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Research Paper

Regularization of the Movement of a Material Point Along a Flat Trajectory: Application to Robotics Problems

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Abstract. A control problem of the robot's end-effector movement along a predefined trajectory is considered. The aim is to reduce the work against resistance forces and improve the comfortability of the motion. The integral of kinetic energy and weighted inertia forces for the whole period of motion is introduced as a cost functional. By applying variational methods, the problem is reduced to a system of quasilinear ordinary differential equations of the fourth order. Numerical examples of solving the problem for movement along straight, circular and elliptical trajectories are presented. For the sake of clarity, the problem is studied for a specific kind of a 3D printer in the 2DoF approximation. However, in the case of negligible masses of moving elements compared the mass of an end-effector, the solution is universal, i.e., it remains the same for given trajectories.

Keywords: Numerical computing, Optimal velocities, Law of motion, Predefined trajectory, Minimal inertia.

1. Introduction

In many practical problems the trajectory of the robot's end-effector is known in advance and is defined by the performed operation. To improve the quality of work and robot control, there are problems of precise positioning and following a given trajectory. However, in order to improve the technical parameters of the system, it is also necessary to take into account the control of the speed of the movement along the same trajectory. But in comparison with the control of kinematic indicators, the study of indicators of dynamic characteristics is developing slowly due to the wide variety of robots and the complexity of their dynamic models. Though, due to the widespread use of robots in recent years, the study of dynamic performance indicators has expanded.

Optimization problems of the movement of robots' working tools have many different statements. Part of the formulations concerns determining the optimal trajectory, see, for instance the review [1] and references therein. Another direction of research concerns improvement of different performance indices and some key mathematical expressions of the indices are provided as well [2, and references therein]. Large number of articles are devoted to the kinematic performance evaluation and optimization as well, for instance, articles [4]-[11]. To improve the quality of work and control of the robot there are also problems of accurate positioning and following the trajectory [12, and references therein].

Compared to control of kinematic indices, the study of dynamic performances indices are developed slowly because of high diversity of robots and complexity of their dynamic models. However, because of wide applications of robots in recent years, the study of the dynamic performance indices is increasing [13]-[16].

Planar movements of the working tool are implemented by various robots, both sequential and parallel structure. An example is robot of the DexTAR type (dexterous twin-arm robot) [17], [21], as well as parallel robots with three degrees of freedom of 3RPR type. Several types of planar robots are described in Merlet's monograph [22]. The 3D printers can be partially considered as planar robots as well, cause the printer head is moving mostly in a horizontal plane and is followed by the movement of the platform or of the instrument along vertical axis. For instance, 3D printers' brands Anet, Anycubic, Wanhao can be named. The scope of application of 3D printers is constantly expanding, now they are applied not only in printing of small-volume details, but also in the production of large-sized structural elements. For robots regularly used in these cases, energy consumption indicators are of great importance. Energy consumption is determined by the work of external forces, which in turn depends on their work against friction and the medium resistance, as well as on kinetic and potential energies of the mechanical system. That is the main reason motivating us to state the proposed control problem.

In this work, we will consider the problem of optimizing the movement of the 2DoF robot's working tool along a given trajectory, in which the energy consumption is minimal for friction and inertia forces over a given period of time. It will be shown below that the cost functional in this case is expressed through a combination of the integral of the kinetic energy and of the root-mean-square norm of inertia forces. For the best of our knowledge, the proposed approach has not been applied for the robotic problems yet.

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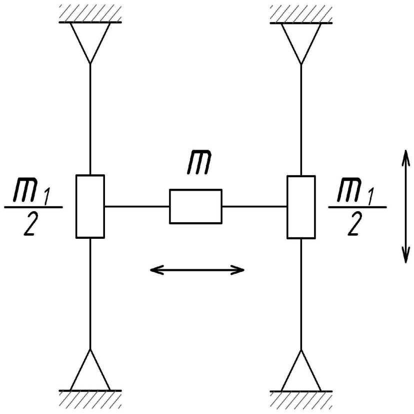

Fig. 1. A structure scheme of the robot (left picture) and an example of its application(3d printer Creality Ender 3).

2. Mathematical Formulation and Numerical Procedure

We describe here a method of constructing the optimal law of motion along a given planar trajectory with minimizing work of viscous friction and inertia forces for a given time interval. For the sake of clarity a widespread kinematic scheme of 3D printers is considered.

For many printer brands, the working tool is operated by moving actuators along three mutually perpendicular rectilinear guides. Here, we consider a robot of a similar structure, the kinematic diagram of it is shown in Figure 1. Here m denotes the total mass of the working tool plus the part of the robot moving along the Ox axis, m_1 is the total mass of the moving part of the two actuators located on two parallel guides along the Oy axis. Suppose that the these actuators are moving synchronously.

Let the Oxy plane of the Cartesian coordinate system coincide with the plane of the robot's movement. The following assumptions are accepted:

a) the movement is carried out by moving the actuators along the guides located parallel to the axes Ox, Oy the Cartesian system;

b) the trajectory of the working tool in the Oxy plane is predefined;

c) the time of movement along the trajectory is fixed;

d) the module of resistance forces in the mechanism is proportional to the product of the mass of the moving elements by the module of velocity;

e) we neglect the change in the potential energy of the system.

Let the position of the working tool be defined by the coordinates of the center of mass (x_C, y_C) in the motion plane.

Let the trajectory of the center of gravity of the moving working tool be known and is given in the coordinate plane Oxy by the following parameterization:

$$\begin{aligned} x_C &= x(p), \\ y_C &= y(p). \end{aligned} \tag{1}$$

Then, the dependence of the function $p(t)$ on time t defines the movement of the working tool along the trajectory:

$$\begin{aligned} x_C(t) &= x(p(t)), \\ y_C(t) &= y(p(t)). \end{aligned} \tag{2}$$

For instance, in a straight-line movement, the parameter p can be set equal to some coordinate; in a circular movement, it can be set equal to the rotation angle. In general case, the parameter can be selected based on any convenient way of the trajectory definition.

Let us assume that the positions and velocities of the platform at the initial and final time moments are given, and the time scale is chosen in such a way that the time changes in the interval [0,1]. We also assume that all values are reduced to a dimensionless form, by selecting appropriate length and velocity scales.

Suppose that at the initial and final moments the values of the function p (t) are given:

$$\begin{aligned} p(0) &= p_0, \\ p(1) &= p_1, \end{aligned} \tag{3}$$

and at that moments the velocities are zero, i.e. these points are stopping points of the movement:

$$\begin{aligned} &x(p_0) = x_0, \quad x(p_1) = x_1, \\ &y(p_0) = y_0, \quad y(p_1) = y_1, \\ &\dot{x}(p_0) = 0, \quad \dot{x}(p_1) = 0, \\ &\dot{y}(p_0) = 0, \quad \dot{y}(p_1) = 0. \end{aligned} \tag{4}$$

The dots above functions' symbol, as usual, denote time derivatives. Prime symbols near the functions x(p), y(p) denote differentiation with respect to the parameter p.

a) To state the optimization problem, let us derive the work of the friction forces. Actually, the resistance to motion depends on the design of the drive. By the assumption d) above, the resistance to the motion is proportional to the velocity and the mass of the moving part. Denote by f the coefficient of resistance to movement. Let dx and dy be replacements of the working tool in the time interval dt. Note that the movement of dx is performed only by the mass m, and the movement of dy is simultaneously performed not only by the mass m but also by other structural elements (rod AB, movable elements of the guides CD, EF). The total mass of these structural elements has been denoted by $m1$. Then, taking into account expressions $dx = \dot{x}dt, dy = \dot{y}dt$, the elementary work of the friction forces is written in the form:

$$dA = F_x \cdot dx + F_y \cdot dy = fm\dot{x}dx + f(m_1 + m)\dot{y}dy = f\left(m\dot{x}^2 + (m_1 + m)\dot{y}^2\right)dt.$$

It follows from above that the elementary work of the friction forces is proportional to the kinetic energy of the moving parts of the mechanism:

$$E_k = \frac{m\dot{x}^2}{2} + \frac{(m_1 + m)\dot{y}^2}{2}$$

The total work of the friction forces for the whole period of motion is expressed by the integral over time of the elementary work:

$$A = f\int_0^1 \left(m\dot{x}^2 + (m + m_1)\dot{y}^2\right)dt = fm\int_0^1 \left(\dot{x}^2 + \mu\dot{y}^2\right)dt = 2f\int_0^1 E_k(t)dt. \tag{5}$$

Here $\mu = (m + m_1)/m \geq 1$ is the ratio of the masses of moving parts to the mass of the tool. In this formulation, we have neglected the static friction force, which is important only at the start of the movement.

By the formula (5), the work of the resistance forces is proportional to the average kinetic energy of the mechanism, so the task of minimizing this work is equivalent to minimizing the integral of the kinetic energy for the entire period of movement. To improve the quality of the movement, we will also include in the cost functional a measure of inertia forces. Such a formulation makes sense if it is aimed to reduce the power consumption of the robot and to obtain the law of motion with sufficiently low inertia forces.

Note that a similar problem has been formulated in our preprint [18], but the relationship between kinetic energy and the work of the resistance forces, as well as the structure of the mechanism, has been not considered in [18].

Now let us consider and estimate inertia forces and deduce the formula for the cost functional. Let $a_C = \sqrt{\ddot{x}_C{}^2 + \ddot{y}_C{}^2}$ be a module of the acceleration of the working tool. As a measure of the magnitude of inertia forces applied to the moving masses m and m_1 for the whole movement time, we consider the integral of the squared inertia force, taken over the time interval [0,1]:

$$\begin{aligned}\|F_i\|^2 &= \int_0^1 \left(m^2 a_C^2 + m_1^2 \ddot{y}_C{}^2\right)dt = m^2\int_0^1 \left(\ddot{x}_C{}^2 + \ddot{y}_C{}^2\right)dt + m_1^2\int_0^1 \ddot{y}_C{}^2 dt \\ &= m^2\int_0^1 \ddot{x}_C{}^2 dt + \left(m^2 + m_1^2\right)\int_0^1 \ddot{y}_C{}^2 dt = m^2\left(\int_0^1 \ddot{x}_C{}^2 dt + \nu\int_0^1 \ddot{y}_C{}^2 dt\right).\end{aligned}$$

here $\nu = \left(m^2 + m_1^2\right)/m^2$.

Further, for the sake of brevity, we omit the “C” index near the mass center coordinates of the working tool; as well as the scale of the mass is chosen to be equal to m, i.e. m = 1.

Finally, the cost functional in the form of the sum of kinetic energy and weighted squared norm of inertia forces is defined as follows:

$$J(p) = \frac{1}{2}f\int_0^1 \left(\dot{x}^2(t) + \mu\dot{y}^2(t)\right)dt + \frac{\alpha}{2}\left(\int_0^1 \ddot{x}_C{}^2 dt + \nu\int_0^1 \ddot{y}_C{}^2 dt\right). \tag{6}$$

Here we have taken into account that m =1. Here $\alpha > 0$ is a positive parameter that determines the weight of inertia forces in the cost functional (6). The larger the parameter α is, the greater be the influence of inertia forces on the cost functional. Parameters $\mu > 1$ and $\upsilon > 1$ depends on masses m and m_1. Then, we have the following

Formulation of the optimization problem.

It is necessary to minimize the functional (6) with respect to the function p(t) that define the law of motion along the trajectory (x(t), y(t)) according to the formulas (1)-(2).

Formula (6) is a quadratic norm of the two-dimensional vector function (x(t), y(t)) in the Sobolev’s space $W_2^2[0,1]$ formed by functions with quadratically summed derivatives up to second order [23].

It can be shown that this space is a Hilbert’s space [23], [24] and functional (6), as the norm of a function in a Hilbert’s space, is strongly convex [25]. However, we will consider it as a functional defined on a subset of quadratically summed functions $L_2[0,1]$. Then any bounded and closed set of $W_2^2[0,1]$ is compact by virtue of the embedding theorems [23]. According to [25], the functional reaches its minimum on a convex compact set. Besides, if the functional is differentiable, then at the minimum point its Frechet derivative is equal to zero. Moreover, this condition is sufficient for the minimality of the cost functional [25].

To determine the Frechet derivative of the cost functional, let us first derive the formula for the first variation of the functional (6). We assume that all arguments p(t) and increments of the function $\delta p(t)$ lie in some bounded convex set from $W_2^2[0,1]$. Then the increment of the functional (6) corresponding to the increment of the argument p is written as follows:

$$\begin{aligned}\delta J(p) = J(p + \delta p) - J(p) &= \frac{f}{2}\int_0^1 \left(\dot{x}^2(p + \delta p) - \dot{x}^2(p) + \mu\dot{y}^2(p + \delta p) - \mu\dot{y}^2(p)\right)dt \\ &+ \frac{\alpha}{2}\int_0^1 \left(\ddot{x}^2(p(t) + \delta p(t)) - \ddot{x}^2(p(t)) + \nu\ddot{y}^2(p(t) + \delta p(t)) - \nu\ddot{y}^2(p(t))\right)dt.\end{aligned} \tag{7}$$

Taking into account that

$$\dot{x} = \frac{dx(p)}{dp}\dot{p} = x'\dot{p},\ \dot{y} = \frac{dy(p)}{dp}\dot{p} = y'\dot{p}\ , \tag{8}$$

the linear part of the following variation is computed:

$$\begin{aligned}\delta\dot{x}^2(p) &= 2\dot{x}\delta\dot{x}(p) = 2\dot{x}\left(x'(p+\delta p)(\dot{p}+\delta\dot{p}) - x'(p)\dot{p}\right) = \\ &= 2\dot{x}\left(\left(x'(p) + x''(p)\delta p\right)\dot{p} + x'(p+\delta p)\delta\dot{p} - x'(p)\dot{p}\right) = \\ &= 2\dot{x}\left(x''(p)\dot{p}\delta p + x'(p)\delta\dot{p}\right),\end{aligned} \tag{9}$$

i.e.

$$\delta\dot{x}^2(p) = 2\dot{x}\left(x''\dot{p}\delta p + x'\delta\dot{p}\right).$$

The similar formula is valid for increment $\delta\dot{y}^2(p)$. For the second derivative, we have

$$\begin{aligned}\ddot{x} &= \frac{d}{dt}(x'\dot{p}) = x''(p)\dot{p}^2 + x'(p)\ddot{p}, \\ \ddot{y} &= y''(p)\dot{p}^2 + y'(p)\ddot{p}.\end{aligned} \tag{10}$$

Taking into account expressions (10), for the increment of the second derivative $\delta\ddot{x}^2(p)$, we obtain:

$$\begin{aligned}\delta\ddot{x}^2(p) &= 2\ddot{x}\delta\ddot{x}(p) = 2\ddot{x}\left(\frac{d^2x(p+\delta p)}{dp^2}(\dot{p}^2 + \delta\dot{p}^2) + \frac{dx(p+\delta p)}{dp}(\ddot{p} + \delta\ddot{p})\right. \\ &\left. - \frac{d^2x(p)}{dp^2}\dot{p}^2 - \frac{dx(p)}{dp}\ddot{p}\right) = 2\ddot{x}\left(x'''(p)\dot{p}^2\delta p + x''(p)(2\dot{p}\delta\dot{p} + \ddot{p}\delta p) + x'\delta\ddot{p}\right).\end{aligned} \tag{11}$$

The formula for increment of the second derivative $\delta\ddot{y}^2(p)$ is written similarly. Taking into account formulas (9) - (11) for the linear part of the functional increment, we have:

$$\begin{aligned}\delta J(p) &= f\int_0^1 \dot{x}\left(x''\dot{p}\delta p + x'\delta\dot{p}\right) + \mu\dot{y}\left(y''\dot{p}\delta p + y'\delta\dot{p}\right)dt \\ &+ \alpha\int_0^1 \ddot{x}\left(x'''\dot{p}^2\delta p + x''(2\dot{p}\delta\dot{p} + \ddot{p}\delta p) + x'\delta\ddot{p}\right)dt \\ &+ \alpha\nu\int_0^1 \ddot{y}\left(y'''\dot{p}^2\delta p + y''(2\dot{p}\delta\dot{p} + \ddot{p}\delta p) + y'\delta\ddot{p}\right)dt.\end{aligned} \tag{12}$$

Considering functional (12) as a functional defined on a subset of quadratically summable functions $W_2^2[0,1]$ in $L_2[0,1]$, we represent its increment in the form

$$\delta J(p) = \langle J'(p), \delta p\rangle = \int_0^1 J'(p(t))\delta p(t)dt. \tag{13}$$

Let the functions $x(p)$ and $p(t)$ be sufficiently smooth, so that all our derivatives are legitimate. The necessaire smoothness order, for which the representation (13) is acceptable, will be clear from the calculations below. In this way, the corresponding subspace of functions from $L_2[0,1]$ will be defined, where the functional (5) is differentiable in the Frechet sense; then the condition of equality to zero of the Frechet derivative will be applied to find the minimum point.

Collecting the coefficients at $\delta p, \delta\dot{p}, \delta\ddot{p}$ in (12), we rewrite expression (12) in the following form:

$$\delta J(p) = \int_0^1 f_1(t)\delta p dt + \int_0^1 f_2(t)\delta\dot{p} dt + \int_0^1 f_3(t)\delta\ddot{p} dt \equiv J_1(p) + J_2(p) + J_3(p), \tag{14}$$

where the functions $f_i(p), i = \overline{1,3}$ are defined by the formulas:

$$\begin{aligned}f_1(t) &= f\left(\dot{x}x'' + \mu\dot{y}y''\right)\dot{p} + \alpha\left(\ddot{x}x''' + \nu\ddot{y}y'''\right)\dot{p}^2 + \alpha\left(\ddot{x}x'' + \nu\ddot{y}y''\right)\ddot{p}, \\ f_2(t) &= f\left(\dot{x}x' + \mu\dot{y}y'\right) + 2\alpha\left(\ddot{x}x'' + \nu\ddot{y}y''\right)\dot{p}, \\ f_3(t) &= \alpha\left(\ddot{x}x' + \nu\ddot{y}y'\right).\end{aligned} \tag{15}$$

In formulas (15), we should substitute the time derivatives of the functions $x(t)$, $y(t)$ from formulas (8) and (10), which we did not do in order not to clutter the description.

Integrating by parts in (14), we obtain:

$$\begin{aligned}J_2(p) &= \int_0^1 f_2(p)\delta\dot{p} dt = f_2(p)\delta p\Big|_{t=0}^{t=1} - \int_0^1 \dot{f}_2\delta p dt = -\int_0^1 \dot{f}_2\delta p dt, \\ J_3(p) &= \int_0^1 f_3(p)\delta\ddot{p} dt = f_3(p)\delta\dot{p}\Big|_{t=0}^{t=1} - \int_0^1 \dot{f}_3\delta\dot{p} dt = -\dot{f}_3\delta p\Big|_{t=0}^{t=1} + \int_0^1 \ddot{f}_3\delta p dt = \int_0^1 \ddot{f}_3(t)\delta p dt.\end{aligned}$$

Hence

$$\delta J(p) = \left\langle \left(f_1(t) - \dot{f}_2(t) + \ddot{f}_3(t) \right), \delta p \right\rangle.$$

Then, if the functions $f_2(t)$, $f_3(t)$ are sufficiently smooth, the functional $J(p)$ is differentiable in the Frechet sense with respect to p, and its gradient is given by the function

$$J'(p) = f_1(t) - \dot{f}_2(t) + \ddot{f}_3(t).$$

The equation that allows to determine the minimum point of the functional has the form

$$f_1(t) - \dot{f}_2(t) + \ddot{f}_3(t) = 0, \tag{16}$$

with boundary conditions following from formulas (3) and (4):

$$p(0) = p_0,\ \dot{p}(0) = 0,\ p(1) = p_1,\ \dot{p}(1) = 0.$$

For the convenience of applying standard mathematical packages, we reduce equation (15) to a system of quasilinear equations. For this aim, the following notations are used:

$$\begin{aligned} z_0(t) &= p(t), \\ z_1(t) &= \dot{p}(t), \\ z_2(t) &= \ddot{p}(t), \\ z_3(t) &= \dddot{p}(t). \end{aligned} \tag{17}$$

Then equalities (8) and (10) can be rewritten in the form:

$$\begin{aligned} &\dot{x} = x'(z_0(t))z_1(t),\ \dot{y} = y'(z_0(t))z_1(t), \\ &\ddot{x} = x''(z_0(t))z_1^2(t) + x'(z_0(t))z_2(t), \\ &\ddot{y} = y''(z_0(t))z_1^2(t) + y'(z_0(t))z_2(t). \end{aligned} \tag{18}$$

In the future, we will omit the argument $(z_0(t))$ to shorten the representation. Next we need expressions for the following time derivative of third order:

$$\begin{aligned} \dddot{x} &= x'''z_1^3 + 3x''z_1z_2 + x'z_3, \\ \dddot{y} &= y'''z_1^3 + 3y''z_1z_2 + y'z_3, \end{aligned} \tag{19}$$

Taking into account the notations (17) and expressions (18), the summand $f_1(t)$ in equation (16) has the form:

$$\begin{aligned} f_1(t) &= f\left(\dot{x}x'' + \mu\dot{y}y''\right)z_1(t) + \alpha\left(\ddot{x}x''' + \nu\ddot{y}y'''\right)z_1^2(t) + \alpha\left(\ddot{x}x'' + \nu\ddot{y}y''\right)z_2(t) \\ &= f\left(x'x'' + \mu y'y''\right)z_1^2(t) + \alpha\left(x''x''' + \nu y''y'''\right)z_1^4(t) + \alpha\left(x'x''' + \nu y'y'''\right)z_1^2(t)z_2(t) + \alpha\left(x''^2 + \nu y''^2\right)z_1^2(t)z_2(t) + \alpha\left(x'x'' + \nu y'y''\right)z_2^2(t) \\ &= f\left(x'x'' + \mu y'y''\right)z_1^2(t) + \alpha\left(x''x''' + \nu y''y'''\right)z_1^4(t) + \alpha\left(\left(x'x''' + \nu y'y'''\right) + \left(x''^2 + \nu y''^2\right)\right)z_1^2(t)z_2(t) + \alpha\left(x'x'' + \nu y'y''\right)z_2^2(t). \end{aligned} \tag{20}$$

Now calculate the time derivative of the function $f_2(t)$:

$$\begin{aligned} \dot{f}_2(t) &= f\frac{d}{dt}\left(\dot{x}x' + \mu\dot{y}y'\right) + 2\alpha\frac{d}{dt}\left(\left(\ddot{x}x'' + \nu\ddot{y}y''\right)\dot{p}\right) \\ &= f\left(\ddot{x}x' + \mu\ddot{y}y'\right) + f\left(\dot{x}x'' + \mu\dot{y}y''\right)\dot{p} + 2\alpha\left(\dddot{x}x'' + \nu\dddot{y}y''\right)\dot{p} + 2\alpha\left(\ddot{x}x''' + \nu\ddot{y}y'''\right)\dot{p}^2 + 2\alpha\left(\ddot{x}x'' + \nu\ddot{y}y''\right)\ddot{p}. \end{aligned} \tag{21}$$

Let's write separately the components of expression (21) in terms of notations (17):

$$\begin{aligned} &f\left(\ddot{x}x' + \mu\ddot{y}y'\right) + f\left(\dot{x}x'' + \mu\dot{y}y''\right)\dot{p} = f\left((x''z_1^2 + x'z_2)x' + \mu(y''z_1^2 + y'z_2)y'\right) + f\left(x'x''z_1 + \mu y'y''z_1\right)z_1 \\ &\qquad = f\left(2(x''x' + \mu y''y')z_1^2 + (x'^2 + \mu y'^2)z_2\right), \\ &\left(\dddot{x}x'' + \nu\dddot{y}y''\right)\dot{p} + \left(\ddot{x}x''' + \nu\ddot{y}y'''\right)\dot{p}^2 = (x''x''' + \nu y''y''')z_1^4 + 3(x''^2 + \nu y''^2)z_1^2z_2 + (x'x'' + \nu y'y'')z_1z_3 + (x''x''' + \nu y''y''')z_1^4 + (x'x''' + \nu y'y''')z_1^2z_2 \\ &\qquad = 2(x''x''' + \nu y''y''')z_1^4 + \left(3(x''^2 + \nu y''^2) + x'x''' + \nu y'y'''\right)z_1^2z_2 + (x'x'' + \nu y'y'')z_1z_3, \\ &\left(\ddot{x}x'' + \nu\ddot{y}y''\right)\ddot{p} = (x''^2z_1^2z_2 + x'x''z_2^2) + \nu(y''^2z_1^2z_2 + y'y''z_2^2) = (x''^2 + \nu y''^2)z_1^2z_2 + (x'x'' + \nu y'y'')z_2^2. \end{aligned} \tag{22}$$

Now, taking into account expressions (22), we can rewrite the formula for the coefficient $\dot{f}_2(t)$ in equation (16)

$$\begin{aligned} \dot{f}_2(t) &= f\left(2(x''x' + \mu y''y')z_1^2 + (x'^2 + \mu y'^2)z_2\right) \\ &\quad + 4\alpha(x''x''' + \nu y''y''')z_1^4 + 2\alpha\left(4(x''^2 + \nu y''^2) + x'x''' + \nu y'y'''\right)z_1^2z_2 + 2\alpha(x'x'' + \nu y'y'')z_1z_3 + 2\alpha(x'x'' + \nu y'y'')z_2^2. \end{aligned} \tag{23}$$

We calculate the derivatives of the function $f_3(t)$ in (15):

$$\ddot{f}_3(t) = \alpha \frac{d^2}{dt^2}\left(\ddot{x}x' + \nu\ddot{y}y'\right),$$
$$\frac{d}{dt}\left(\ddot{x}x' + \nu\ddot{y}y'\right) = \dddot{x}x' + \nu\dddot{y}y' + \ddot{x}x''\dot{p} + \nu\ddot{y}y''\dot{p}, \tag{24a}$$
$$\frac{d^2}{dt^2}\left(\ddot{x}x' + \nu\ddot{y}y'\right) = \ddddot{x}x' + \nu\ddddot{y}y' + 2\dddot{x}x''\dot{p} + 2\nu\dddot{y}y''\dot{p} + \ddot{x}x'''\dot{p}^2 + \nu\ddot{y}y'''\dot{p}^2 + \ddot{x}x''\ddot{p} + \nu\ddot{y}y''\ddot{p}.$$

Using the notations (17) and equalities (18), (19), we have:

$$\begin{aligned}\frac{d^2}{dt^2}\left(\ddot{x}x' + \nu\ddot{y}y'\right) &= \ddddot{x}x' + \nu\ddddot{y}y' + 2(\dddot{x}x'' + \nu\dddot{y}y'')z_1 + (\ddot{x}x''' + \nu\ddot{y}y''')z_1^2 + (\ddot{x}x'' + \nu\ddot{y}y'')z_2 \\ &= \ddddot{x}x' + \nu\ddddot{y}y' + 3(x''x''' + \nu y''y''')z_1^4 + 7(x''^2 + \nu y''^2)z_1^2z_2 + 2(x'x'' + \nu y'y'')z_1z_3 + (x'x''' + \nu y'y''')z_1^2z_2 + (x'x'' + \nu y'y'')z_2^2.\end{aligned} \tag{24b}$$

In formula (24b), we need formulas for the fourth time derivative of the coordinates $x(t)$ and $y(t)$. Differentiating (19), we get:

$$\begin{aligned}\ddddot{x} &= \frac{d}{dt}\left(x'''z_1^3 + 3x''z_1z_2 + x'z_3\right) \\ &= x^{(IV)}z_1^4 + 3x'''z_1^2z_2 + x''z_1z_3 + 3x'''z_1^2z_2 + 3x''z_2^2 + 3x''z_1z_3 + x'\dot{z}_3 \\ &= x^{(IV)}z_1^4 + 6x'''z_1^2z_2 + 4x''z_1z_3 + 3x''z_2^2 + x'\dot{z}_3.\end{aligned} \tag{25}$$

The formulas for the fourth time derivative of the function $y(t)$ have the similar form.
Substitute (25) into (24b):

$$\begin{aligned}\frac{d^2}{dt^2}\left(\ddot{x}x' + \nu\ddot{y}y'\right) &= \left(x^{(IV)}z_1^4 + 6x'''z_1^2z_2 + 4x''z_1z_3 + 3x''z_2^2 + x'\dot{z}_3\right)x' + \nu\left(y^{(IV)}z_1^4 + 6y'''z_1^2z_2 + 4y''z_1z_3 + 3y''z_2^2 + y'\dot{z}_3\right)y' \\ &+ 3(x''x''' + \nu y''y''')z_1^4 + 7(x''^2 + \nu y''^2)z_1^2z_2 + 2(x'x'' + \nu y'y'')z_1z_3 + (x'x''' + \nu y'y''')z_1^2z_2 + (x'x'' + \nu y'y'')z_2^2.\end{aligned} \tag{26}$$

Reducing similar terms in (26), we get:

$$\begin{aligned}\frac{d^2}{dt^2}\left(\ddot{x}x' + \nu\ddot{y}y'\right) &= \left(x^{(IV)}x' + \nu y^{(IV)}y' + 3(x''x''' + \nu y''y''')\right)z_1^4 + \left(x'^2 + \nu y'^2\right)\dot{z}_3 + 7(x''^2 + \nu y''^2 + x'x''' + \nu y'y''')z_1^2z_2 \\ &+ 6(x'x'' + \nu y'y'')z_1z_3 + 4(x'x'' + \nu y'y'')z_2^2.\end{aligned} \tag{27}$$

By substituting (20), (23), (27) into (16), we rewrite equation (16) in expanded form in the notations (2) and (17):

$$\begin{aligned}& f\left(x'x'' + \mu y'y''\right)z_1^2(t) + \alpha\left(x''x''' + \nu y''y'''\right)z_1^4(t) + \alpha\left(x'x''' + \nu y'y''' + x''^2 + \nu y''^2\right)z_1^2(t)z_2(t) + \alpha\left(x'x'' + \nu y'y''\right)z_2^2(t) \\ & -f\left(2(x''x' + \mu y''y')z_1^2 + (x'^2 + \mu y'^2)z_2\right) - 4\alpha(x''x''' + \nu y''y''')z_1^4 - 2\alpha\left(4(x''^2 + \nu y''^2) + x'x''' + \nu y'y'''\right)z_1^2z_2 \\ & -2\alpha(x'x'' + \nu y'y'')z_1z_3 - 2\alpha(x'x'' + \nu y'y'')z_2^2 + \alpha\left(x^{(IV)}x' + \nu y^{(IV)}y' + 3(x''x''' + \nu y''y''')\right)z_1^4 + \alpha\left(x'^2 + \nu y'^2\right)\dot{z}_3 \\ & +7(x''^2 + \nu y''^2 + x'x''' + \nu y'y''')z_1^2z_2 + 6\alpha(x'x'' + \nu y'y'')z_1z_3 + 4\alpha(x'x'' + \nu y'y'')z_2^2 = 0.\end{aligned} \tag{28}$$

By regrouping and collecting similar items in (28), we get:

$$\begin{aligned}& \alpha\left(x'^2 + \nu y'^2\right)\dot{z}_3 - f(x'^2 + \mu y'^2)z_2 - f\left(x'x'' + \mu y'y''\right)z_1^2 + \alpha\left(x'x'' + \nu y'y''\right)\left(3z_2^2 + 4z_1z_3\right) + 6\alpha\left(x'x''' + \nu y'y'''\right)z_1^2z_2 \\ & +\alpha\left(x^{(IV)}x' + \nu y^{(IV)}y'\right)z_1^4 = 0.\end{aligned} \tag{29}$$

Equation (29) can be written in the form resolved with respect to $\dot{z}_3$:

$$\dot{z}_3 = \frac{f}{\alpha}\frac{\left(x'^2 + \mu y'^2\right)}{\left(x'^2 + \nu y'^2\right)}z_2 + \frac{f}{\alpha}\frac{\left(x'x'' + \mu y'y''\right)}{\left(x'^2 + \nu y'^2\right)}z_1^2 - \frac{\left(x'x'' + \nu y'y''\right)}{\left(x'^2 + \nu y'^2\right)}\left(3z_2^2 + 4z_1z_3\right) - \frac{6\left(x'x''' + \nu y'y'''\right)}{\left(x'^2 + \nu y'^2\right)}z_1^2z_2 - \frac{\left(x^{(IV)}x' + \nu y^{(IV)}y'\right)}{\left(x'^2 + \nu y'^2\right)}z_1^4. \tag{30}$$

Definitions (17) can be written in the form of equations:

$$\begin{aligned}\dot{z}_0(t) &= z_1(t), \\ \dot{z}_1(t) &= z_2(t), \\ \dot{z}_2(t) &= z_3(t).\end{aligned} \tag{31}$$

Equation (30) together with (31) forms a quasilinear system of ordinary differential equations, which has a canonical form.

The specific type of system (30) - (31) depends on the parameterization of the trajectory. Let's consider a few special cases. In the presented examples, the mass of the tool is taken as a unit of mass, the motion time is a time unit, and all values of length are also selected in a certain unit of length.

3. Numerical Solution

Example 1. Let the tool trajectory be straight line and parameterized by the relations

$$\begin{aligned}x &= p, \\ y &= kp + b = kx + b.\end{aligned}$$

The case of a straight trajectory is important for applications, since it is often implemented in practice. For example, in 3D printing with filling of some plane area, the printer head moves along a set of parallel straight lines.

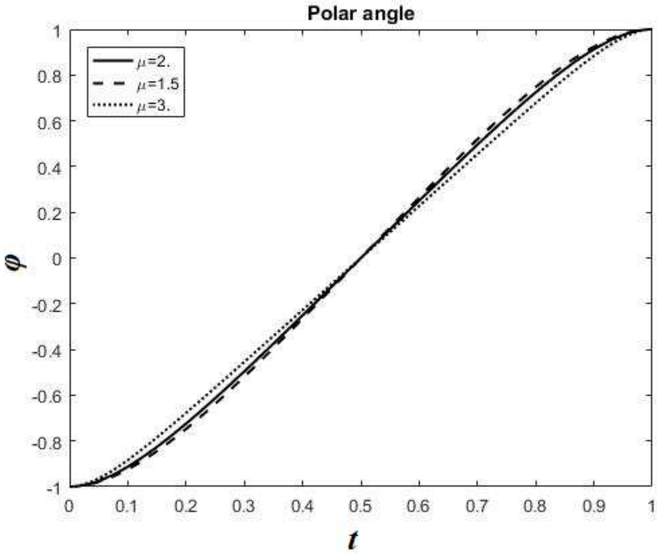

Fig. 2. The law of change of the abscissa at different weight coefficients α in the cost functional.

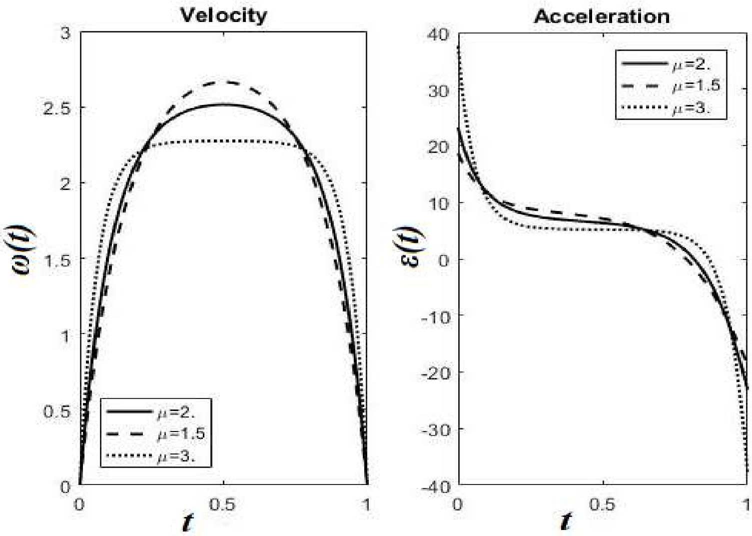

Fig. 3. Velocity and acceleration components on the Ox axis for straight-line movement along the trajectory at optimal motion mode.

Obviously $x' = 1, y' = k$, the remaining derivatives are equal to zero. Therefore, equation (30) takes a simple form

$$\dot{z}_3 = \frac{f\,(1+\mu k^2)}{\alpha\left(1+\nu k^2\right)} z_2.$$

Returning to definitions (17), we have:

$$\ddddot{p} = \frac{f\,(1+\mu k^2)}{\alpha\left(1+\nu k^2\right)} \ddot{p}.$$

The solution to this problem is obtained in analytical form and is given by the formulas:

$$x(t) = p(t) = A\sinh(\gamma t) + B\cosh(\gamma t) + Ct + D,$$

where

$$\gamma = \sqrt{\frac{f\,(1+\mu k^2)}{\alpha\left(1+\nu k^2\right)}},\ A = \frac{(x_1 - x_0)\sinh(\gamma)}{\Delta},\ B = \frac{(x_1 - x_0)(1-\cosh(\gamma))}{\Delta},$$
$$\Delta = \sinh(\gamma)(\sinh(\gamma)-\gamma) - (\cosh(\gamma)-1)^2 = 2(\cosh(\gamma)-1) - \gamma\sinh(\gamma),$$
$$D = x_0 - B,\ C = -A\gamma.$$

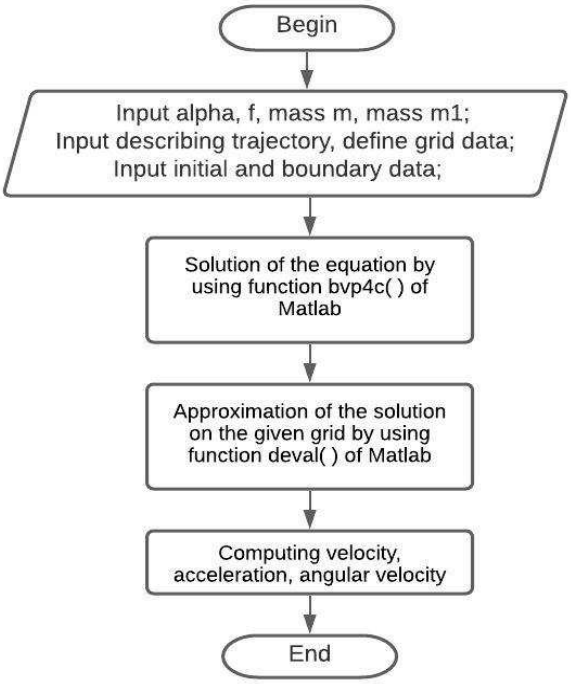

Fig. 4. The flow chart of the numerical solution.

Figure 2 shows the optimal law of change of the abscissa for different parameters a) α, f = 1, μ=2, 1.5, 3, υ=2, 1.25, 5. A trajectory of the form y=-2x+1 passing between points (0,1) and (1, -1) is used for calculations.

Figure 2 shows the calculation results — the law of the change in abscissa x(t) at different weight coefficients α in the quality functional. Figure 3 shows the velocities on the Ox axis at different weights α (left) and the acceleration on the Ox axis depending on the time for different α (right). It can be seen that the velocities change more smoothly at greater values of α.

In the examples below the main spets of numerical modeling are presented in the following shart:

The algorithm of numerical implementation is schematically presented in the following pseudo code:

```
Begin
    Define parameters: mu, nu, alpha, f
    Define specific trajectory's parameters (k, R, a, b) for each example
    Define functions in Matlab that contain the information on the coefficient of the system and the structure of
    boundary data (for instance, @MyODE,@MyBC)
    Define initial and final positions (p0, p1)
    Define time step and grid points of time (tgrid)
    Define initial approximation to the solution of the system(solinit)
    Define options and tolerances for Matlab function bvp4c(...)
    Solve the main system via bvp4c(...) of Matlab:
        solX = bvp4c(@MyODE,@MyBC,solinit,options);
    Approximate the solution on the grid points via the function deval(...) of Matlab:
        solXgrid=deval(solX,tgrid);
    Compute velocity, acceleration, energy on grid points
    Made graphical representation of results
    Save results on text and graphical files for future analysis
End
```

Example 2. Circular motion. In this case the trajectory parameterization and parameter derivatives are set by the relations:

$$\begin{aligned} &x = R\cos p, y = R\sin p, \\ &x' = -R\sin p, y' = R\cos p, \\ &x'' = -R\cos p, y'' = -R\sin p, \\ &x''' = R\sin p, y''' = -R\cos p, \\ &x^{(IV)} = R\cos p, y^{(IV)} = R\sin p. \end{aligned}$$

Substitute these expressions in equation (30) and get:

$$\dot z_3 = \frac{f}{\alpha}\frac{\sin^2 z_0 + \mu\cos^2 z_0}{\sin^2 z_0 + \nu\cos^2 z_0} z_2 + 6z_1^2 z_2.$$

$$\dot z_3 = \frac{f}{\alpha}\frac{\sin^2 z_0 + \mu\cos^2 z_0}{\sin^2 z_0 + \nu\cos^2 z_0} z_2 + \frac{f}{\alpha}\frac{(1-\mu)\sin z_0\cos z_0}{\sin^2 z_0 + \nu\cos^2 z_0} z_1^2 - \frac{(1-\nu)\sin z_0\cos z_0}{\sin^2 z_0 + \nu\cos^2 z_0}\left(3z_2^2 + 4z_1 z_3\right) + 6z_1^2 z_2 + \frac{(1-\nu)\sin z_0\cos z_0}{\sin^2 z_0 + \nu\cos^2 z_0} z_1^4.$$

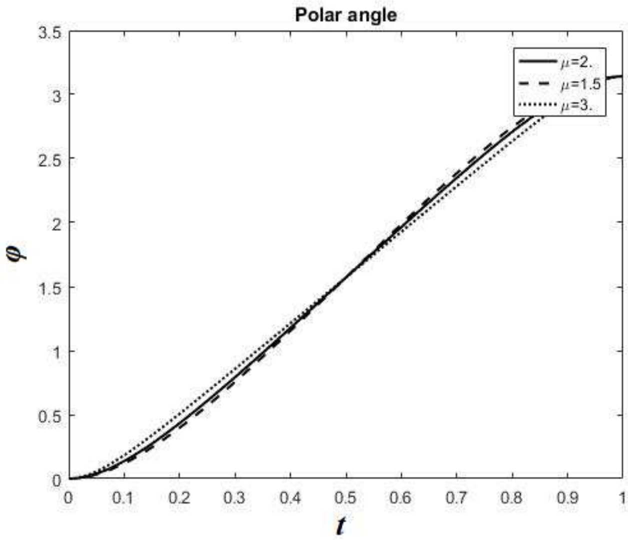

Fig. 5. The optimal function of changing the rotation angle for movement in a semicircle.

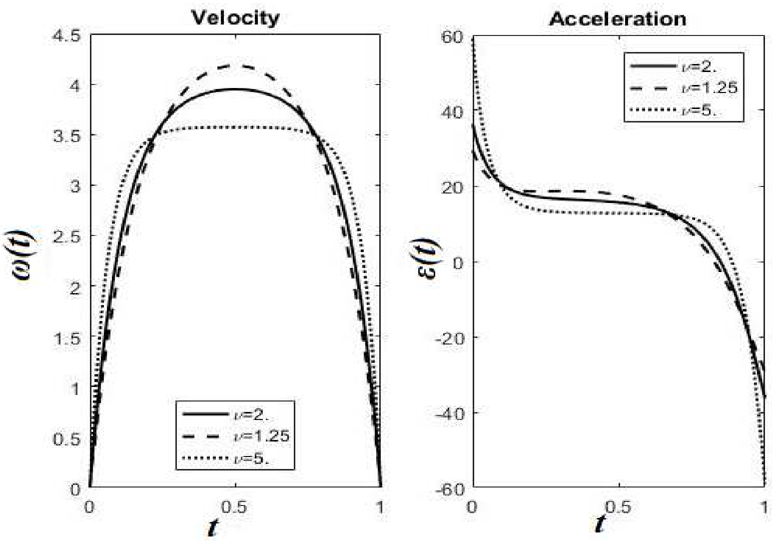

Fig. 6. Angular velocity and acceleration at different weights α for inertia forces at the optimal mode of movement along the semi-circle.

The solution to this equation is obtained numerically. As an example, we considered the case of movement in a semicircle. In this case, the parameter describing the trajectory is the polar angle that changes in the interval $p_0 = 0,\ p_1 = \pi$.

The boundary conditions are set according to (4).The movement is made from a point with coordinates (R, 0) along a semicircle of radius R, to a point with coordinates (-R, 0). In the calculation, the length scale was taken equal to R (m)

Figure 5 shows the law of rotation angle change in the interval [0, π] for movement along the semicircle for different values of weight α in the cost functional. Increasing α leads to a smoother mode of movement. Figures 6 show the corresponding profiles of angular velocities and accelerations for the optimal solution.

Example 3. The trajectory is described by a parabola, for example,

$$x = p, y = kp^2 + b,$$
$$x' = 1, y' = 2kp,$$
$$x'' = 0, y'' = 2k.$$

In this case, equation (30) together with (30) is reduced to a canonical form:

$$\begin{cases} \dot{z}_0(t) = z_1(t), \\ \dot{z}_1(t) = z_2(t), \\ \dot{z}_2(t) = z_3(t), \\ \dot{z}_3 = \dfrac{1}{\alpha m} z_2 - \dfrac{4k^2 z_0}{\alpha m\left(1+4k^2\right)}\left(z_1 - 2z_1^2\right) - \dfrac{4k^2 z_0}{\left(1+4k^2\right)}\left(4z_1 z_3 + 3z_2^2\right). \end{cases}$$

Example 4. The trajectory is an ellipse:

$$
\begin{aligned}
&x = a\cos p, y = b\sin p,\\
&x' = -a\sin p, y' = b\cos p,\\
&x'' = -a\cos p, y'' = -b\sin p,\\
&x''' = a\sin p, y''' = -b\cos p,\\
&x^{(IV)} = a\cos p, y^{(IV)} = b\sin p.
\end{aligned}
$$

Given the notation (31), we get a system of quasilinear equations:

$$
\begin{cases}
\dot{z}_0(t) = z_1(t),\\
\dot{z}_1(t) = z_2(t),\\
\dot{z}_2(t) = z_3(t),\\
\dot{z}_3 = \dfrac{1}{\alpha m}z_2 - \dfrac{(a^2-b^2)\sin p\cos p}{(a^2\sin^2 p + b^2\cos^2 p)}\left(\dfrac{z_1 - 2z_1^2}{\alpha m} + 4z_1z_3 + 3z_2^2\right) + 6z_1^2z_2 + \dfrac{5(a^2-b^2)\sin p\cos p}{(a^2\sin^2 p + b^2\cos^2 p)}z_1^4.
\end{cases}
\tag{32}
$$

4. Conclusion

For robots performing the same repetitive movements along a given trajectory, the law of motion of the end-effector can be computed once in advance and then implemented according to a given table of successive tool positions. Therefore, despite the complicated, nonlinear nature of the resulting equations, solutions to these equations can be built in advance, based on available numerical methods. We have shown what should be like the preferred law of motion for some straight, circular, and elliptical trajectories over a given time interval to minimize kinetic energy and weighted inertia. In the case of an arbitrary trajectory defined by a table of points, it is possible to generalize the proposed method by uniformly approximating the trajectory with analytical expressions. However, in this case, it will be necessary to solve system (30) - (31), which has a rather complicated form. Note that if we consider only the motion of the robot's center of mass, by neglecting other elements, the approach described above is universal for all robots that implement movements along a predefined trajectory. In particular, the same consideration can be applied to the DexTar robot [1] - [2]. The presented optimization method can also be applied in 3D printing tasks. For example, when the printed layer is completely filled with polymer material, the printer makes many movements along straight lines parallel each other. Optimization of the straight line movement will reduce the forces of inertia. It will naturally also be necessary to control the supply of material, which is regulated by the current speed of the device head movement. But this problem is still beyond our consideration.

Let us list here some possible directions of future studies. First of all, the described method can be checked in practice for a specific robot. Might be the structure of the mechanism should be taken into account to obtain the corresponding optimal law of motion. Then the cost functional can include the energy and inertia of all movable elements of the robot. Here we have considered movement along a plane trajectory only. Further possible step is to reformulate and solve the problem for a spatial trajectory. Additionally, the problem of calculating the law of motion for an arbitrary trajectory given in the form of a sequence of points, remains open. Another possible direction of research is to solve the control problem for a quality functional that is reformulated for different models of resistance and frictions forces.

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Author Contributions

B. Mukanova stated the problem and proposed the idea of the solution. D. Azimova and M. Akhmetzhanov have derived the equations, D. Azimova has conducted the numerical simulations and processed the data. The manuscript has been written through the contribution of all authors. All authors have discussed the results, reviewed and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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Nomenclature

m	Mass of the tool [kg]	I	Moment of inertia [kg.m^2]
$\varphi(t)$	Angular velocity of the tool [rad/s]	ρ	Surface density [kg/m^2]

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