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Research Paper

Frequency Separation in Architected Structures using Inverse Methods

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Abstract. A major goal in the design of architected structures for low frequency vibration applications (also called mechanical metamaterials, metastructures, elastic metamaterials, auxetic structures) is the creation of regions in the frequency domain where vibration amplitudes are minimal, regardless of the source of excitation. The idea is to provide vibration suppression in manmade structures. The proposed effort is to examine approaches to produce straightforward methods of designing a given mechanical metamaterial to have a specified gap in the frequency spectrum by adjusting its local mass and stiffness values of the individual cells. Previous work in mechanical metamaterial design has focused on using optimization procedures concerned with global vibration suppression. Here our efforts are focused on frequency separation using two direct approaches by interpreting techniques from the areas of model updating and inverse eigenvalue solutions. Rather than examining the overall suppression of vibration, creating specific bandgaps eliminates the possibility of resonance occurring in a given range of excitation frequencies.

Keywords: Inverse eigenvalue problems, model updating, frequency separation, vibration suppression, metastructures.

1. Introduction

Architected structures, or mechanical metamaterials, have received increasing attention since the advent of 3D printing allowing easy manufacturing and construction of such structures. Mechanical metamaterials as used here refers to a structure composed of repeated cells, each of which constitutes a vibration absorber. Fig. 1 illustrates several such mechanical metamaterials. The goal and type of mechanical metamaterials considered here is to arrange the internal cells to create a system that creates a frequency region in which little or no vibration amplitude occurs for harmonic excitations of frequency in that region. Borrowing language from acoustic metamaterials such regions are called bandgaps. Introductions to and reviews of metamaterials, mechanical metamaterials and architected structures have been recently published [1-4] capturing their main features, construction, and applications. The engineering application of mechanical metamaterials ranges from vehicles to civil infrastructures.

Mechanical metamaterials are artificially made structures designed to have properties not easily obtained by conventional structures at the macro scale. The area was originally an outgrowth of literature dealing with rearranging atoms and molecules to manipulate electromagnetic waves. Acousticians took the repeated lattice concept to a larger scale and applied it to creating regions in the acoustic spectrum where no waves would pass, leading to the concept of bandgaps. Ruzzene [5] and others took this approach down to the vibration scale and introduced the concept of materials that notch out low frequency spectrum regions where very little amplitude is transmitted, providing a solution to the age-old problem of avoiding resonance. Essentially the mechanical metamaterial problem can be thought of as the classic multiple vibration absorber problem as the equations of motion are similar, the difference being the structural configuration. The multiple absorber problem considers adding absorbers externally to the structure whereas the mechanical metamaterial approach is to arrange the absorber like inserts internally to the structure using a cell like configuration. As will be shown here, the approach has the potential to provide a solution without increasing the total system mass as in the case of a traditional absorber approach. Increasing the mass of a structure by adding traditional vibration absorbers is an issue in both ground and aerospace vehicles as it generally increased fuel consumption. The approach here offers an alternative by providing absorber like performance without increasing mass.

The currently dominant method of designing mechanical metamaterials is to create bandgaps using optimization schemes to minimize a structures frequency response magnitude. The approach presented here is to use two different scenarios, one borrowed from the field of model updating and the other from inverse eigenvalue theory.

Model updating, also called model correction, is a field that grew out of the use of vibration testing results to validate analytical models. The basic idea being that the differences between the analytically predicted frequencies and mode shapes and the measured frequencies and mode shapes would be minor and small corrections to the analytical model would result in a predictive model for further design and analysis.

Inverse eigenvalue problems for second order matrix differential equations resulting from the modeling of structures using Newton's Laws or Hamilton's principal have mostly been studied in the context of mathematics [6, 7]. Inverse eigenvalue

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problems have been used in the solution of a number of problems including pole placement and eigenstructure assignment from the field of control theory [8] using feedback control to obtain a more favorable response. The research reported here is to use the mathematics of inverse eigenvalue theory to create specific bandgaps in architected structures by examining a previously published metastructure solution [9] designed such that the overall mass remained constant.

The forward problem for a linear damped structure is *given the equations of motion, initial conditions and applied forces determine the displacement of the system*. For linear, time invariant systems the solution exists and is unique. On the other hand, the inverse problem is *given the response to the system determined the coefficients in the equation of motion.* Unfortunately, the solution to an inverse problem may or may not exist and if it does exist, it is not necessarily unique. The inverse eigenvalue problem is *given eigenvalues and eigenvectors reconstruct the system's mass, damping and stiffness matrices.*

The idea of existence and uniqueness in inverse eigenvalue problems is conceptionally explained by considering a simple one-degree-of freedom example because it has only one eigenvalue. The inverse problem consisting of given the natural frequency of the system determine its mass and stiffness. Because the frequency is the square root of the stiffness k divided by the mass m, there is no single, unique solution to this inverse eigenvalue problem. In addition, if m and k are constrained to be values in certain ranges a solution may not exists. For example, if the stiffness is limited to lie between 300 N/m < k < 400 N/m, the mass is limited to 1 kg and the design calls for a 21 rad/s frequency, then no solution exists because the largest frequency available with the parameters given is 20 rad/s. When solving inverse eigenvalue problems, one must keep in mind the properties of existence and uniqueness of solutions. In the cases considered here, uniqueness is not as important as existence.

2. Model Updating Approach

Model updating (see [10] for example) is the concept that analytical models often to not faithfully reproduce measured frequencies. Updating methods were devised to adjust an analytical model in such a way that the updated model would reproduce the frequencies as measured in a vibration test. Here we use this approach to adjust the model to produce a bandgap defined as a range of frequencies where resonance will not occur.

Consider an undamped mechanical metamaterial defined by the equations of motion:

$$M\ddot{\mathbf{x}}(t)+K\mathbf{x}(t)=0 \tag{1}$$

Here M is the n x n symmetric, positive definite mass matrix, K is the corresponding n x n symmetric positive semi- definite stiffness matrix, and $\mathbf{x}(t)$ is the n x 1 vector of displacements. In order to capitalize on the symmetry properties of matrices consider the coordinate transformation defined by

$$\mathbf{x}(t)=M^{-1/2}\boldsymbol{q}(t) \tag{2}$$

Substitution of eq. (2) into eq. (1) and multiplying by $M^{-1/2}$ yields:

$$I\ddot{\boldsymbol{q}}(t)+\hat{K}\boldsymbol{q}(t)=0 \tag{3}$$

where $\hat{K}$=$M^{-1/2}KM^{-1/2}$ is symmetric and positive definite, thus having positive real eigenvalues corresponding to the squares of the natural frequencies of vibration. Because of the symmetry, eq. (3) can be further transformed into a diagonal system of modal equations by solving the corresponding eigenvalue problem for the eigenvalues and eigenvectors of $\hat{K}$. Let λ_i the n eigenvalues and $\mathbf{u}_i$, i = 1, 2 … n, the n eigenvectors. To complete the modal analysis, form the eigenvector matrix U = [$\mathbf{u}_1$ $\mathbf{u}_2$ $\mathbf{u}_3$ … $\mathbf{u}_n$] and use the additional coordinate transformation $\mathbf{q}(t)$ =$U\mathbf{r}(t)$ substituted into equation (3) and multiply by U^T to get the *diagonal* system of modal equations:

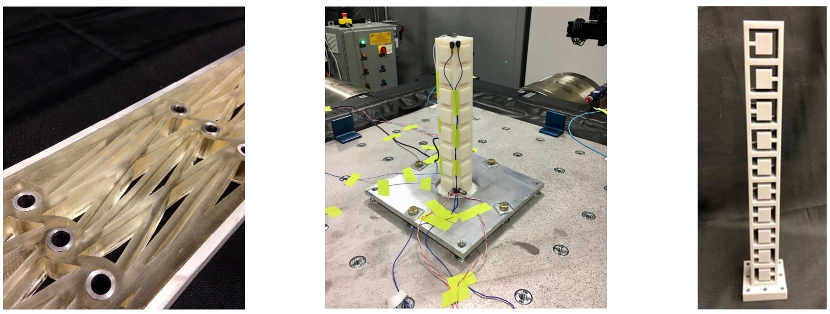
$$\ddot{\mathbf{r}}(t)+\Lambda\mathbf{r}(t)=0 \tag{4}$$

The eigenvectors are orthonormal so that $U^TU = I$, and the matrix Λ is the diagonal matrix of eigenvalues, i.e. the squares of the natural frequencies.

Next, consider the n x n diagonal matrix Γ consisting of all zeros except where it is desired to increase the bandgap. Then create a new system in the modal coordinates of eq. (4) of the form:

$$\ddot{\mathbf{r}}(t)+(\Lambda+\Gamma)\mathbf{r}(t)=0 \tag{5}$$

These new modal equations will have the desired gap in frequencies. However, to be useful in the application of mechanical metamaterials, eq. (5) needs to be transformed back into the physical coordinate system so that adjustments in the stiffness matrix can be made to approach the desired frequency shifts. Solving for $\mathbf{r}(t)$ in terms of $\mathbf{x}(t)$ yields:



(a) (b) (c)

Fig. 1. Examples of several 3D printed mechanical metamaterials. Such structures can reduce the mass and increase the vibration suppression of a variety of engineering structures.

$$\mathbf{r}(t)=U^T M^{1/2}\mathbf{x}(t) \tag{6}$$

Substitution of $\mathbf{r}(t)$ back into eq. (5) and multiplying from the left by $M^{1/2}U$ yields:

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{M}^{1/2}\mathbf{U}(\Lambda+\Gamma)\mathbf{U}^T\mathbf{M}^{1/2}\mathbf{x}(t)=0 \tag{7}$$

Thus, the updated stiffness matrix becomes:

$$\mathbf{K}_U=\mathbf{M}^{1/2}\mathbf{U}(\Lambda+\Gamma)\mathbf{U}^T\mathbf{M}^{1/2} \tag{8}$$

The new system $\mathbf{M}\ddot{\mathbf{x}}(t)+K_U\mathbf{x}(t)=0$ will have a larger frequency separation (e.g. a larger bandgap) than the original system.

2.1 Example

Many of the initial mechanical metamaterial designs focused on identical mass and stiffness values for the absorbers and sought optimal values to decrease the global amplitude. Later methods allowed individual mass and stiffness values vary to find the minimum global displacement. In this example the mass is fixed, and the focus is on creating a range of frequencies where no excitation will occur, that is the creation of a bandgap.

To mimic a repeated lattice mechanical metamaterial, consider the 5-degree-of-freedom (DOF) system given in Fig. 2 minus the incasing structure. Each mass and stiffness are assumed to have the same value so the equations of motion are represented as

$$\begin{bmatrix}3&0&0&0&0\\0&3&0&0&0\\0&0&3&0&0\\0&0&0&3&0\\0&0&0&0&3\end{bmatrix}\ddot{\mathbf{x}}(t)+\left(1\times10^4\right)\begin{bmatrix}8&-4&0&0&0\\-4&8&-4&0&0\\0&-4&8&-4&0\\0&0&-4&8&-4\\0&0&0&-4&8\end{bmatrix}\mathbf{x}(t)=0$$

The eigenvalues for the system given in eq. 8 are 0.3573, 1.3333, 2.6667, 4.0000, 4.9761 $(\text{rad/s})^2 \times 10^4$. In Hertz these frequencies are:

$$f_1=9.5130,\quad f_2=18.3776,\quad f_3=25.9899,\quad f_4=31.8310,\quad f_5=35.5029\text{ Hz}$$

Note that the gap between the third and fourth frequencies is f_4 - f_3 = 5.8411 Hz. Next chose a set of adjustments in the eigenvalues with the goal of producing a larger gap between these two frequencies. In this case, the following numbers are chosen to decrease λ_3 and increase λ_4:

$$\Delta\lambda_3=-0.666\ \text{and}\ \Delta\lambda_4=0.5$$

in an attempt to make a larger gap between the 3rd and 4th frequency. Note that no attempt is made to change the 1st, 2nd and 5th eigenvalues although they will change as the result of coupling. Referring to eq. (8), the correction matrix becomes

$$\Gamma=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&-0.666&0&0\\0&0&0&0.5&0\\0&0&0&0&0\end{bmatrix}$$

Following the steps outlined in eq. (7) to compute the required stiffness, the modified system results in the following set of frequencies:

$$f_1=9.513,\quad f_2=18.3776,\quad f_3=22.5117,\quad f_4=33.7619,\quad f_5=35.5029\text{ Hz}$$

The change in frequency between f_3 and f_4 is originally f_4 - f_3 = 5.84 Hz while the gap in the updated system $(f_4 - f_3)_{new}$ = 11.25 Hz, a 93% increase.

Unfortunately, the new value of the stiffness matrix when transformed back into the physical coordinate system, while still symmetric loses its connectivity. In this case it becomes:

$$K_U=\begin{bmatrix}7.7090&-4.3750&0.6660&0.3750&-1.0410\\-4.3750&8.3750&-4&-0.3750&0.3750\\0.6660&-4&7.3340&-4&0.6660\\0.3750&-0.3750&-4&8.3750&-4.3750\\-1.0410&0.3750&0.6660&-4.3750&7.7090\end{bmatrix}\times10^4$$

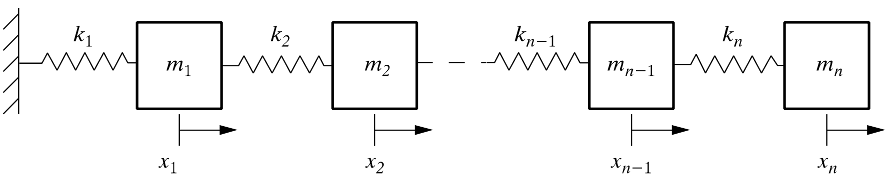


Fig. 2. A representation of a repeated lattice structure as a series of springs and masses.

Unfortunately, this stiffness matrix does not correspond to the physical connectivity of the original stiffness matrix, which is banded. In an attempt to translate this to a physical system the off band elements are set to zero to produce:

$$K_{UC} = \begin{bmatrix} 7.7090 & -4.3750 & 0 & 0 & 0 \\ -4.3750 & 8.3750 & -4 & 0 & 0 \\ 0 & -4 & 7.3340 & -4 & 0 \\ 0 & 0 & -4 & 8.3750 & -4.3750 \\ 0 & 0 & 0 & -4.3750 & 7.7090 \end{bmatrix} \times 10^4$$

The corresponding frequencies become

$$f_1 = 7.9931, \quad f_2 = 17.5657, \quad f_3 = 25.2804, \quad f_4 = 32.3958, \quad f_5 = 35.6973 \text{ Hz}$$

Note that in this case all of the frequencies have shifted some. The new band gap between the 3rd and 4th frequency is also reduced to 7.1154 Hz but is still a 28% increase in gap compared to the original frequencies. In this case, K_{UC} indicates how the stiffness values of the absorbers can be adjusted to improve the bandgap. Thus, while this is not an ideal approach, it does produce a physically realizable solution capable of being constructed.

However most systems have damping. In particular most experimental examples of metastructures are 3D printed [5] and most 3D printers use polymers. Of course, polymers introduce damping. To include damping in the above analysis, the equations end up having to be decoupled complicating the expressions. Thus, a second method is proposed taking advantage of inverse eigenvalue analysis set forth by Starek [11].

3. Inverse Eigenvalue Approach

To distinguish between the inverse approach and the above updating approach the vibration problem of interest is defined in terms of the following matrix equation of motion:

$$M\ddot{\mathbf{x}}(t) + C\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = 0 \tag{9}$$

Here the matrices M, C and K are the symmetric positive definite n x n mass, damping and stiffness matrices respectively, $\mathbf{x}(t)$ is the n x 1 vector of displacements and the over dots denote differentiation with respect to time. Multiplying by the inverse of the mass matrix yields:

$$I\ddot{\mathbf{x}}(t) + H_2\dot{\mathbf{x}}(t) + H_3\mathbf{x}(t) = 0 \tag{10}$$

where the matrix I is the identity matrix, $H_2 = M^{-1}C$ and $H_3 = M^{-1}K$. Equation (10) has associated *Lambda Matrix* or second order matrix polynomial [7]:

$$\mathbf{L}(\lambda)\mathbf{x} = (\lambda^2\mathbf{I} + \lambda\mathbf{H}_2 + \mathbf{H}_3)\mathbf{x} = 0 \tag{11}$$

Here $\mathbf{x}$ is a right eigenvector, complex valued for underdamped systems [12], and $\mathbf{y}$ where $\mathbf{y}L(\lambda) = 0$ is the left eigenvector. The scalars λ are the complex eigenvalues taking on the form

$$\lambda_i, \lambda_{i+1} = -\zeta_i\omega_i \pm \omega_i\sqrt{1-\zeta_i^2}\,i \tag{12}$$

Here ω_i is the i^{th} natural frequency, ζ_i is the i^{th} modal damping ratio and $i = \sqrt{-1}$. Likewise, the frequencies and damping ratios are determined from the complex values of λ by

$$\omega_i = \sqrt{\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2}, \quad \zeta_i = \frac{-\mathrm{Re}(\lambda_i)}{\sqrt{\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2}} \qquad i = 1, 3, \ldots 2n\text{-}1 \tag{13}$$

The state space formulation for eq. (10) is

$$\dot{\mathbf{z}}(t) = A\mathbf{z}(t), \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -H_3 & -H_2 \end{bmatrix} \tag{14}$$

Equations (13) and (14) provide an easy route to numerical calculation of the mode shapes, natural frequencies and damping ratios for a damped system. A second form of the state equations can be formed by pre-multiplying eq. (14) by the matrix N defined by

$$N = \begin{bmatrix} H_2 & I \\ I & 0 \end{bmatrix}$$

This forms the equation

$$N\dot{\mathbf{v}}(t) - P\mathbf{v}(t) = 0, \text{ where } P = \begin{bmatrix} -H_3 & 0 \\ 0 & I \end{bmatrix} \tag{15}$$

The eigenvalue problems for eqs. (14) and (15) are

$$(A - \lambda I)\mathbf{z} = 0, \quad \mathbf{z} \neq 0 \tag{16}$$

and

$$(P - \lambda N)\mathbf{v} = 0, \quad \mathbf{v} \neq 0 \tag{17}$$

This last expression is a first order matrix pencil and is sometimes used for numerically ill conditioned state matrices. Here however, the matrix pencil provides the key to solving the inverse eigenvalue problem. Both eqs. (16) and (17) will yield the same eigenvalues and eigenvectors when solved. Thus eqs. (11), (16) and (17) are three ways to solve for the same thing. Also note that **x, z, v** and λ are in general all complex numbers for an underdamped system.

Using the various eigenvectors the following matrices can be constructed:

$$\begin{aligned} X &= [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_{2n}], \quad \text{which is } n \times 2n \\ Z &= [\mathbf{z}_1 \quad \mathbf{z}_2 \quad \cdots \quad \mathbf{z}_{2n}], \quad \text{which is } 2n \times 2n \\ V &= [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_{2n}], \quad \text{which is } 2n \times 2n \end{aligned} \tag{18}$$

In addition, the eigenvalues can be collected into the $2n$ x $2n$ diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{2n} \end{bmatrix} \tag{19}$$

Using the four matrices X, Z, V and Λ, the three eigenvalue problems of interest can be written as:

$$X\Lambda^2 + H_2 X\Lambda + H_3 X = 0 \tag{20}$$

$$AZ = Z\Lambda \tag{21}$$

$$PV = NV\Lambda \tag{22}$$

The matrix of physical eigenvectors X and the matrix of state space eigenvectors Z are related by:

$$\mathbf{z}_i = \begin{bmatrix} \mathbf{x}_i \\ \lambda_i \mathbf{x}_i \end{bmatrix} \Rightarrow Z = \begin{bmatrix} X \\ X\Lambda \end{bmatrix} \tag{23}$$

Left eigenvectors satisfy the equation $W^T A = \Lambda W^T$ where the columns of the matrix W are the left eigenvectors of the matrix A. The left eigenvectors can be computed directly in a code (such as the Matlab command ([Z, D, W] =eig(A)) or from the formula relating the left and right eigenvectors:

$$W = \left(Z^{-1}\right)^T = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \cdots \quad \mathbf{w}_{2n}] \tag{24}$$

Thus, $W^T Z = S^{-1} Z = I$, and

$$W^T A = \Lambda W^T \tag{25}$$

Obtaining the left eigenvectors in the physical space (i.e. for $L(\lambda)$) requires a bit of manipulation. To that end, define the nonsingular matrix Q by:

$$Q = Z^{-1} N^{-1} \quad \text{so that} \quad QNZ = I \tag{26}$$

Next define the product matrix B_1 by

$$B_1 = NAN^{-1} = NZ\Lambda Z^{-1} N^{-1} = \begin{bmatrix} H_2 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -H_3 & -H_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -H_2 \end{bmatrix} = \begin{bmatrix} 0 & -H_3 \\ I & -H_2 \end{bmatrix} \tag{27}$$

Recognizing that Q=NZ, B_1 can be written as

$$B_1 = NAN^{-1} = NZ\Lambda Z^{-1} N^{-1} = Q^{-1}\Lambda Q \tag{28}$$

Partitioning of Q into two $2n$ x n matrices Q_1 and Q_2 such that Q = [Q_1 Q_2], allows the first partition to be calculated from

$$Q_1 = Q\begin{bmatrix} I \\ 0 \end{bmatrix} = Z^{-1} N^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} = Z^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = W^T \begin{bmatrix} 0 \\ I \end{bmatrix} = Y^T \tag{29}$$

Here the matrices I and 0 are both n x n and Y is the matrix of left eigenvectors of the physical system. Solving for Q_2 in a similar way yields that $Q_2 = \Lambda Q_1$. Thus, the matrix Q takes on the form

$$Q = \left[Y^T \quad \Lambda Y^T\right] \tag{30}$$

With these various matrices and partitions, inverse eigenvalue problems can be formulated in such a way as to represent the coefficient matrices in terms of the partitioned sets of eigenvectors and eigenvalues. From eq. (26), the matrix Z^{-1} can be written as

$$Z^{-1} = QN = \left[Y^T \quad \Lambda Y^T\right] \begin{bmatrix} H_2 & I \\ I & 0 \end{bmatrix} = \left[Y^T H_2 + \Lambda Y^T \quad Y^T\right] \tag{31}$$

This relationship along with eq. (21) in the form $A = Z\Lambda Z^{-1}$ can be expressed as

$$Z\Lambda Z^{-1} = \begin{bmatrix} X \\ X\Lambda \end{bmatrix} \Lambda Z^{-1} = \begin{bmatrix} X\Lambda Z^{-1} \\ X\Lambda^2 Z^{-1} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -H_3 & -H_2 \end{bmatrix} \tag{32}$$

Replacing Z^{-1} in eq. (22) with its formulation given in eq. (31) yields:

$$\begin{bmatrix} X\Lambda(Y^TH_2+\Lambda Y^T) & X\Lambda Y^T \\ X\Lambda^2(Y^TH_2+\Lambda Y^T) & X\Lambda^2Y^T \end{bmatrix} = \begin{bmatrix} 0 & I \\ -H_3 & -H_2 \end{bmatrix} \tag{33}$$

Equating the four partitions of the left side to those in the right side yields the inverse eigenvalue solution (put forth by Starek, [11]). They are

$$X\Lambda(Y^TH_2+\Lambda Y^T)=0, \qquad X\Lambda Y^T = I \tag{34}$$

and

$$H_2 = -X\Lambda^2Y^T \text{ and } H_3 = -X\Lambda^2(Y^TH_2+\Lambda Y^T) \tag{35}$$

Combining these 4 equations into three yields that the coefficient matrices are determined by

$$\begin{aligned} H_{22} &= -X\Lambda^2Y^T \\ H_3 &= H_2^2 - X\Lambda^3Y^T \end{aligned} \tag{36}$$

where the modal eigenvectors are normalized according to

$$X\Lambda Y^T = I \tag{37}$$

The diagonal matrix Λ contains the complex eigenvalues represented in terms of the natural frequencies and damping ratios as given in eq. (13).

4. Eigenvalue Separation by Inverse Equations

In this section the inverse eigenvalue problem is modified by partitioning the inverse formulas into those eigenvalues corresponding to resonances that are acceptable and those that are not acceptable. Consider the diagonal matrix of eigenvalues Λ, and segregate the eigenvalues into those to be kept and those to me moved. Then, partition these into two groups arranged in the form:

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{38}$$

At the same time move and partition the associated right eigenvectors of the physical system X, and the matrix of left eigenvectors Y corresponding to the moved eigenvalues denoted

$$X = [X_1 \quad X_2] \text{ and } Y = [Y_1 \quad Y_2] \tag{39}$$

Here the subscript 1 denotes those eigenvalues that are to be kept and the subscript 2 indicates the replacement eigenvalues chosen to insure the desired bandgap. Substitution of the partitioned matrices of eqs. (38) and (39) into eqs. (36) and (37) yields:

$$\begin{aligned} H_{2c} &= -\left(X_1\Lambda_1^2Y_1^T + X_2\Lambda_2^2Y_2^T\right) \\ H_{3c} &= H_{2c}^2 - \left(X_1\Lambda_1^3Y_1^T + X_2\Lambda_2^3Y_2^T\right) \\ I &= X_1\Lambda_1Y_1^T + X_2\Lambda_2Y_2^T \end{aligned} \tag{40}$$

Here the n x n "corrected" coefficient matrices H_{2c} and H_{3c} are related to the physical matrices of eq. (9) by $C = MH_{2c}$ and $K = MH_{3c}$. In principle this set of damping and stiffness matrices along with the original mass matrix will produce a system with the desired separation in frequencies, and hence the desired bandgap. By comparing these new damping and stiffness matrices to the original matrices can provide insight into how to arrange the absorbers in a structural metamaterial.

4.1 Example

This example illustrates the procedure as well as the fact that there is no guarantee that the newly constructed matrices retain the same physical connections corresponding to the original device. A low order model is again used to illustrate the point. Consider the following system:

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 5 & -3 & 0 \\ 0 & -3 & 7 & -4 \\ 0 & 0 & -4 & 4 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 0 & 0 \\ -3 & 5 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{F}(t)$$

The natural frequencies are: f_1 = 3.0346 Hz, f_2 = 10.4270 Hz, f_3 = 19.7024 Hz, and f_4 = 22.2831 Hz. Suppose it is desired to have a bandgap that prevents resonance from 0 to 9 Hz. To remove the 3 Hz frequency, chose a new eigenvalue, say $\lambda_{1,2}$ = - 0.002 ± 0.6i corresponding to a natural frequency of 9.5 Hz. Following the procedure outlined above in eq. (40) the new damping and stiffness matrices become:

$$H_{2c} = \begin{bmatrix} 3.1117 & -1.7658 & 0.3353 & 0.0850 \\ -1.6388 & 5.7328 & -1.8948 & -0.2815 \\ 0.4440 & -2.0847 & 8.3460 & -3.6579 \\ 0.1174 & 0.2414 & -3.6438 & 4.0906 \end{bmatrix}, \quad H_{3c} = \begin{bmatrix} 4.1332 & -2.7432 & 0.4189 & 0.1073 \\ -2.7432 & 5.4949 & -1.1927 & -2.2068 \\ 0.4189 & -1.1927 & 6.3170 & -2.6626 \\ 0.1073 & -2.2069 & -2.6625 & 3.0865 \end{bmatrix}$$

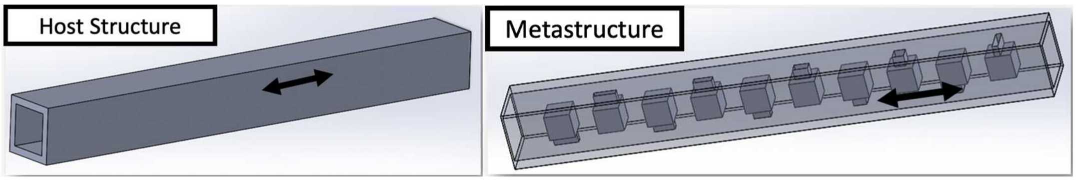


Fig. 3. Drawing of the mechanical metamaterial pictured in Fig. 1c, showing the host structure and the structure as printed with 10 absorbers.

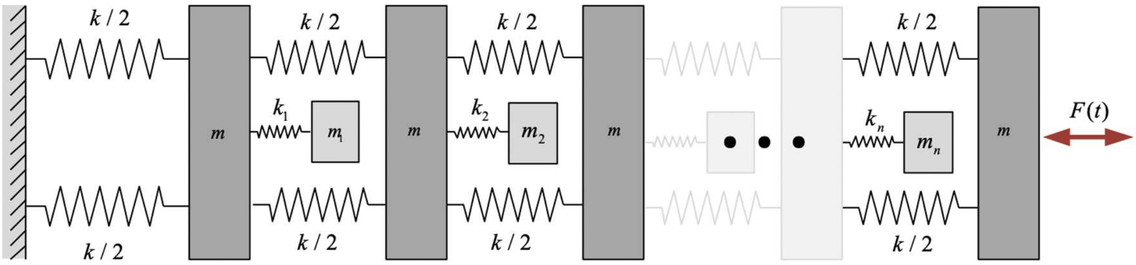


Fig. 4. A lumped mass model of the mechanical metamaterial of Figs. 1c and 3. The large masses labeled, *m*, and stiffnesses, *k*, model the host structure. The absorbers are the masses m_i and stiffness k_i.

The resulting natural frequencies become f_1 = 9.5493 Hz, f_2 = 10.4270 Hz, f_3 = 19.7024 Hz, and f_4 = 22.2831 Hz. Thus, there will be no resonance for any harmonic input with frequency between 0 and 9 Hz and the remaining 3 natural frequencies remain unchanged.

Unfortunately, the solution to this example has lost symmetry in the damping matrix and along with the stiffness matrix is no longer banded indicating that the solution although it exists mathematically may not be physically realizable. It is possible in some situations to find a transformation to a symmetric form [13] and new connections may be found to agree with the numerical solutions. However, additional assumptions provide a way forward as described in the following section.

5. Lightly Damped Solution

The inverse problem approach of Section 3 can be improved by adding constraints or additional assumptions provided the existence of solution is not violated. Here we assume small proportional damping. This allows us to use real eigenvectors consisting of the normal modes of the undamped system which we chose to remain unchanged as the metamaterial is updated. This essentially is akin to solving a pole placement problem in control theory by using an inverse approach. Thus, in the formulation outlined in the previous section, the vectors $\mathbf{x}_i$ remain unchanged and only the diagonal matrix of eigenvalues, Λ, is changed to accommodate the desired bandgap. While the matrix, *X*, of right eigenvectors remains unchanged, its columns are rearranged to agree with the partitioning of Λ. Likewise, the matrix of left eigenvectors, *Y*, is also partitioned accordingly. This set of assumptions also allows us to develop a solution that results in a symmetric stiffness and damping matrix, which has a better chance of producing a physical solution. The total mass is also held constant. The following two examples illustrate the procedure.

5.1 Examples

Two examples are presented in this section to illustrate the modified inverse eigenvalue approach for lightly damped structures. The examples are simple models of the 1-dimensional mechanical metamaterial depicted in Fig. 3 and printed as pictured in Fig. 1c. Note that the block masses pictured on the right in Fig. 3 move back and forth in the direction of the arrow. The small attachment beams are stiff in the transverse direction by flexible and spring like in the longitudinal direction.

The longitudinal vibrations of the system of Fig. 3 are modeled as illustrated in Fig. 4. The first example is a low order system presented to illustrate the procedure while the second example is for the system of Fig. 3. Consider a host structure with just three masses and two absorber masses all connected in series. While not a good mechanical metamaterial example, the low order allows an easy display of the results. The host structure is assumed to have the following mass and stiffness matrices:

$$M=\begin{bmatrix}1.990&0&0\\0&1.990&0\\0&0&1.990\end{bmatrix},K=\begin{bmatrix}2.364&-1.1820&0\\-1.1820&2.364&-1.1820\\0&-1.1820&1.1820\end{bmatrix}\times10^{7}$$

The proportional damping matrix is given by C = 10^{-5}K. The units are kg, N/m and Ns/m and the resulting natural frequencies are in Hz are f_1 =545.9, f_2 =1,529.5, and f_3 =2,210.3.

Adding absorbers to the first and second mass results in the 5 degree of freedom system. The metastructure design given in [9] is used to keep the total mass constant and minimizes the H_2 norm of the frequency response. The resulting mass and stiffness matrices become:

$$M=\begin{bmatrix}0.1393&0&0&0&0\\0&0.1393&0&0&0\\0&0&0.1393&0&0\\0&0&0&0.0895&0\\0&0&0&0&0.0895\end{bmatrix},\quad K=\begin{bmatrix}2.4693&-1.1820&0&-0.1053&0\\-1.1820&2.4693&-1.1820&0&-0.1053\\0&-1.1820&1.1820&0&0\\-0.1053&0&0&0.1053&0\\0&-0.1053&0&0&0.1053\end{bmatrix}\times10^{7}$$

The damping matrix is again C = 10^{-5}K. The new frequencies become (in Hz)

$$f_1 = 456.2,\ f_2 = 536.2,\ f_3 = 769.0,\ f_4 = 1{,}865.7,\ \text{and}\ f_5 = 2{,}675.4\ .$$

The design shifts the frequencies either side of the fundamental frequency of the host structure creating a bandgap of 107 Hz.

Next the inverse eigenvalue approach is applied to create a larger bandgap and completely eliminate any resonance between 0 and 600 Hz. The concept is to change the first two natural frequencies to 600 and 700 Hz respectively. Using eqs. (12) with these new natural frequencies, the new eigenvalues become

$$\lambda_{1,2} = -70 \pm 3770i\ \text{ and }\ \lambda_{3,4} = -95 \pm 4399i$$

Using these new eigenvalues and the original eigenvectors from the constant mass solution and following the steps outlined in eqs. (40) yields the following new stiffness and damping matrices (note the mass stays constant):

$$K = \begin{bmatrix} 2.4719 & -1.1780 & 0.0044 & -0.0973 & 0.0070 \\ -1.1780 & 2.4736 & -1.1743 & 0.0070 & -0.0897 \\ 0.0044 & -1.1743 & 1.1905 & 0.0076 & 0.0174 \\ -0.0973 & 0.0070 & 0.0076 & 0.1699 & -0.0125 \\ 0.0070 & -0.0897 & 0.0174 & -0.0125 & 0.1546 \end{bmatrix} \times 10^7,\ \ C = \begin{bmatrix} 247.1815 & -117.8166 & 0.4240 & -9.7615 & 0.6744 \\ -117.8166 & 247.6055 & -117.4573 & 0.6744 & -9.0224 \\ 0.4240 & -117.4573 & 119.0226 & 0.7391 & 1.6792 \\ -9.7615 & 0.6744 & 0.7391 & 16.7104 & -1.1753 \\ 0.6744 & -9.7915 & 1.6792 & -1.1753 & 15.2694 \end{bmatrix}$$

With these adjusted coefficient matrices, the new natural frequencies (in Hz) become:

$$f_1 = 600.1,\ f_2 = 700.3,\ f_3 = 769.0,\ f_4 = 1{,}865.7,\ \text{and}\ f_5 = 2{,}675.4$$

Note that the resulting system has the desired 0 to 600 Hz frequency gap and the remaining three frequencies remain unchanged. Also note that the new system retains symmetry in the stiffness and damping matrix, but the physical connections have not been retained. To see the visual difference in bandgap between the three different designs, the frequency response of the tip displacement divided by the tip excitation is plotted for all three cases in Fig. 5. Note that the inverse approach has improved the bandgap substantially compared with both the base structure and the optimized structure given in [8] with not additional increase in mass.

The frequency response plots are made by using the modal information to compute the receptance matrix, α, given by

$$\alpha_{ij} = \sum_{k=1}^{n} \frac{\left[\mathbf{u}_k \mathbf{u}_k^T\right]}{\left(\omega_k^2 - \omega^2\right) + \left(2\zeta_k \omega_k \omega\right)}$$

Here i is the output location, j is in coordinate were the driving force of frequency ω us applied, ω_k is the k^{th} natural frequency and $\mathbf{u}_k$ is the k^{th} right eigenvector.

Because the chosen bandgap is between 0 and 600 Hz, one would also expect the impulse response to be improved by the new design. The impulse responses for the three systems to the same unit impulse are plotted in Fig. 6. Comparing the three plots shows that the impulse response is also improved. The inverse approach results in a response that has a lower settling time and lower amplitudes than the other two designs.

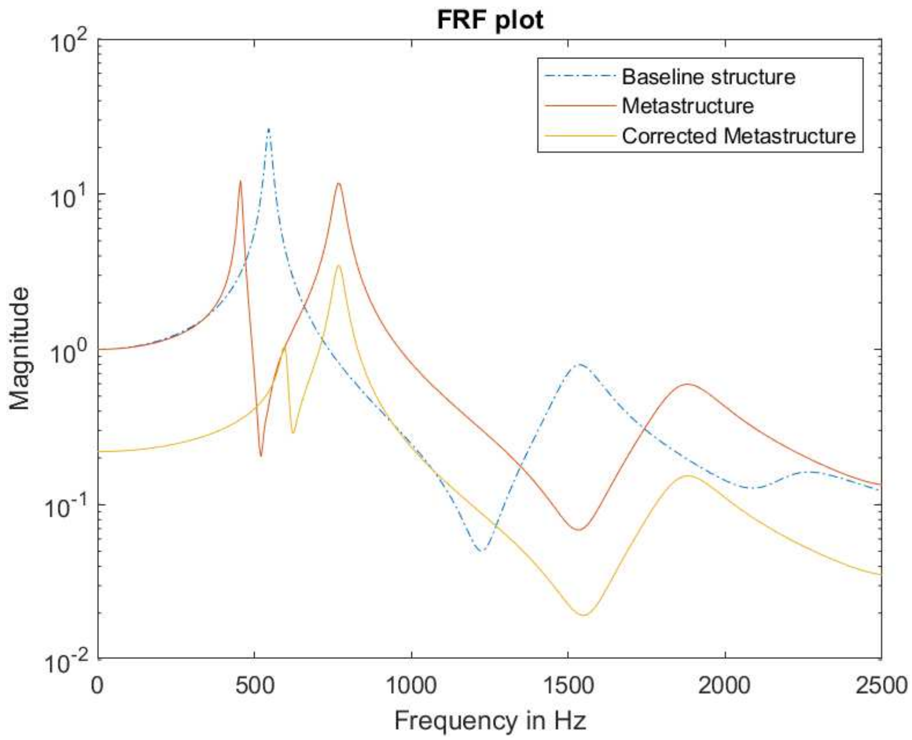


Fig. 5. The frequency response of the tip displacement divided by the force applied at the tip for the three cases of the base structure (dashed blue line), the constant mass mechanical metamaterial from [8] (the red line) and the bandgap solution presented here (the yellow line).

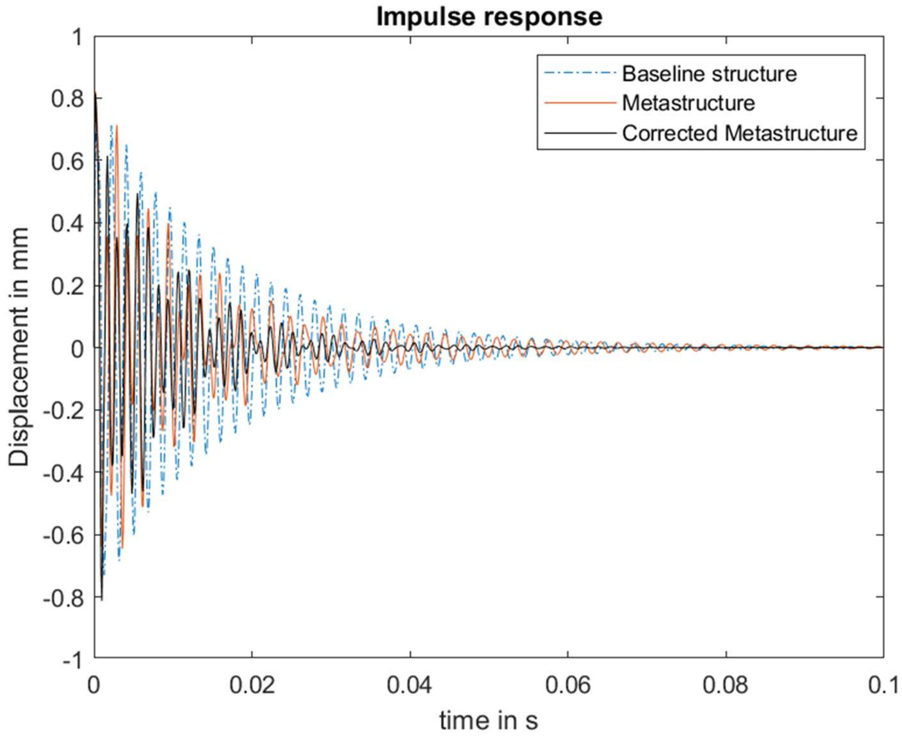


Fig. 6. The impulse responses of the 3 mass 2 absorber system showing that the system with the largest bandgap designed by the inverse eigenvalue method has a shorter settling time and lower overall deflection than either the host or the mechanical metamaterial design by optimization.

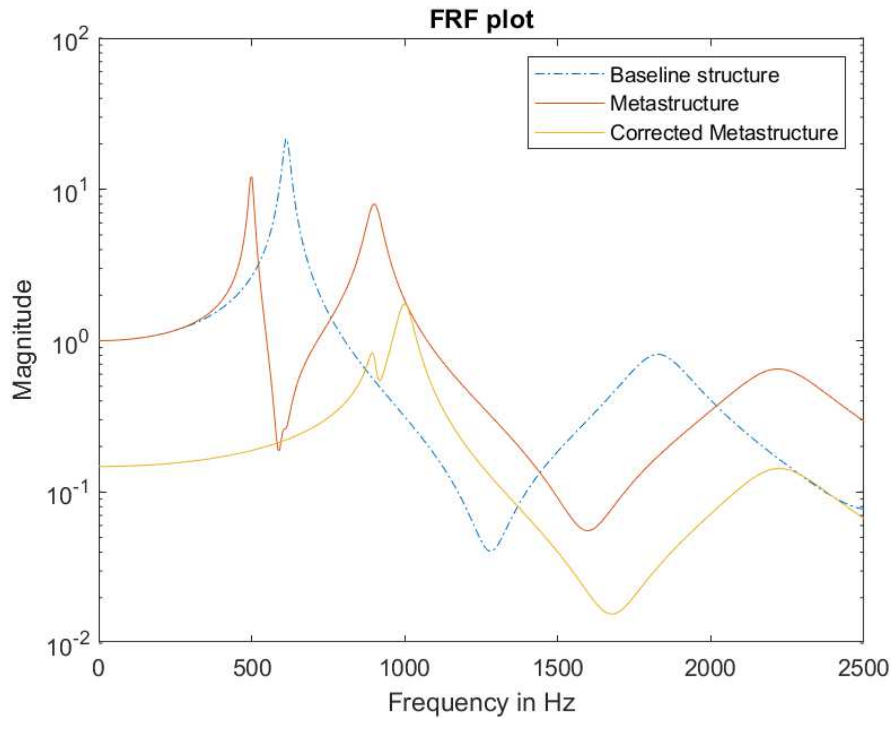


Fig. 7. The frequency response of the tip displacement divided by the force applied at the tip for the three cases of the base structure (dashed blue line), the constant mass mechanical metamaterial from [8] (the red line) and the bandgap solution presented here (the yellow line).

Next consider the ten degree of freedom model suggested in [9] as a base structure with 10 absorbers added, presenting a reasonable model of the longitudinal mechanical metamaterial pictured in Fig. 1c and Fig. 4. The system was designed to reduce the magnitude of the tip deflection by minimizing the H_2 norm of the response which is equivalent to minimizing the total energy. The system was also designed to keep the mass of the original structure constant so that the base structure (no absorbers) and the mechanical metamaterial with absorbers have the same mass. Essentially this represents designing a structure to have better vibration suppression properties by redistributing its mass. The analytical model given in [8] was experimentally validated and is used here as a starting point. We start with the H_2 solution and apply the proposed inverse method to create a specific bandgap designed to avoid resonance between 0 and 900 Hz. Fig. 7 shows the resulting frequency response functions (FRF) of all three systems: base structure, H_2 design (labeled "Metastructure") and the bandgap design (labeled "Corrected Metastructure"). Note the large low frequency bandgap in the inverse solution along with the much lower amplitude compared with the baseline solution and optimized solution presented in [9].

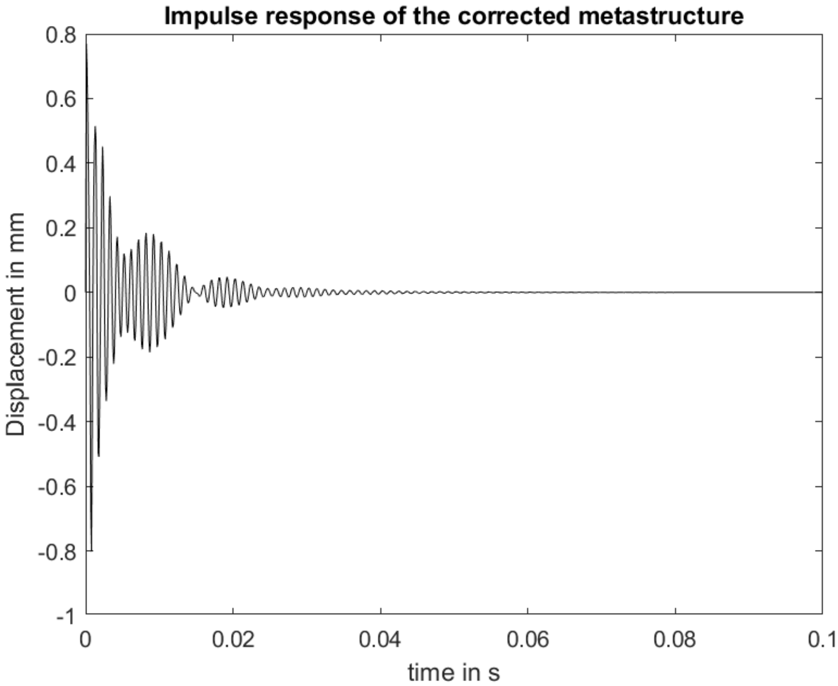


Fig. 8. The impulse response of the system with a 900 Hz bandgap.

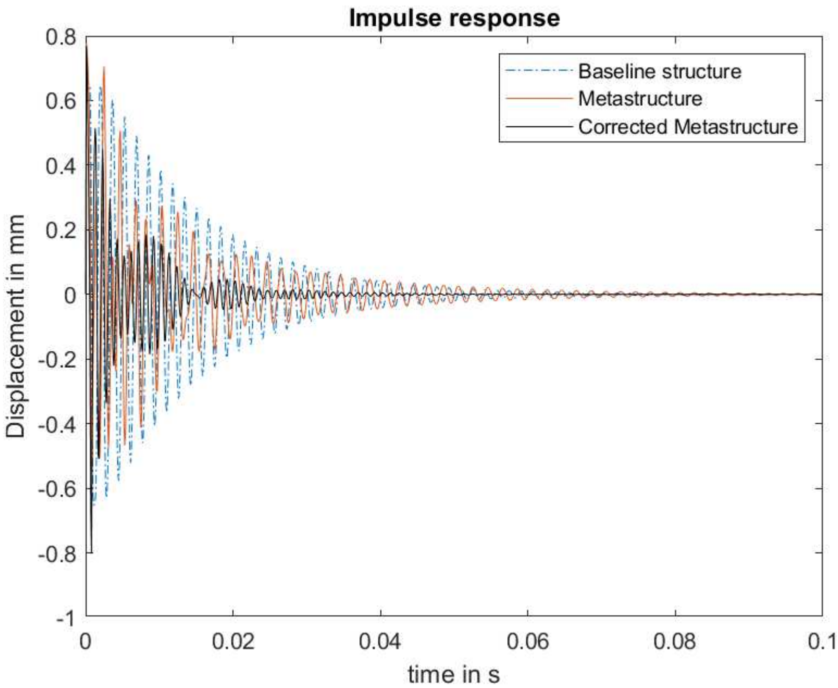


Fig. 9. The impulse responses of all three versions.

The fundamental frequency of the base structure is 613.8 Hz. The effect of the metastructure design was to split this first peak into two peaks a 501 Hz and a second peak at 901.8 Hz. To obtain the desired bandgap between 0 and 900 Hz the first two frequencies need to be changed. These were changed to 900 Hz and 1000 Hz by replacing the first four eigenvalues (first two frequencies) with

$$\lambda_{1,2} = -70 \pm 5{,}655i \Rightarrow f_1 = 5656 \text{ rad/s} = 900 \text{ Hz } (\zeta_1 = 0.0159)$$
$$\lambda_{3,4} = -115 \pm 6{,}280i \Rightarrow f_2 = 6282 \text{ rad/s} = 1000 \text{ Hz } (\zeta_2 = 0.0199)$$

These new values produced the 0 to 900 Hz bandgap illustrated in Fig. 7 without adding mass. While the impulse response of the bandgap structure was not the focus of the research it is of interest to note that the impulse response is also improved by the shifting of resonance away from zero. The impulse response for the system designed by the inverse method corresponding 900 Hz bandgap is plotted in Fig. 8. Note the beating behavior in the impulse response which is the result of the repeated frequencies of the added absorbers. Figure 9 superimposes the impulse responses of all three structures. The baseline response is represented by the blue dashed line, the constant mass metastructure response is represented by the red line and the black line is the repeat of Fig. 8 of the structure with a 900 Hz bandgap. Note that the settling time, i.e., the time for the system to approach zero response is shortest (about 0.05 sec) for the system with the large bandgap compared to about 0.07 sec for the base structure. The peak amplitudes are also reduced.

In addition to providing a shorter settling time for the impulse response, note in Fig. 9 that the magnitude of the response is clearly lower for the system with the large bandgap. This is because the bandgap chosen was to strike out the low frequency response and an impulse response excites mostly the low frequency modes. Unlike traditional vibration absorber systems which add substantial mass to a structure, the approach outlined here does not add any mass to the system, but rather redistributes mass and adjusts stiffness to provide substantial vibration suppression and avoid resonance.

6. Conclusion

Two methods of creating bandgaps in architected structures have been presented to directly obtain specified gaps. The idea is to specify a specific range of frequencies through which an architected structure will not exhibit resonance. The first method was to mimic the method of model updating taken from the discipline of vibration testing and focused on an undamped system. The second technique employs elements of inverse eigenvalue theory which allows a given mechanical metamaterial design to be altered to have a bandgap in its frequency spectrum corresponding to excitation frequencies to be avoided. By employing an earlier theory of inverse eigenvalue solutions, fixing the mode shapes and assuming proportional damping, the inverse result produces real symmetric matrices. Several examples are given to illustrate the procedure as applied to a model of a mechanical metamaterial which was previously validated by experiments.

While the design proposed here focused on bandgaps in the frequency response, impulse responses were also plotted revealing that in general the structure with a significant bandgap produced an impulse response with lower overall displacement and faster settling times compared to both the original structure and the previously designed mechanical metamaterial. The original metastructure designed in [9] solved a unique problem because it did not involve increasing the mass of the original structure. Typical absorber type solutions require a mass increase of up to 25%. Traditionally the mass ratio of absorber mass to primary mass varies from 0.05 to 0.25 such that the larger the mass ratio the broader the frequency band of absorption. The design proposed here increased the bandgap of absorption substantially, again without any increase in mass by combining the results of [9] with inverse eigenvalue theory set forth in [11], modified by assuming normal mode solutions via proportional damping.

There are several issues which remain to be solved using this technique and these were pointed on and summarized here. First there is no general existence or uniqueness of solution for inverse problems. While uniqueness is not too much of an issue for a designer, as it can lead to search for an optimal solution, existence is important to understand when using the technique. As shown by example physical constraints on mass and stiffness values may render the mathematical solution to the bandgap problem physically unrealizable. That is, the solution may require stiffness or mass values out of the range of possibility for a given application. The second issue is that while the procedure presented here produces symmetric matrices, the physical connections of the original problem can be lost. For the example given the original equations of motion render the stiffness matrix and damping matrix banded, but the inverse solution's matrices are not banded. This means that to be able to physically print a mechanical metamaterial with the given properties new connections between masses by adding additional springs must be added, and these may or may not be possible. The approach outlined here solves the usual problem with designing vibration absorber solutions to remove resonance problems by avoiding adding mass and being able to specify exact ranges of frequencies where no resonance will occur.

Author Contributions

Daniel J. Inman developed the research, initiated the project, suggested the examples and wrote the paper; Aishwarya Gunasekar wrote the supporting Matlab codes and made the plots for the figures. All authors discussed the results, reviewed and approved the final version of the manuscript.

Conflict of Interest

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Nomenclature

M	Mass matrix	K_U	Updated stiffness matrix
C	Damping matrix	K_{UC}	Banded version of K_U
K	Stiffness matrix	H_2	$= M^{-1}C$
λ_i	i^{th} eigenvalue	H_3	$= M^{-1}K$
$\mathbf{u}_i$	right i^{th} eigenvector	H_{2c}	Corrected damping matrix
U	matrix of eigenvectors	H_{3c}	Corrected stiffness matrix
Λ	diagonal matrix of eigenvalues	ω_i	Natural frequency (rad/sec)
Γ	matrix of eigenvalue changes	f_i	Natural frequency (Hz)
$L(\lambda)$	Lambda matrix polynomial	ζ_i	Modal damping ratio
A, P, N	Various state matrices	$\mathbf{x}_i$	Right eigenvector of L
$\mathbf{z}_1$	Right eigenvector of A	$\mathbf{y}_i$	Left eigenvector of L
$\mathbf{v}_1$	Right eigenvector of P, N	Y	Matrix of $\mathbf{y}_i$
$\mathbf{w}_i$	Left eigenvector of A	Z	Matrix of $\mathbf{z}_1$
X	Matrix of $\mathbf{x}_i$	V	Matrix of $\mathbf{v}_1$
Q	$=Z^{-1}N^{-1}$	W	Matrix of $\mathbf{w}_i$
		B_1	$= NAN^{-1}$

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