., Ekici, M. Alzahrani, A.K., Belic, M.R., Optical soliton perturbation with polynomial and triple-power laws of refractive index by semi-inverse variational principle, *Chaos, Solitons & Fractals*, 135, 2020, 109765.
[37] He, J.H., El-Dib, Y.O., Periodic property of the time-fractional Kundu–Mukherjee–Naskar equation, *Results in Physics*, 19, 2020, 103345.
[38] He, J.H., Variational principle and periodic solution of the Kundu–Mukherjee–Naskar equation, *Results in Physics*, 17, 103031
[39] He, J.H., Kou, S.J., He, C.H., Zhang, Z.W., Khaled, A. Gepreel., Fractal oscillation and its frequency-amplitude property, *Fractals*, 2021, DOI: 10.1142/S0218348X2150105X.
[40] He, C.H., Liu, C., He, J.H., Shirazi, A.H., Sedighi, H.M., Passive Atmospheric water harvesting utilizing an ancient Chinese ink slab and its possible applications in modern architecture, *Facta Universitatis Series-Mechanical Engineering*, 2021, doi: 10.22190/FUME201203001H.

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