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Research Paper

Alternative Integration Approaches in the Weight Function Method for Crack Problems

Martin A. Eder, Xiao Chen

Department of Wind Energy, Technical University of Denmark, Frederiksborgvej 399, 4000 Roskilde, Denmark

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Corresponding author: X. Chen (xiac@dtu.dk)

Abstract. This study proposes two alternative approaches to complement existing integration strategies used in the weight function method for linear elastic crack problems. The first approach is based on an interpolation type integration scheme and the second approach is based on Gauss quadrature. The proposed approaches enable a computationally efficient numerical integration for computing stress intensity factors in 2D fracture problems. The efficiency is gained through a comparatively low number of integration points facilitated by higher-order approximation. The integration weights only need to be computed once for a given crack length-to-width ratio and can be applied to arbitrary continuous and smooth stress distributions. The proposed approaches show excellent accuracy. In particular, the Gauss quadrature approach exhibits several orders of magnitude more accuracy compared to the most commonly used trapezoidal integration.

Keywords: Stress intensity factor; fracture mechanics; crack length; singularity; weight function integration.

1. Introduction

The fracture behavior of cracked structures is dominated by the near-tip stress field. The weight function method allows fracture parameter estimation with an integral over the crack plane. There are many studies on integration strategies used in the weight function method, see a recent review work by Wu [1]. This study proposes two alternative approaches to complement existing integration strategies for linear elastic crack problems. The Mode-I stress intensity factor K_I of a linear elastic fracture problem can be obtained with the Green function based weight function method [2] as follows:

$$K_I = \int_0^a h(x)\sigma(x)dx \quad \text{for} \quad 0 \le x \le a \in \mathbb{R} \tag{1}$$

where $h(x)$, $\sigma(x)$, and a are the weight function, the stress distribution function, and the crack length respectively. Eq. 1 is the archetypical mathematical definition of a weight function-based integration problem with the significance that the weight function in this particular case bears a physical meaning. The weight function $h(x)$ is available analytically for different geometries in 2D and 3D. In the case of the 2D compact tension (CT) specimen a series of different weight function solutions was derived where the solution proposed by Wu and Carlsson [3] is reproduced here:

$$h(x) = \frac{1}{\sqrt{2\pi a}}(\beta_1(1-\frac{x}{a})^{-\frac{1}{2}} + \beta_2(1-\frac{x}{a})^{\frac{1}{2}} + \beta_3(1-\frac{x}{a})^{\frac{3}{2}} + \beta_4(1-\frac{x}{a})^{\frac{5}{2}}) \tag{2}$$

where β_i (a/W) are coefficients that depend on the CT specimen geometry and are provided in Table 1.

The form of Eq. 2 can be seen as a generic representation of 2D weight functions as pointed out by Moftakhar and Glinka [5]. For the application of Eq. 1, the function $\sigma(x)$ in the un-cracked situation is required which e.g. in the case of residual stress is typically obtained from numerical stress analysis for which Eq. 1 cannot be solved analytically. Therefore, numerical integration schemes are required to solve the integral approximately. Eq. 2 however, has an inverse square root singular term, which makes the integral improper leading to potential difficulties in dealing with the singular point $\lim_{x\to a} h(x) = \infty$.

In the most commonly used trapezoidal rule, the issue is dealt with an approach known as 'ignoring the singularity' by simply excluding the last data point located at the singularity. However, the convergence of this approach to the exact value is slow as the contribution of the ignored trapezoidal adjacent to the singularity is comparatively high. Moreover, additional integration errors arise in cases where $\sigma(x)$ has one or more roots in the interval [0, a] – representing rather ill-conditioned situations for the trapezoidal rule.

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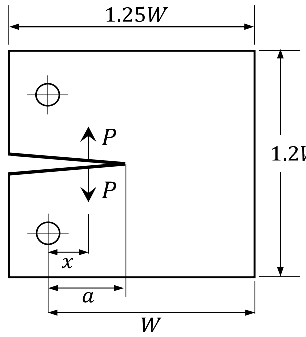

Fig. 1. Compact tension specimen with a definition of the key parameters for the weight function given by Eq.2; the weight function provides the stress intensity factor for a couple of concentrated forces (Dirac deltas) acting perpendicular to the crack faces in opposite directions. Note: the methods are demonstrated based on the CT specimen, but can easily be adapted to other geometries.

Moftakhar and Glinka [5] have proposed a weighted sum approach with a piecewise linear approximation of both the weight function and the stress distribution function. The utilization of a natural coordinate system facilitates an efficient computation of the integral. An analytical solution for the sub-interval next to the singularity was derived, providing a significantly improved accuracy. Anderson and Glinka [6] derived an interpolation-based weight function integration method for surface and corner cracks. A quadratic interpolation between discretely known integrand values provides excellent accuracy for rather large intervals.

Inferring from the successful integration strategies in the literature, the incentive of this work is to present two alternative weight-based integration approaches for the three following prerequisites: First, the approach should be able to explicitly deal with the singularity. Second, it should provide a high level of accuracy and computational efficiency by avoiding discretization with a vast number of integration intervals. Third, interpolation between the discrete interval points should be enabled to any desired order.

In this study, the proposed integration approaches are demonstrated on the example of Eq. 2 but can straightforwardly be extended to other weight functions under the proviso that the moments (Eq. 4) can be obtained analytically. The general idea is to conduct the integration of Eq. 1 in natural coordinate space allowing tabulation of the weights which can then be applied to any arbitrary Cartesian crack length a. In this way, the computation of the weights is required only once for a specific *a/w* ratio, facilitating a computationally efficient integration.

2. Two alternative integration approaches

2.1 Interpolation (IP) approach

The first approach is based on an interpolation-type integration scheme. As pointed out previously, it is convenient to define a natural coordinate $\xi = x/a$ for $0 \le \xi \le 1 \in \mathbb{R}$, and to substitute the corresponding terms in equation 2 rendering the following weight function:

$$h(\xi) = \tfrac{1}{\sqrt{2\pi}}\left(\beta_1(1-\xi)^{-\frac{1}{2}} + \beta_2(1-\xi)^{\frac{1}{2}} + \beta_3(1-\xi)^{\frac{3}{2}} + \beta_4(1-\xi)^{\frac{5}{2}}\right) \tag{3}$$

Equation 4 is the analytical solution for the moment for any arbitrary order of approximation.

$$\int_0^1 h(\xi)\,\xi^n\,d\xi = \frac{1}{\sqrt{2\pi}}\left(\frac{\beta_1\sqrt{\pi}\,\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} + \frac{\beta_2\sqrt{\pi}\,\Gamma(n+1)}{2\Gamma(n+\frac{5}{2})} + \frac{3\beta_3\sqrt{\pi}\,\Gamma(n+1)}{4\Gamma(n+\frac{7}{2})} + \frac{15\beta_4\sqrt{\pi}\,\Gamma(n+1)}{8\Gamma(n+\frac{9}{2})}\right) = \frac{1}{\sqrt{2\pi}}C_n \tag{4}$$

where n=0,1,2,... $\in \mathbb{N}$ and the following established definitions for the gamma-function are reproduced for the sake of convenience:

$$\Gamma(n+1) = n! \tag{5}$$

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{4^n n!}\sqrt{\pi} \tag{6}$$

Table 1. Coefficients β_i for the weight function proposed by [3] and reproduced in Eq. 2; values taken from [4]

a/w	β_1	β_2	β_3	β_4
0.2	2.00	3.3270	1.4351	-0.4652
0.3	2.00	4.9886	1.7280	-0.4130
0.4	2.00	7.2610	2.7054	-0.4570
0.5	2.00	10.4356	5.2943	-0.7632
0.6	2.00	15.1033	11.3700	-1.6671
0.7	2.00	22.6834	26.0237	-4.0924
0.8	2.00	37.2393	69.1970	-11.7568

For a given sequence of abscissae in natural coordinates along with the crack length interval $0 \le \xi_0, \xi_1, \dots, \xi_{1n} < 1 \in \mathbb{R}$, a corresponding sequence of weights w_i can be determined such that the following training function holds exactly:

$$\int_0^1 h(\xi)\, p(\xi)\, d\xi = \sum_{i=1}^{n} w_i p(\xi_i) \tag{7}$$

The improper integral Eq. 1 is then be approximated (theoretically) to any degree of accuracy with an interpolation based integral whose weights are obtained from Eq. 7 for any arbitrary continuous stress distribution function $\sigma(x)$, as follows:

$$K = \int_0^a h(x)\sigma(x)\, dx = \frac{1}{\sqrt{a}} \int_0^1 h(\xi)\sigma(\xi) a\, d\xi \cong \sqrt{a} \sum_{i=1}^{n} w_i \sigma(\xi_i) \tag{8}$$

Note that integration of $\sigma(\xi)$ in the natural coordinate system evokes the 1D *Jacobian* $J = dx / d\xi = d / d\xi(\xi a) = a$ where $dx = a d\xi$. To compute the weights, Eq. 7 is cast into a set of n linear equations with the utilization of Eq. 4 for the right-hand side vector and solved directly for the left-hand side vector w_i.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \xi_0 & \xi_1 & \cdots & \xi_n \\ \vdots & \vdots & \ddots & \vdots \\ \xi_0^n & \xi_1^n & \cdots & \xi_n^n \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{Bmatrix} \tag{9}$$

where the corresponding evenly spaced abscissae in natural coordinates are $\xi_i = xi / a = (0, \frac{1}{n+1}, \frac{2}{n+1}, .., \frac{n}{n+1})$. Note that the integration approach does per se not impose any restrictions on the spacing. For most practical purposes in engineering, a fourth-order approximation with n=4 suffices in which case the weights for the problem are given by Eq. 10 and 11 expressed in fractional numbers for accuracy.

$$w_i = \frac{1}{\sqrt{2\pi}} W_{ij} \beta_j \ \text{ for } \ i = 1,2,..n+1 \text{ and } \ j = 1,2,..n+1 \in \mathbb{N} \tag{10}$$

$$W_{ij} = \begin{bmatrix} 103/189 & 23/297 & 8429/135135 & 1597/27027 \\ -400/189 & 368/2079 & 496/2456 & 4720/27027 \\ 320/63 & 8/33 & 496/9009 & 40/3003 \\ -880/189 & 16/2079 & 1712/27027 & 80/2079 \\ 85/27 & 337/2079 & 67/3861 & 5/27027 \end{bmatrix} \tag{11}$$

Equation 8 is theoretically exact for any polynomial function $\sigma(\xi)$ up to order n and below. The integration approach is straightforward to use manually e.g. with a pocket calculator and conveniently implemented in any numerical fracture analysis platform. On the other hand, a limitation of this approach is the introduction of significant rounding errors for n>10 due to its weights wildly oscillating between large- and small numbers as can be directly inferred from W_{ij} (Eq. 11).

2.2 Gauss Quadrature (GQ) approach

The second integration approach - Gauss Quadrature (GQ) - represents a considerably more accurate method due to a significantly reduced variation of the weights. The price to be paid for the higher fidelity is the more elaborate procedure and computation effort. The basis of the GQ is a sequence of monic orthogonal polynomials of Hermite type, e.g. defined by Corteel et al. [7] with the following recurrence relation in natural coordinates as follows:

$$\begin{aligned} &p_{-1}(\xi) \equiv 0 \\ &p_0(\xi) \equiv 1 \\ &p_{i+1}(\xi) \equiv (\xi - \bar{a}_i) p_i(\xi) - \bar{b}_i p_{i-1}(\xi) \quad i = 1,2,..n \end{aligned} \tag{12}$$

The coefficients of the polynomials (Eq. 12) in natural coordinates are indicated by an overbar and their associated Cartesian expressions are notably:

$$a_i = a\bar{a}_i = a \frac{\langle \xi p_i | p_i \rangle}{\langle p_i | p_i \rangle} \quad i = 0,1,2,..n \tag{13}$$

$$b_0 = \bar{b}_0 \equiv 0 \tag{14}$$

$$b_i = a^2 \bar{b}_i = a^2 \frac{\langle p_i | p_i \rangle}{\langle p_{i-1} | p_{i-1} \rangle} \quad i = 1,2,..n \tag{15}$$

and the inner products in Eq. 13 and 15 are generically defined with Eq. 3 as follows:

$$\langle p_i | p_j \rangle = \int_0^1 p_i(\xi)\, p_j(\xi)\, h(\xi) d\xi = K_i \delta_{ij} \tag{16}$$

where $K_i \neq 0$ is a non-zero constant and δ_{ij} is the well-known Kronecker delta.

Eq. 16 shows that the polynomials have the Kronecker delta property making them orthogonal per definition. The abscissae of the GQ, are defined by the roots of the polynomials and the corresponding weights can ensuing be obtained in closed form with the following expression:

$$w_i = \frac{\langle p_{n-1} | p_{n-1} \rangle}{p_{n-1}|_{\xi_i} \left.\frac{\mathrm{d}p_n}{\mathrm{d}\xi}\right|_{\xi_i}} \tag{17}$$

It needs to be mentioned that a set of monic orthonormal polynomials of order n can alternatively be obtained formally using the Hankel determinant of the moment sequence (Eq. 3) as outlined by Corteel et al. [7]. However, in this approach, the error accumulates in the process of sequentially computing the coefficients and weights. Furthermore, the roots of the polynomials need to be obtained iteratively entailing a significant rounding error for n>10 compromising precision.

In order to mitigate this issue, the approach proposed by Fukuda et al. [8] is applied due to its superior ability to maintain precision for n>10. It is important to compute the recursive coefficients given by Eq. 13 and 14 symbolically e.g. in Maple [9] or Matlab [10] before converting them into numbers. Instead of numerically seeking the roots and subsequently computing the weights with Eq. 17, use is made of the tridiagonal *Jacobi* matrix given by Eq.18 populated with the coefficients (Eq. 13 through 15).

$$J_{n\times n} = a\bar{J}_{n\times n} = a\begin{bmatrix} \bar{a}_0 & \sqrt{\bar{b}_1} & & & & \\ \sqrt{\bar{b}_1} & \bar{a}_1 & \sqrt{\bar{b}_2} & & & \\ & \sqrt{\bar{b}_2} & \bar{a}_2 & \sqrt{\bar{b}_3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \sqrt{\bar{b}_{n-2}} & \bar{a}_{n-2} & \sqrt{\bar{b}_{n-1}} \\ & & & & \sqrt{\bar{b}_{n-1}} & \bar{a}_{n-1} \end{bmatrix} \tag{18}$$

The abscissae and the weights of the GQ are then given by the eigenvalues $\bar{\lambda}_i$ and the eigenvectors $\boldsymbol{\upsilon}_i$ of $\bar{J}$ respectively. The Cartesian abscissae vector components are $x_i = a\xi_i = a\,\bar{\lambda}_i$ and the weights are eventually obtained with the following expression:

$$w_i = \sqrt{a}\,\bar{w}_i = \sqrt{a}\,\frac{\bar{\mu}_0(\upsilon_i \cdot e_i)^2}{\upsilon_i \cdot \upsilon_i} \tag{19}$$

where $\boldsymbol{e}_i=(1,0,..0)_{(1\times n)}$ is a unit vector of length n and $\bar{\mu}_0$ is the moment for n=0 given by Eq. 4.

Eq. 20 is the weighted sum GQ approximation of Eq. 1. The GQ procedure outlined above was implemented into a Matlab script.

$$K = \int_0^a h(x)\sigma(x) dx \cong \sqrt{a}\sum_{i=1}^{n} \bar{w}_i\, \sigma|_{a\xi_i} \tag{20}$$

Table 2 lists the numerical values for the abscissae and the weights for the weight function given in Eq. 2. Table 2 can be used to compute K (Eq. 20) for any regular function $\sigma(x)$ within the interval $[0, a] \in \mathbb{R}$. The coefficients for Eq. 2 [4] are listed in Table 1 for the sake of reproducibility.

3. Comparison of results

Both integration approaches proposed in this work are now tried against the analytical solutions for three different stress distribution functions as shown in Eq. 21 - 23:

$$\sigma(x)_I = -35 - 1.143e3\,x + 6.375e5\,x^2 - 2.854e7\,x^3 + 3.691e8\,x^4 \tag{21}$$

$$\sigma(x)_{II} = -\cos\frac{x\pi}{a} \tag{22}$$

$$\sigma(x)_{III} = e^{20x} \tag{23}$$

Figure 2 depicts Eq. 21 - 23 where the coefficients of the polynomial are chosen in such a way that it emulates a typical distribution from numerical stress analysis with a root along with the interval. Both, the cosine function and the exponential function were chosen for the purpose to investigate the approximation error of non-polynomial functions.

Table 2. Abscissae ξ_i and weights $\bar{w}_i$ for GQ approximation for n=8 for different a/W ratios corresponding to the source data [4] reproduced in Table 1.

	1	2	3	4	5	6	7	8	$\frac{a}{W}$
ξ_i	1.9949E-02	1.0229E-01	2.3929E-01	4.1326E-01	6.0121E-01	7.7711E-01	9.1447E-01	9.9018E-01	0.2
$\bar{w}_i$	1.2712E-01	2.7402E-01	3.7490E-01	4.1961E-01	4.1515E-01	3.8180E-01	3.4383E-01	3.2018E-01	
ξ_i	1.9759E-02	1.0136E-01	2.3726E-01	4.1011E-01	5.9739E-01	7.7358E-01	9.1244E-01	9.8988E-01	0.3
$\bar{w}_i$	1.6572E-01	3.5435E-01	4.7761E-01	5.2217E-01	4.9871E-01	4.3602E-01	3.6914E-01	3.2750E-01	
ξ_i	1.9548E-02	1.0031E-01	2.3496E-01	4.0654E-01	5.9301E-01	7.6944E-01	9.0993E-01	9.8949E-01	0.4
$\bar{w}_i$	2.2677E-01	4.8071E-01	6.3744E-01	6.7920E-01	6.2402E-01	5.1555E-01	4.0543E-01	3.3742E-01	
ξ_i	1.9314E-02	9.9143E-02	2.3238E-01	4.0249E-01	5.8799E-01	7.6455E-01	9.0680E-01	9.8897E-01	0.5
$\bar{w}_i$	3.2956E-01	6.9220E-01	9.0184E-01	9.3422E-01	8.2249E-01	6.3791E-01	4.5957E-01	3.5130E-01	
ξ_i	1.9065E-02	9.7895E-02	2.2960E-01	3.9806E-01	5.8236E-01	7.5889E-01	9.0291E-01	9.8825E-01	0.6
$\bar{w}_i$	5.1289E-01	1.0676E+00	1.3666E+00	1.3755E+00	1.1582E+00	8.3891E-01	5.4567E-01	3.7179E-01	
ξ_i	1.8803E-02	9.6576E-02	2.2661E-01	3.9322E-01	5.7607E-01	7.5226E-01	8.9793E-01	9.8716E-01	0.7
$\bar{w}_i$	8.7795E-01	1.8121E+00	2.2811E+00	2.2321E+00	1.7964E+00	1.2103E+00	6.9939E-01	4.0553E-01	
ξ_i	1.8513E-02	9.5103E-02	2.2325E-01	3.8766E-01	5.6864E-01	7.4408E-01	8.9118E-01	9.8527E-01	0.8
$\bar{w}_i$	1.7895E+00	3.6641E+00	4.5381E+00	4.3173E+00	3.3158E+00	2.0655E+00	1.0383E+00	4.7355E-01	

Table 3. Comparison of trapezoidal rule (TR), the proposed interpolation integration (IP), and Gaussian quadrature (GQ) with the exact solution for Eq. 1 evaluated for the functions given by Eq. 21 through 23 for a crack length of a=0.03; the last three columns are the relative errors ε in [%] relative to the exact solution; the coefficients for a/w=0.3 (see Table 1) were used in this study.

	Exact	TR	Proposed #1 IP	Proposed #2 GQ	ε_{TR}	ε_{IP}	ε_{GQ}
$\sigma(x)_I$	3.9256e0	3.9167e0	3.9280e0	3.9256e0	-2.27e-1	6.07e-2	6.04e-4
$\sigma(x)_{II}$	2.8525e-2	2.8254e-2	2.8543e-2	2.8525e-2	-9.59e-1	6.37e-2	-6.70e-4
$\sigma(x)_{III}$	7.6234e-1	7.6184e-1	7.6255e-1	7.6234e-1	-6.48e-2	2.81e-2	2.25e-5

Table 3 shows the numerical values for the stress intensity factor obtained with the different methods for a=0.03. Both, the interpolation approach and the GQ are using an n=8 approximation. Comparison of the relative error in the last three columns of Table 3 shows that the trapezoidal integration (Matlab command 'trapz') for a chosen number of 1×10^6 integration points produces the largest error especially in cases $\sigma(x)_I$ and $\sigma(x)_{II}$ exhibiting a root along [0, a]. The interpolation-based method provides at least one order of magnitude more accurate results, whereas the GQ provides several orders of magnitude more accuracy for the investigated cases.

Figure 3(a) shows that the error of the trapezoidal rule oscillates in the case of the polynomial stress distribution because both, the shape of the polynomial and the discretization change with the crack length. The error of the interpolation method is more steady and significantly smaller indicating that this method is not affected by the former. The error in Figure 3(b) appears to be constant for all three methods since the shape of the cosine function is preserved. Figure 3(c) shows that the error of the trapezoidal rule increases with increasing crack length. Contrariwise, the error of the proposed methods is smaller and remains constant. Figure 3(a-c) shows that in all cases the error of the GQ method is distinctly smaller compared to the other methods.

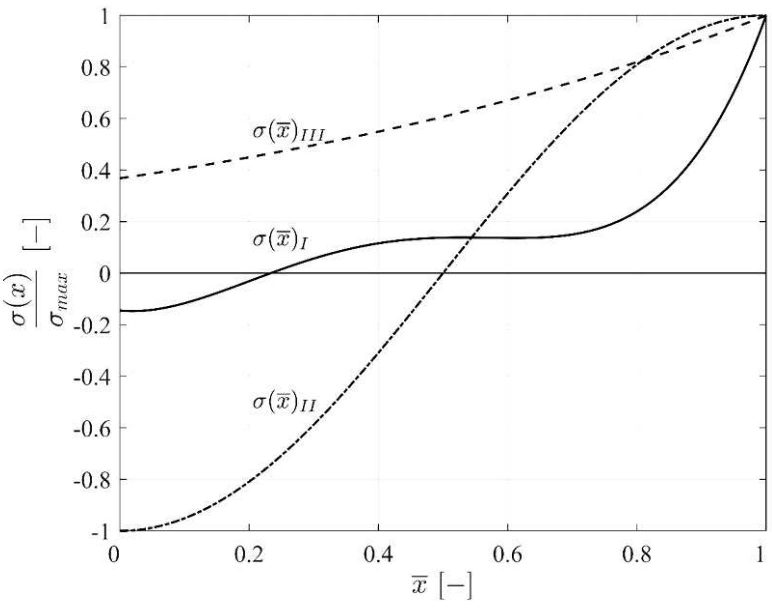

Fig. 2. Three different normalized stress distributions as a function of the normalized crack length where $\sigma(x)_I$, $\sigma(x)_{II}$ and $\sigma(x)_{III}$ are a fourth-order polynomial, a cosine function, and an exponential function.

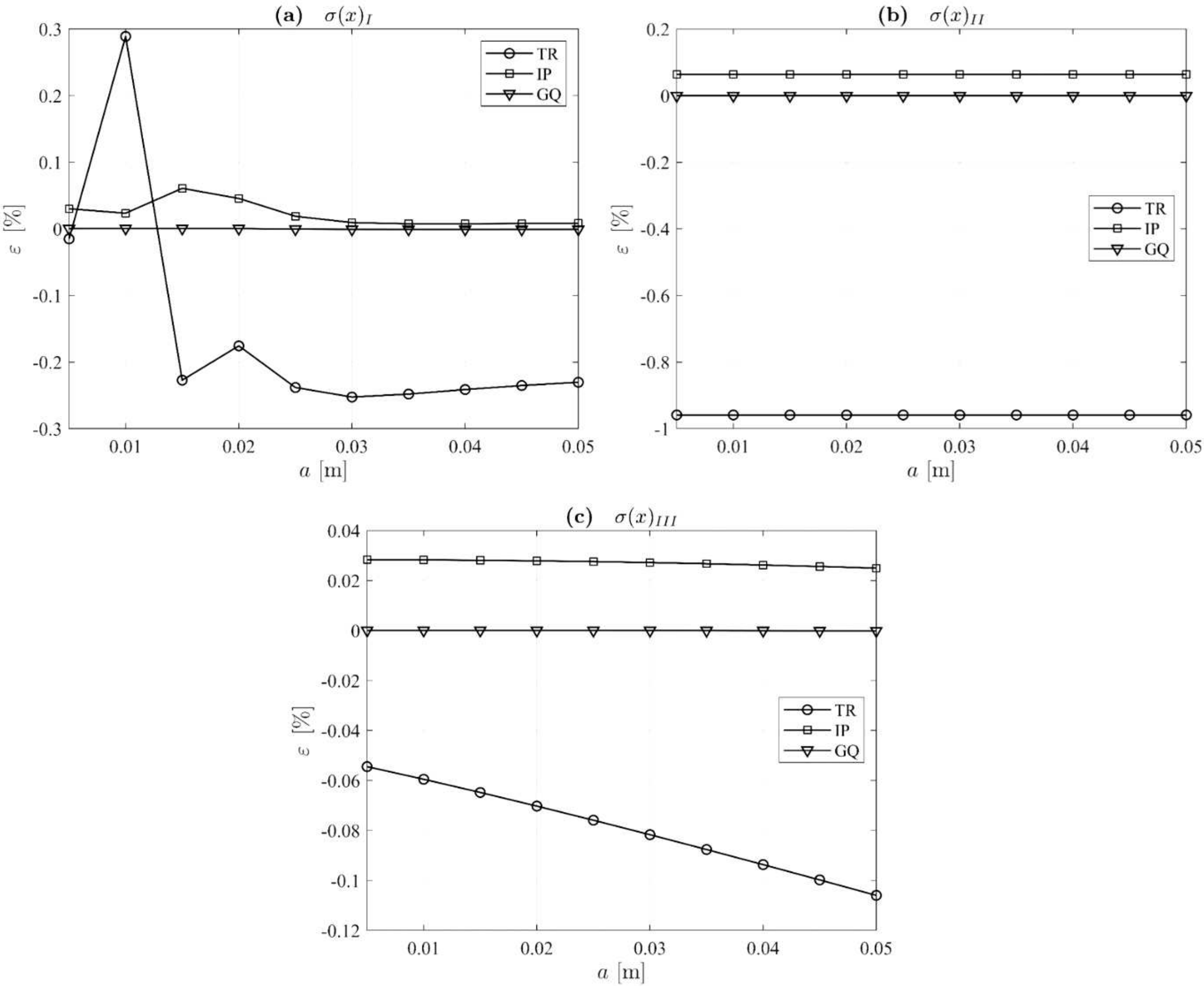

Fig. 3. Relative error [%] of the stress intensity factors using three integration methods namely the trapezoidal rule (TR), the interpolation method (IP), and the Gauss quadrature (GQ) evaluated for ten different crack lengths for (a) the fourth-order polynomial (Eq. 21) and (b) the cosine function (Eq. 22) and (c) the exponential function (Eq. 23)

4. Concluding remarks

The failure of cracked components is governed by the stresses in the vicinity of the crack tip. The weight function method represents an important tool for structural integrity assessment, for example, of cracked large offshore structures, due to its ability to predict the stress intensity factor without the necessity to model the crack discretely. Two weight-based integration approaches based on interpolation integration and Gaussian quadrature integration are proposed in this study to complement integration strategies in the existing literature for the weight function method. Comparing to the exact solutions of the three investigated cases, the proposed two approaches show excellent accuracy. Particularly, the Gaussian quadrature integration proves to be superior by providing several orders of magnitude more accuracy than the trapezoidal integration. It needs to be emphasized that a piecewise differentiable and continuous $\sigma(x)$ distribution suffices for two proposed approaches. Since the stress values are most accurately predicted in the integration points which requires extrapolation to nodal stresses that are located on the crack faces. Extrapolation of stresses from Gauss points to nodal stresses is done by utilizing the isoparametric shape functions. Depending on the smoothness of the stress distribution, a piecewise linear interpolation between nodal stress values can be used. That is to say, a piecewise linear interpolation of the nodal stress from finite element analysis can be used for most practical purposes provided that the variation is reasonably smooth. Implementation and demonstration of the proposed method within the framework of the finite element method is future research.

Author Contributions

M.A. Eder planned the scheme and made theoretical derivation; X. Chen analyzed and assessed results and revised the manuscript. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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ORCID iD

Martin A. Eder https://orcid.org/0000-0002-5306-365X
Xiao Chen https://orcid.org/0000-0001-6726-4068

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