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# Body of Optimal Parameters in the Weighted Finite Element Method for the Crack Problem

Viktor A. Rukavishnikov

Computing Center of the Far Eastern Branch of the Russian Academy of Sciences, Kim Yu Chen St., 65, Khabarovsk, 680000, Russia

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Corresponding author: V.A. Rukavishnikov (vark0102@mail.ru)


**Abstract.** In this paper, a high-accuracy weighted finite element method is constructed and investigated for finding an approximate solution of the crack problem. We consider an approximation of the Lamé system in the domain with the reentrant corner $2\pi$ at the boundary. A new concept of definition of the solution of the problem is introduced. It allows us to suppress the influence of the singularity on the accuracy of finding an approximate solution, in contrast to the classical approach. We have introduced a weight function into the basis of the finite element method. The accuracy of finding an approximate solution by the weighted finite element method depends on three input parameters. We created an algorithm and establish the body of optimal parameters in the weighted finite element method for the crack problem. The choice of parameters from this set allows us to accurately and stability find an approximate solution with the smallest deviation from the best error. This is required to generate industrial codes.

**Keywords:** Elasticity problem with a crack, weighted finite element method, body of optimal parameters.

## 1. Introduction

Numerical methods for finding solutions to problems in the theory of elasticity with a singularity (tearing, sliding modes) play an essential role in fracture mechanics (see, for example, [1, 2]). For the system of Lamé equations on a nonconvex bounded polygonal domain with the Dirichlet conditions, it is known [3–6] that the solution to this problem can be written in the form

$$\mathbf{u}(x)=\sum_{j=0}^{m} r_j^{\pi/\omega_j} C_j \boldsymbol{\chi}(r_j,\theta_j)\Psi_j(r_j)+\boldsymbol{\psi}(x), \quad \boldsymbol{\psi}\in \mathbf{W}_2^2(\Omega) \tag{1}$$

Here $\mathbf{u}(x)=(u_1,u_2)$, $\boldsymbol{\chi}(r_j,\theta_j)=(\chi_1,\chi_2)$, $\boldsymbol{\psi}(x)=(\psi_1,\psi_2)$, coefficient $C_j$ is the stress intensity factor (SIF), $\chi_1,\chi_2$ are sufficiently smooth functions, $\omega_j$ is the internal angle, $\pi\le\omega_j\le 2\pi$, at the singularity $p_j$, $(r_j,\theta_j)$ are the polar coordinates at the point $p_j$, and $\Psi_j(r_j)$ is the cut-off function. The first term in eq. (1) determines the singular component of the solution and the second is its regular component.

The generalized solution of the boundary value problem for the Lamé system in a two-dimensional domain with a boundary containing reentrant corners $\omega_j$, $j=0,\dots,m$ belongs to the space $\mathbf{W}_2^{1+\gamma-\varepsilon}(\Omega)$, $\gamma=\min_{j=0,\dots,m}\gamma_j$, where $\gamma_j=\pi/\omega_j$ for the Dirichlet or Neumann problem and $\gamma_j=\pi/2\omega_j$ for the mixed boundary value problem, $\varepsilon$ is an arbitrary positive number (see e.g. [3]). According to the principle of coordinated estimations (see e.g. [7]) an approximate generalized solution to the problem obtained by the classic finite-element method converges to the generalized solution with the rate $O(h^\gamma)$ $(\gamma<1)$ in the norm of the space $\mathbf{W}_2^1(\Omega)$.

By using special methods for extracting the singular part of the solution near corner point or applying grids refined toward the singularity point, it is possible to construct first-order accurate finite-element schemes. But these methods lead to the ill-conditioned systems of linear algebraic equations. These involve complication of computing process and affect the accuracy of the results.

We suggested to define a solution to the boundary value problems with singularity as an $R_\nu$-generalized one in the weighted Sobolev space [8]. The essence of this approach is to include the weight function with a certain exponent in the integral identity of the weak solution. The value of the weighting function coincides with the distance to the vertex of the reentrant corner $p_j$ in some of its neighborhood. The presence of the weight function in the integral identity allows us to suppress the influence of the singularity on the accuracy of finding an approximate solution by the finite element method. This methodology is common for

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different problems with strong and corner singularity [9-12].

We have created and investigated a weighted finite element method with a convergence rate $O(h)$ to find an approximate $R_\nu$-generalized solution to problems of electrodynamics, hydrodynamics and the theory of elasticity [13-17].

We considered a crack problem in the form of the Lamé system in a domain with the corner of $2\pi$ on the boundary in [18]. In recent years a large number of different finite element methods were constructed to approximate the crack problem; among them are extended FEM (XFEM) [19–25], smoothed FEM (see, for example, [26–31]), meshless/meshfree methods [32-38] and field-enriched FEM [39-41]. We proposed a weighted finite element method (WFEM) for finding an approximate solution to the crack problem in [18]. This method is based on the introduction of an $R_\nu$-generalized solution. The reentrant corner $2\pi$ at the boundary of the domain does not affect the accuracy of finding of the approximate solution by the WFEM as compared to the classical FEM and the method with a refined mesh. The rate of convergence of an approximate WFEM solution to the exact one is $O(h)$ in the norm of the space $\mathbf{W}^1_{2,\nu+\beta/2}$ and in the weighted energy norm as demonstrated.

The determining factor that provides high accuracy of the WFEM is the correct choice of parameters: $\nu$ is the exponent of the weight function in an $R_\nu$-generalized solution, $\nu^*$ is the exponent of the weight function in the FEM basis, and $\delta$ is the radius of the neighborhood in which the weight function is specified in the course of calculations as the distance to the point of singularity.

Note that function $\chi(r_0,\theta_0)$ remains unchanged for problems with different initial data: coefficients and right-hand sides of the equation and boundary conditions (see, for example, [42]). It is required to define SIF $C_0$. The optimal parameters $\nu^*$ are contained in the interval [0,0.49], and the suitable parameters $\nu$ are found belonging to the half-interval established in the existence and uniqueness theorem of an $R_\nu$-generalized solution [11].

In this article the body of optimal parameters (BOP) is determined for the WFEM applied to the numerical solution of the crack problem. We have found the body of parameters at which the error of the approximate solution calculated by the WFEM in the norm of the weighted Sobolev space differs from the smallest error by no more than 5%, 10%, and 15%. As far as the stability of the computation process is concerned, we have demonstrated that a small change in the input parameters $\nu$, $\nu^*$, $\delta$ corresponds to a small change in the error. The BOP depends on the dimension of the mesh or the mesh step.

## 2. $R_\nu$-generalized solution for the crack problem

In [18] for finding of a displacement field $\mathbf{u}=(u_1,u_2)$ in the crack problem we considered the first boundary value problem of linear elasticity posed in displacements for isotropic homogeneous media:

$$-\left(2\,\mathbf{div}\left(\mu\varepsilon(\mathbf{u})\right)+\mathbf{grad}\left(\lambda\,\mathrm{div}\,\mathbf{u}\right)\right)=\mathbf{f},\quad x\in\Omega, \tag{2}$$

$$\mathbf{u}=\mathbf{q},\quad x\in\Gamma. \tag{3}$$

Here $\varepsilon(u)$ is a strain tensor, $\mathbf{f}$ is a distributed body force, $\lambda$ and $\mu$ are constant Lamé coefficients.

Without loss of generality, we will assume that $\Omega$ is the rectangle shown in Fig. 1.

Let $\Gamma$ be a boundary of domain $\Omega$ and $\Gamma_C\subset\Gamma$ be a crack with sides $\Gamma_C^+$ and $\Gamma_C^-$. We denote $\bar\Omega$ the closure of $\Omega$, i.e. $\bar\Omega=\Omega\cup\Gamma$.

**Comment 1.** The solution of the problem eqs. (2), (3) has the form ([3])

$$\mathbf{u}(x)=r_0^{1/2}\chi(r_0,\theta_0)\Psi_0(r_0)+\psi(x), \tag{4}$$

where $r_0$ is distance from point $x$ to $O(0,0)$. Therefore $\mathbf{u}\in\mathbf{W}^{1+1/2-\varepsilon}(\Omega)$ $(\varepsilon>0)$ and for regular finite elements methods one obtains an order of at most $O(h^{1/2-\varepsilon})$, where $h$ is the mesh step.

In [18] we proposed the weighted finite element method that allows to find an approximate solution to problem eqs. (2), (3) at a rate of $O(h)$.

Let $O^\delta$ be a disk of radius $\delta>0$ with its centre in the point $(0,0)$, the radius $\delta$ lot smaller than the side of rectangle $\Omega$ and $\Omega'=\Omega\cap O^\delta$.

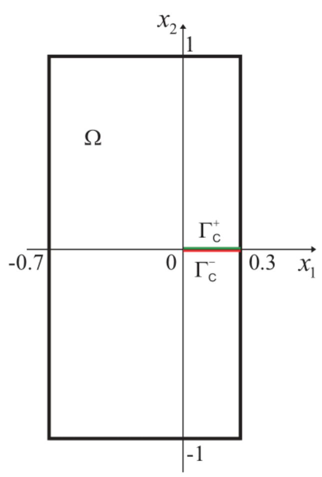


**Fig. 1.** [18] Rectangle domain $\Omega$ with a crack.

Let $\rho(x)$ be a weight function that is positive everywhere, except in $O(0,0)$, and satisfies the following conditions:

a) $\rho(x)=(x_1^2+x_2^2)^{1/2}$ for $x\in\overline{\Omega'}$,

b) $\rho(x)=\delta$ for $x\in\overline{\Omega}\setminus\overline{\Omega'}$.

We introduce the weighted spaces with norms:

$$\|u\|^2_{W^k_{2,\alpha}(\Omega)}=\sum_{|\lambda|\le k}\int_\Omega \rho^{2\alpha}\,|D^\lambda u|^2\,dx,$$
$$\|u\|^2_{L_{2,\alpha}(\Gamma)}=\int_\Gamma \rho^{2\alpha}u^2 ds,\quad \|u\|_{W^k_{2,0}(\Omega)}=\|u\|_{W^k_2(\Omega)}, \tag{5}$$

where $D^\lambda=\partial^{|\lambda|}/(\partial x_1^{\lambda_1}\partial x_2^{\lambda_2})$, $\lambda=(\lambda_1,\lambda_2)$ and $|\lambda|=\lambda_1+\lambda_2$, $k$ is a nonnegative integer, and $\alpha$ is a real number. The space $\overset{\circ}{W}{}^k_{2,\alpha}(\Omega)\subset W^k_{2,\alpha}(\Omega)$ is defined as a closure in the norm eq. (5) of the set of infinitely differentiable and finite in $\Omega$ functions.

We say that $\varphi\in W^{1/2}_{2,\alpha}(\Gamma)$ if there exists a function $\Phi(x)$ from $W^1_{2,\alpha}(\Omega)$ such that $\Phi(x)|_\Gamma=\varphi(x)$ and

$$\|\varphi\|_{W^{1/2}_{2,\alpha}(\Gamma,\delta)}=\inf_{\Phi|_\Gamma=\varphi}\|\Phi\|_{W^1_{2,\alpha}(\Omega)}.$$

Let $W^1_{2,\alpha}(\Omega,\delta)$ be the set of functions satisfying the following conditions:

a) $\left|D^\lambda u(x)\right|\le C_1\left(\dfrac{\delta}{\rho(x)}\right)^{\alpha+|\lambda|}$, $x\in\Omega'$, $|\lambda|=0,1$, $C_1>0$ is a constant;

b) $\|u\|_{L_{2,\alpha}(\Omega\setminus\Omega')}\ge C_2$, $C_2=const$,

with norm eq. (5).

By analogy, one can introduce sets for other spaces.

The spaces and sets for vector-functions are designated with bold letters, for example $\mathbf{W}^1_{2,\alpha}(\Omega)$.

**Definition 1.** [11] *Let the right-hand sides of eqs. (2), (3) satisfy the conditions*

$$\mathbf{f}\in\mathbf{L}_{2,\beta}(\Omega),\quad \mathbf{q}\in\mathbf{W}^{1/2}_{2,\beta}(\Gamma),\quad \beta\ge 0.$$

*A function $\mathbf{u}_\nu=(u_{\nu1},u_{\nu2})$ from the space $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega)$ is called **an $R_\nu$-generalized solution** to the problem eqs. (2), (3) if it satisfies boundary condition eq. (3) almost everywhere on $\Gamma$ and for every $\mathbf{v}$ from $\overset{\circ}{\mathbf{W}}{}^1_{2,\alpha+\beta/2}(\Omega)$ the integral identities*

$$a_1(\mathbf{u}_\nu,v_1)=\int_\Omega\left[(\lambda+2\mu)\frac{\partial u_{\nu1}}{\partial x_1}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_1}+\mu\frac{\partial u_{\nu1}}{\partial x_2}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_2}+\lambda\frac{\partial u_{\nu2}}{\partial x_2}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_1}+\mu\frac{\partial u_{\nu2}}{\partial x_1}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_2}\right]dx=\int_\Omega\rho^{2\nu}f_1v_1dx=l_1(v_1);$$

$$a_2(\mathbf{u}_\nu,v_2)=\int_\Omega\left[\lambda\frac{\partial u_{\nu1}}{\partial x_2}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_1}+\mu\frac{\partial u_{\nu1}}{\partial x_2}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_1}+(\lambda+2\mu)\frac{\partial u_{\nu2}}{\partial x_2}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_2}+\mu\frac{\partial u_{\nu2}}{\partial x_1}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_1}\right]dx=\int_\Omega\rho^{2\nu}f_2v_2dx=l_2(v_2)$$

*holds for any fixed value of $\nu$ satisfying the inequality $\nu\ge\beta$.*

**Comment 2.** We notice that an $R_\nu$-generalized solution has a sheaf of solutions in the neighborhood of the singularity point if it is defined in the weighted space $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega)$ and does not take into account the additional properties of this solution (see, for example, [43]). In [11] we proved the uniqueness of an $R_\nu$-generalized solution if it is defined in the set $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega,\delta)$.

An $R_\nu$-generalized solution satisfies conditions (a), (b) of the set $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega,\delta)$. This follows from the asymptotic of the solution to problem eqs. (2), (3) (see eq. (4)). We use the "special" properties of functions from this set additionally. At the same time, we do not refuse to use the properties of the space $\mathbf{W}^1_{2,\nu+\beta/2}(\Omega)$ (the presence of a zero element, etc.).

**Comment 3.** We proved that the $R_\nu$-generalized solution is the same for different $\nu$ (see [44]).

**Comment 4.** In contrast to the weak solution of problem eqs. (2), (3), the weight function is introduced into the definition of an $R_\nu$-generalized solution. This allows us to suppress the influence of the singularity on the regularity of the solution. In [44] we proved that an $R_\nu$-generalized solution of a boundary value problem for a second-order elliptic equation belongs to the weighted space $W^2_{2,\nu+\beta/2}(\Omega)$. Subsequently this made it possible to establish the convergence of the approximate solution to the $R_\nu$-generalized solution with a rate $O(h)$ ([45]).

## 3. Weighted finite element method

The weighted finite element method for finding an approximate an $R_\nu$-generalized solution of problem eqs. (2), (3) was constructed in [18]. Here we briefly describe construction of the WFEM.

We perform a quasi-uniform triangulation of the domain $\overline{\Omega}$ (see Fig. 2). Let $K$ be the union of all the triangles $K_i$, $i=1,\dots,n$; $h$ is the maximal length of the sides of the triangles and it called mesh step. The vertices $P_i$, $i=1,\dots,M$ of the triangles $K$ are nodes of the triangulation, $\{P^M\}=\{P_1,\dots,P_M\}$ and the point $O\in P^M$. Let $P=\{P_k\}_{k=1}^{k=N}$ is the set of internal triangulation nodes.

To each node $P_i\in P$ we assign the weighted function

$$\widehat{\psi}_i=\rho^{\nu^*}(x)\varphi_i,\quad i=1,\dots,N,$$

where $\varphi_i$ is linear function on each triangle $K$, equal to 1 at the node $P_i$ and zero at all the other nodes, $\nu^*$ is a real number.

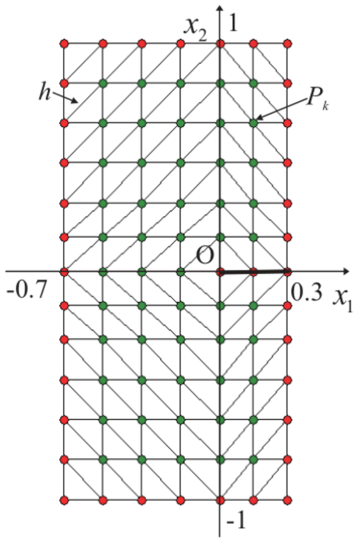


**Fig. 2.** [18] Triangulation of domain $\Omega$ .

We introduce weighted vector basis $\{\boldsymbol{\psi}_k(x)\}_{k=1}^{k=2N}$ , where

$$\boldsymbol{\psi}_k(x)=\begin{cases}(\widehat{\psi}_i(x),0), k=2i-1,\\ (0,\widehat{\psi}_i(x)), k=2i,\end{cases} \quad i=1,\ldots,N.$$

We denote by $\mathbf{V}^h$ the linear span $\{\boldsymbol{\psi}_k(x)\}_{k=1}^{k=2N}$ . In $\mathbf{V}^h$ we denote the subset $\overset{\circ}{V}{}^h=\{\mathbf{v}\in\mathbf{V}^h:\mathbf{v}(P_i)=0, P_i\in\Gamma\}$ .

**Definition 2.** *A function* $\mathbf{u}_\nu^h$ *in the space* $\mathbf{V}^h$ *is called an* ***approximate*** $R_\nu$ ***-generalized solution*** *of the problem eqs. (2), (3) by the weighted finite element method if it satisfies the boundary condition eq. (3) for mesh nodes* $P_i\in\Gamma$ *and the integral identity*

$$a(\mathbf{u}_\nu^h,\mathbf{v}^h)=l(\mathbf{v}^h)$$

*holds for all* $\mathbf{v}^h\in\overset{\circ}{V}{}^h$ *and* $\nu\geq\beta$ *. Here* $a(\mathbf{u}_\nu^h,\mathbf{v}^h)=(a_1(\mathbf{u}_\nu^h,v_1^h),a_2(\mathbf{u}_\nu^h,v_2^h))$ , $l(\mathbf{v})=(l_1(v_1^h),l_2(v_2^h))$ .\

An approximate solution will be found in the form

$$\mathbf{u}_\nu^h=\sum_{k=1}^{2N}d_k\boldsymbol{\psi}_k(x),$$

here $d_k=\rho^{-\nu^*}(P_{[(k+1)/2]})c_k$ , $c_k=\begin{cases}u_{\nu,1}^h(P_{[(k+1)/2]}), k=2i-1\\ u_{\nu,2}^h(P_{[(k+1)/2]}), k=2i\end{cases}$ , $i=1,\ldots,N$ , $[(k+1)/2]$ is an integer part of number $(k+1)/2$ .

**Comment 5.** Note that we have introduced into the basis the weight function raised to some power. The weight basis and an $R_\nu$ -generalized solution made it possible to find an approximate solution without loss of accuracy on quasi-uniform grids.

We proved that the approximate $R_\nu$ -generalized solution by the weighted FEM converges to the exact one with the first rate with respect to the mesh step $h$ [45].

In [18] a numerical analysis was carried out for one model problem on grids of large and small dimensions.

We have obtained experimentally confirmation of the convergence rate of the approximate solution to the exact one $O(h)$ in the norm of the space $\mathbf{W}_{2,\nu}^1(\Omega)$ and in the energy norm. In addition, the smallness of the absolute error $(10^{-7})$ in the overwhelming number of grid nodes was established.

## 4. Body of optimal parameters

### 4.1 Algorithm for determining BOP on grids of various dimensions

For calculation of the approximate $R_\nu$ -generalized solution by the weighted finite element method we need to set the parameters $\nu$ , $\nu^*$ , $\delta$ . These parameters can be arbitrary if they satisfy conditions of the theorem on the existence and uniqueness of the $R_\nu$ -generalized solution and correspond to the asymptotic properties of the solution. But if you want to find an approximate solution to the problem with the smallest error, then these parameters should be close to optimal. Currently, there is no algorithm for theoretical determination of such parameters. We will establish them experimentally for problem eqs. (2), (3), taking into account the invariability of the function $\chi(r_0,\theta_0)$ and information on the admissible intervals of the parameters $\nu$ and $\nu^*$ .

Consider two model problems in the domain $\Omega$ :

(A) Boundary value problem eqs. (2), (3) with a solution containing only a singular component

$$u_1=\frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left(1-\frac{\lambda}{\lambda+\mu}+\sin^2\left(\frac{\theta}{2}\right)\right),$$

$$u_2 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left(2 - \frac{\lambda}{\lambda+\mu} + \cos^2\left(\frac{\theta}{2}\right)\right),$$

Lamé coefficients are $\lambda = 576.923, \mu = 384.615$, and stress intensity factor $K_I = 1.611$.

(B) Boundary value problem eqs. (2), (3) with a solution containing a singular and a regular component from the space $\mathbf{W}_2^2(\Omega)$

$$u_1 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left(1 - \frac{\lambda}{\lambda+\mu} + \sin^2\left(\frac{\theta}{2}\right)\right) + r^2,$$

$$u_2 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left(2 - \frac{\lambda}{\lambda+\mu} + \cos^2\left(\frac{\theta}{2}\right)\right) + r^2.$$

Let us find for problems (A) and (B) the parameters $\nu, \nu^*, \delta$, which allow us to calculate an approximate solution by the weighted finite element method with the best accuracy on quasiuniform meshes of various dimensions. In $\Omega$ we built meshes with a step $h = 0.062, 0.031, 0.015, 0.0077, 0.0038, 0.0019$ and determined the BOP for each of these meshes.

The set of optimal parameters will be discrete, as we form it from the results of numerical experiments carried out for given fixed values $\nu, \nu^*, \delta$.

We chose $\nu^*$ equal to $0, 0.1, 0.2, 0.3, 0.4, 0.49$. The values of $\nu$ were selected from the interval [0.5, 5.5] with a step of 0.1. The radius of the $\delta$-neighborhood $\Omega'$ was equated to $h, 2h, 3h, \ldots$. Calculations were stopped or later disregarded when the error between the exact solution and the found approximate solution became larger than specified limiting error. The relative error was determined for all grids and parameters of WFEM in the weighted Sobolev norm and weighted energy norm with fixed and predetermined parameters $\bar{\nu} = 2.2$, $\bar{\delta} = 0.062$. Note that when choosing other parameters $\bar{\nu}$ and $\bar{\delta}$, there were no significant changes in the results.

For each problem (A), (B) and each mesh we determined three parameters $\nu$, $\nu^*$, $\delta$ for which the relative error in the weighted Sobolev norm and weighted energy norm was the smallest. In addition, we formed sets of parameters $T_1^{(A)}$, $T_2^{(A)}$, $T_3^{(A)}$ and $T_1^{(B)}$, $T_2^{(B)}$, $T_3^{(B)}$ at which the relative errors differed from the best error by no more than $5\%(6.5\%)$, $10\%$ and $15\%$.

**Comment 6.** The ratio of the smallest errors was exactly two in both weighted norms on adjacent meshes for problems (A) and (B). This corresponds to a theoretical estimate of the convergence rate.

For each mesh the body of optimal parameters (BOP) is $T_i = T_i^{(A)} \cap T_i^{(B)}$, $i = 1,2,3$.

We studied the process of choosing the optimal parameters for stability, that is, how much the error deviates with a small change in the parameters.

In addition, we have determined triples of parameters $\nu, \nu^*, \delta$, which allow us to find an approximate solution with an error that differs from the best error by no more than $6.5\%$ on all meshes simultaneously.

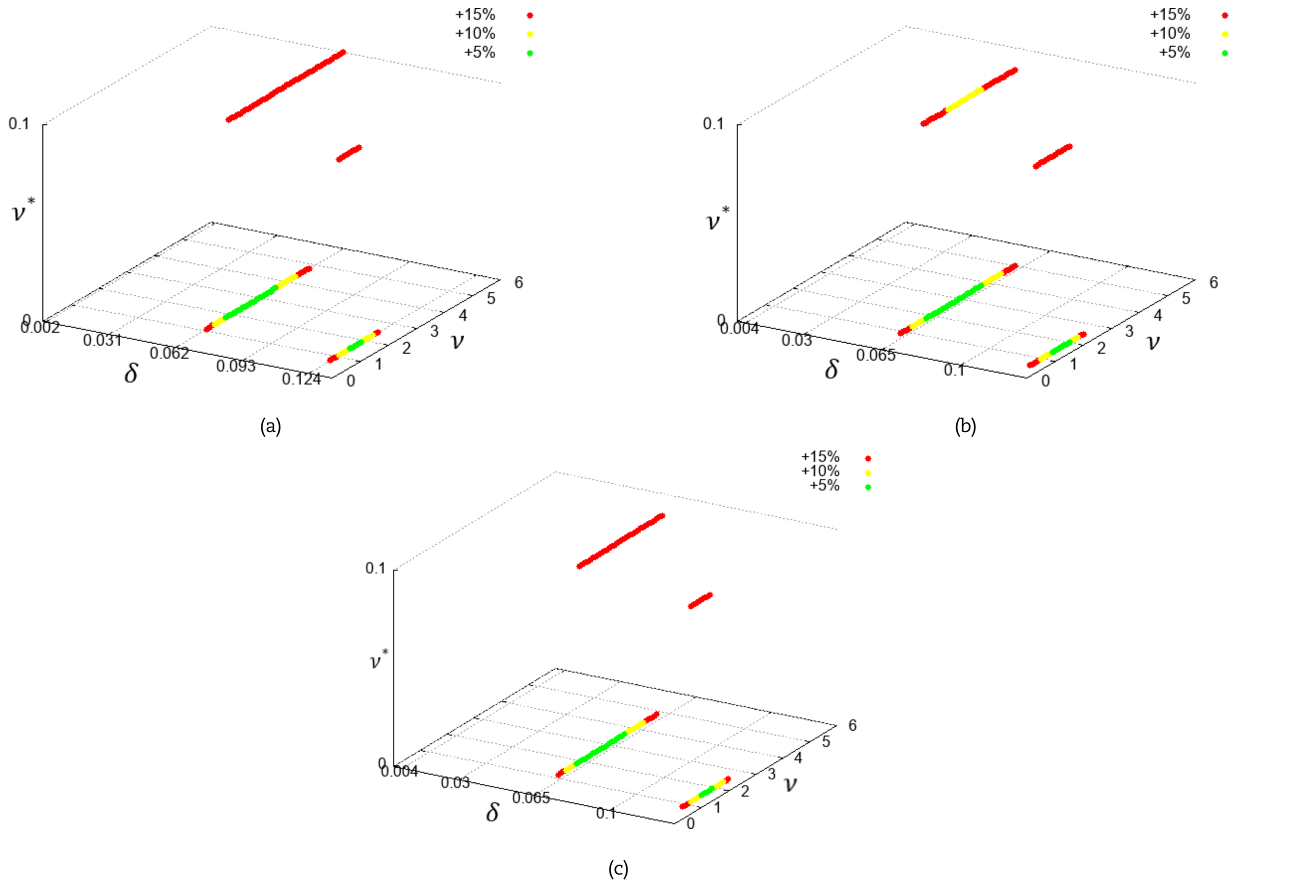


**Fig. 3.** (3a) the sets $T_1^{(A)}$, $T_2^{(A)}$, $T_3^{(A)}$; (3b) the sets $T_1^{(B)}$, $T_2^{(B)}$, $T_3^{(B)}$; (3c) the sets $T_i = T_i^A \cap T_i^B, i = 1,2,3$ for the mesh with step $h = 0.062$.

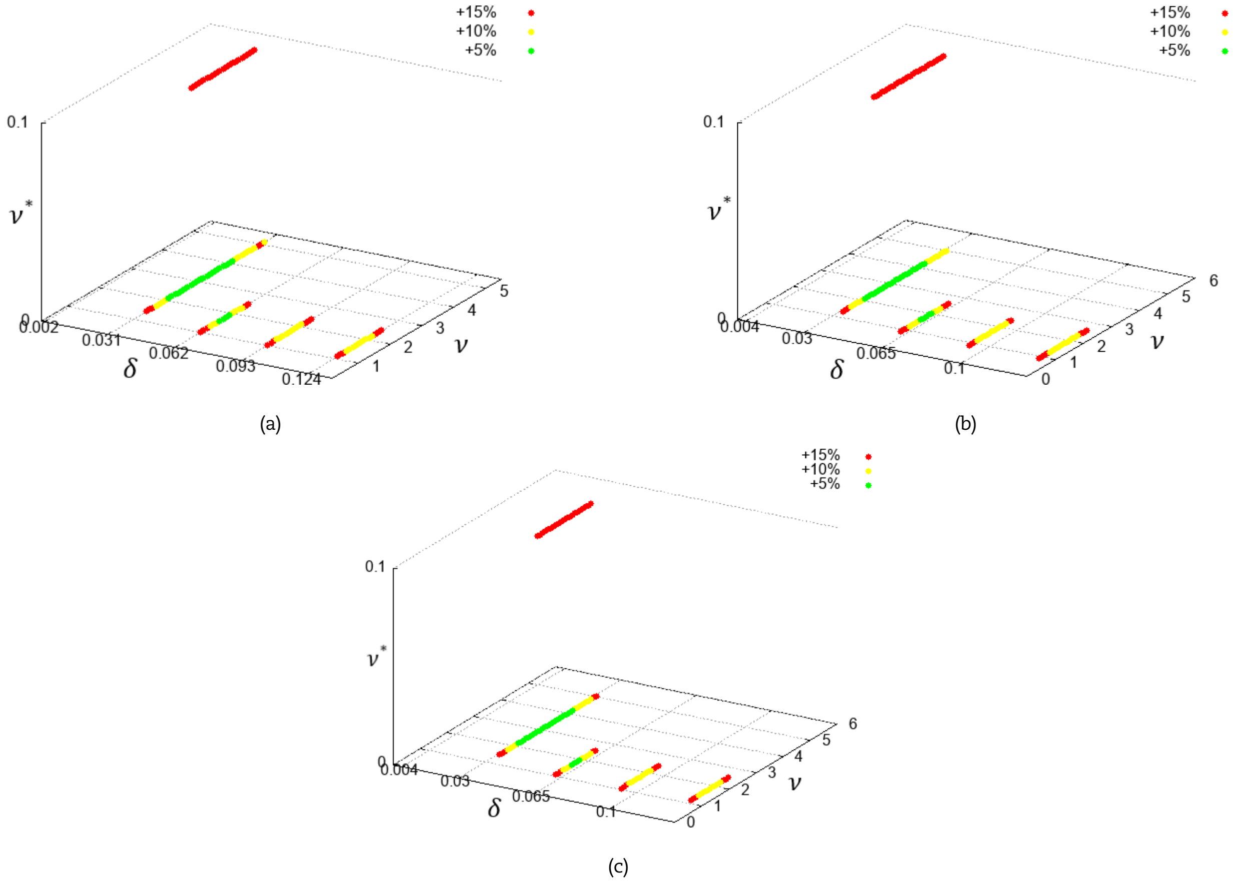


**Fig. 4.** (4a) the sets $T_1^{(A)}$ , $T_2^{(A)}$ , $T_3^{(A)}$ ; (4b) the sets $T_1^{(B)}$ , $T_2^{(B)}$ , $T_3^{(B)}$ ; (4c) the sets $T_i = T_i^A \cap T_i^B, i = 1,2,3$ for the mesh with step $h = 0.031$ .

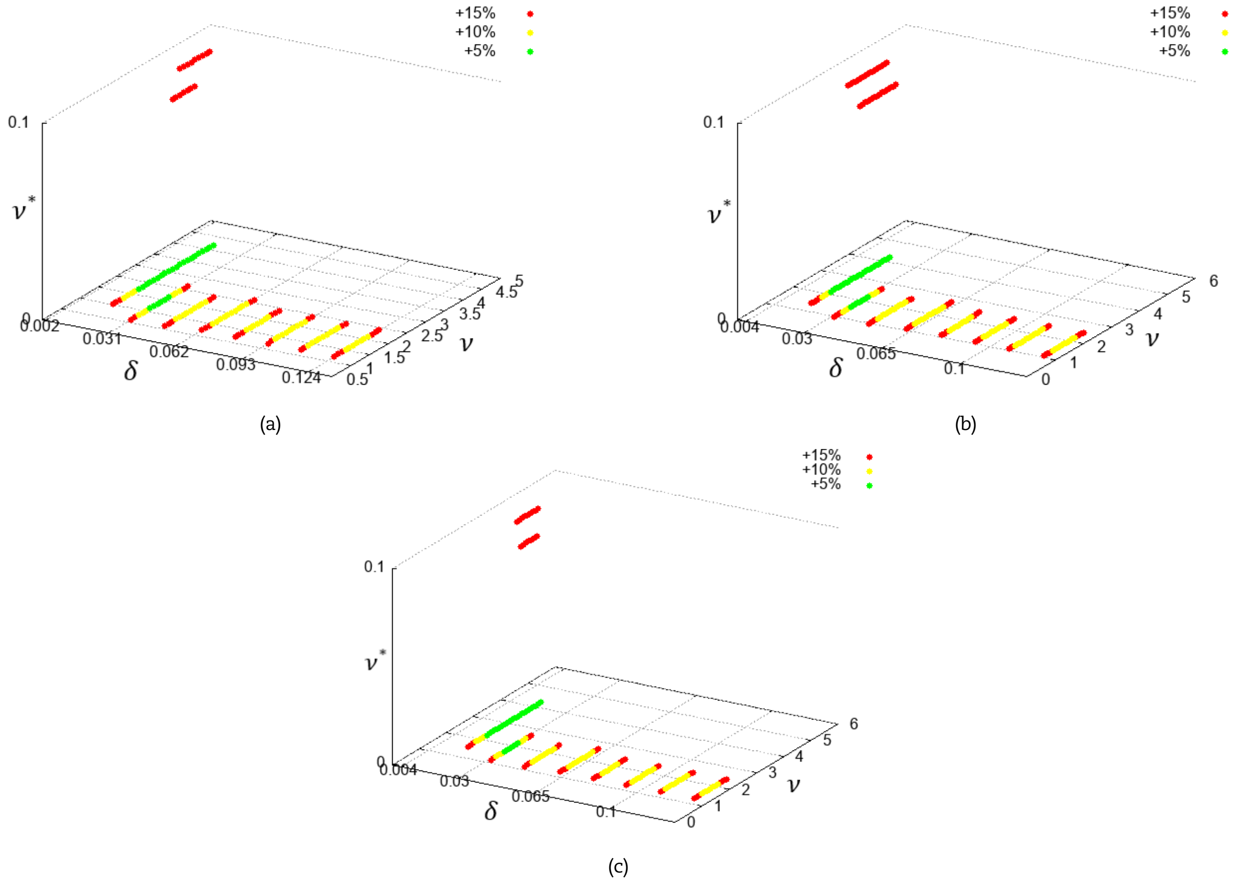


**Fig. 5.** (5a) the sets $T_1^{(A)}$ , $T_2^{(A)}$ , $T_3^{(A)}$ ; (5b) the sets $T_1^{(B)}$ , $T_2^{(B)}$ , $T_3^{(B)}$ ; (5c) the sets $T_i = T_i^A \cap T_i^B, i = 1,2,3$ for the mesh with step $h = 0.015$ .

**Table 1.** Optimal parameters with a given error for the meshes with steps $h = 0.062$ and $h = 0.031$ .

| Percent of error | $\delta$ | $\nu$ | $\nu^*$ | Percent of error | $\delta$ | $\nu$ | $\nu^*$ |
|---|---|---|---|---|---|---|---|
| +5% | 0.06187 | 1.8..3.6 | 0.0 | +5% | 0.03094 | 2.1..4.0 | 0.0 |
| | 0.12374 | 1.5..1.9 | 0.0 | | 0.06187 | 1.6..1.8 | 0.0 |
| +10% | 0.06187 | 1.4..4.3 | 0.0 | +10% | 0.03094 | 1.7..4.8 | 0.0 |
| | 0.12374 | 1.1..2.2 | 0.0 | | 0.06187 | 1.3..2.2 | 0.0 |
| +15% | 0.06187 | 1.1..4.9 | 0.0 | | 0.09281 | 1.3..2.1 | 0.0 |
| | 0.06187 | 1.9..5.0 | 0.1 | | 0.12374 | 1.4..2.2 | 0.0 |
| | 0.12374 | 0.8..2.5 | 0.0 | +15% | 0.03094 | 1.4..5.0 | 0.0 |
| | 0.12374 | 1.1..1.9 | 0.1 | | 0.03094 | 2.8..4.8 | 0.1 |
| | | | | | 0.06187 | 1.0..2.5 | 0.0 |
| | | | | | 0.09281 | 1.0..2.4 | 0.0 |
| | | | | | 0.12374 | 1.1..2.5 | 0.0 |

**Table 2.** Optimal parameters with a given error for the meshes with steps $h = 0.015$ and $h = 0.0077$ .

| Percent of error | $\delta$ | $\nu$ | $\nu^*$ | Percent of error | $\delta$ | $\nu$ | $\nu^*$ |
|---|---|---|---|---|---|---|---|
| +5% | 0.01547 | 2.2..4.0 | 0.0 | +5% | 0.00773 | 2.3..3.6 | 0.0 |
| | 0.03094 | 1.6..2.1 | 0.0 | | 0.01547 | 1.6..2.2 | 0.0 |
| +10% | 0.01547 | 1.8..4.2 | 0.0 | | | 1.8..2.1 | 0.0 |
| | 0.03094 | 1.3..2.4 | 0.0 | | | 1.9..2.2 | 0.0 |
| | 0.0464- 0.12374 | 1.4..2.2 | 0.0 | | | 1.9..2.1 | 0.0 |
| +15% | 0.01547 | 1.5..4.2 | 0.0 | +10% | 0.00773 | 1.9..3.6 | 0.0 |
| | 0.01547 | 3.3..4.1 | 0.1 | | 0.01547 | 1.4..2.6 | 0.0 |
| | 0.03094 | 1.1..2.6 | 0.0 | | 0.01547 | 3.8..3.8 | 0.0 |
| | 0.03094 | 2.2..2.8 | 0.1 | | 0.01547 | 2.7..3.2 | 0.1 |
| | 0.0464- 0.12374 | 1.2..2.4 | 0.0 | | 0.0232 | 1.4..2.4 | 0.0 |
| | | | | | 0.03094 | 1.6..2.7 | 0.0 |
| | | | | | 0.03867- 0.12374 | 1.5..2.2 | 0.0 |
| | | | | | 0.0464 | 1.5..2.2 | 0.0 |
| | | | | +15% | 0.00773 | 1.6..3.6 | 0.0 |
| | | | | | 0.01547 | 1.1..2.8 | 0.0 |
| | | | | | 0.01547 | 3.8..3.8 | 0.0 |
| | | | | | 0.01547 | 2.0..3.6 | 0.1 |
| | | | | | 0.0232 | 1.2..2.6 | 0.0 |
| | | | | | 0.0232 | 3.8..3.8 | 0.0 |
| | | | | | 0.03094 | 1.3..2.9 | 0.0 |
| | | | | | 0.03094 | 3.8..3.8 | 0.0 |
| | | | | | 0.03867- 0.12374 | 1.2..2.5 | 0.0 |

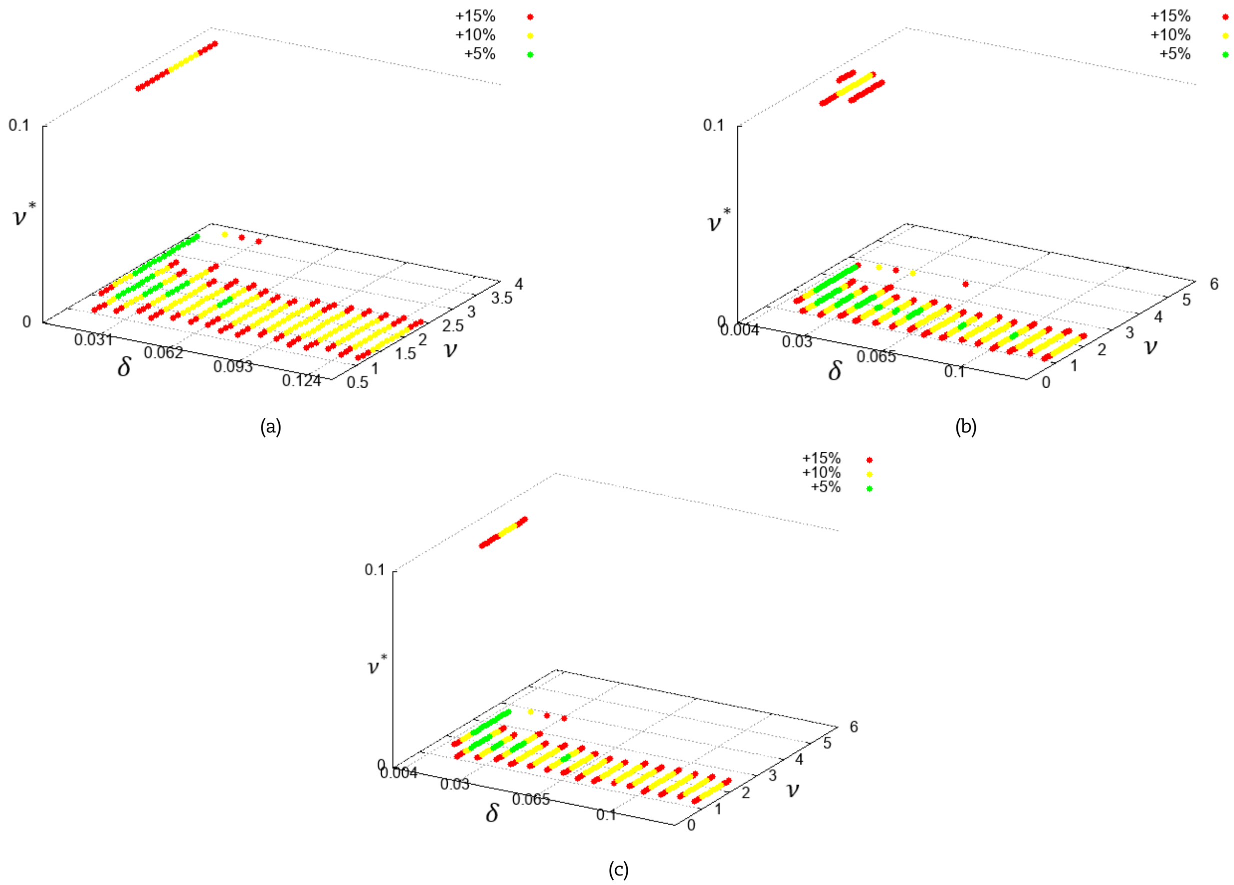


**Fig. 6.** (6a) the sets $T_1^{(A)}$ , $T_2^{(A)}$ , $T_3^{(A)}$ ; (6b) the sets $T_1^{(B)}$ , $T_2^{(B)}$ , $T_3^{(B)}$ ; (6c) the sets $T_i = T_i^A \cap T_i^B, i = 1,2,3$ for the mesh with step $h = 0.0077$ .

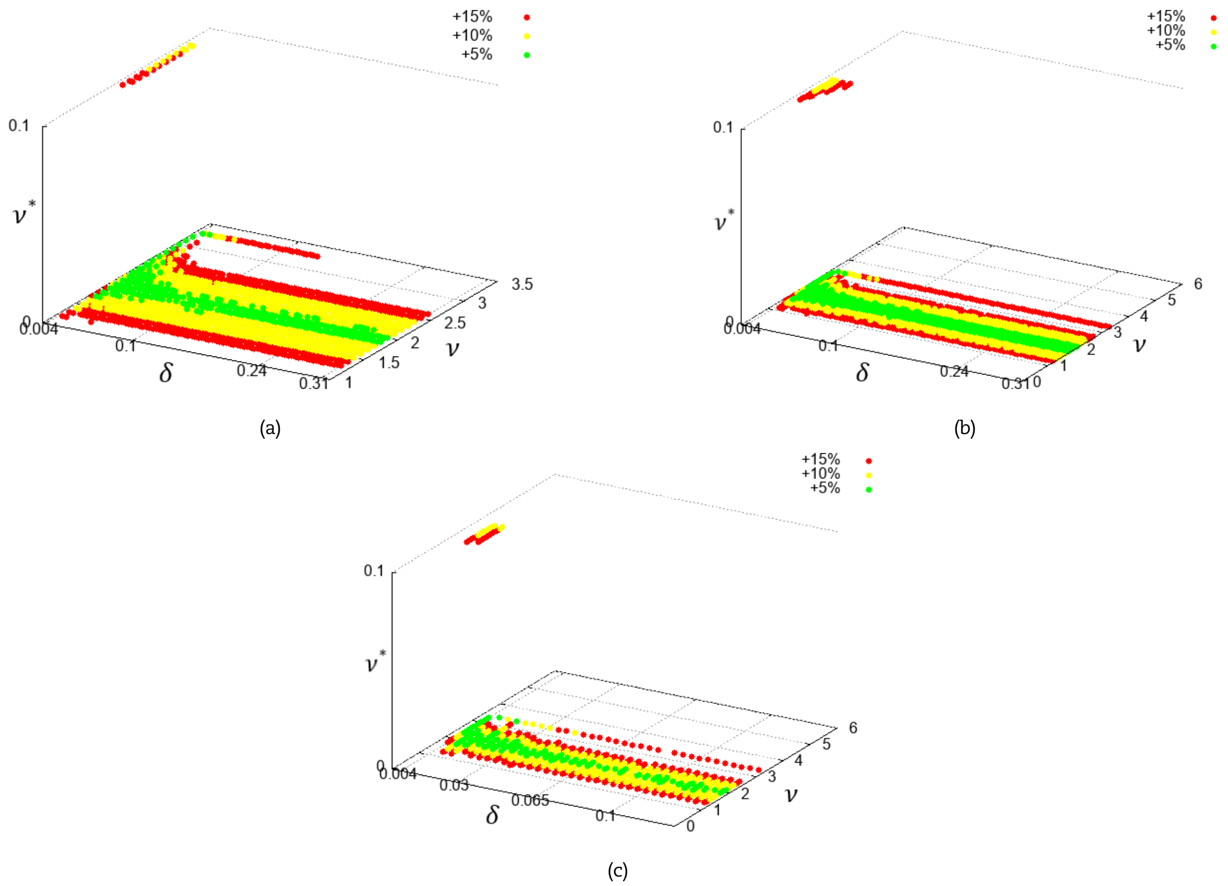


**Fig. 7.** (7a) the sets $T_1^{(A)}$ , $T_2^{(A)}$ , $T_3^{(A)}$ ; (7b) the sets $T_1^{(B)}$ , $T_2^{(B)}$ , $T_3^{(B)}$ ; (7c) the sets $T_i = T_i^A \cap T_i^B, i = 1,2,3$ for the mesh with step $h = 0.0038$ .

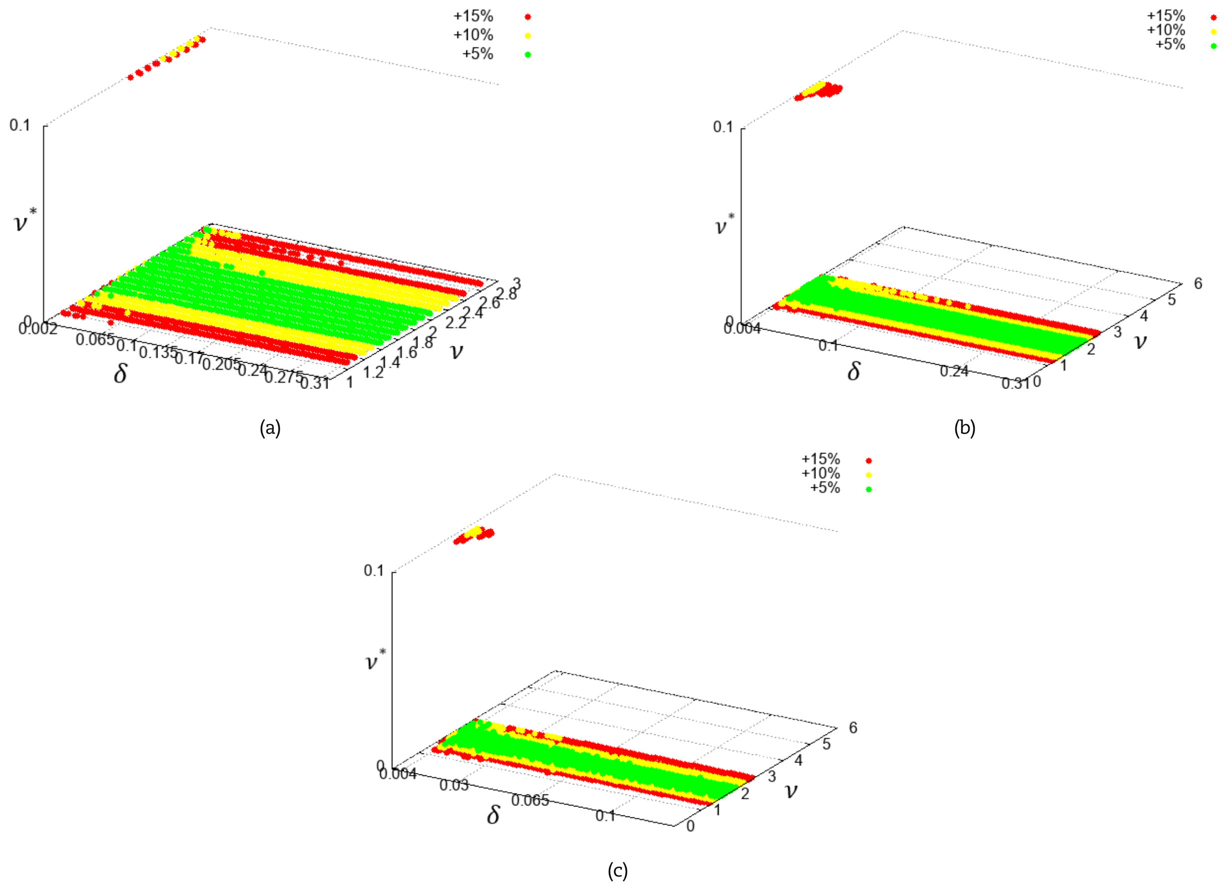


**Fig. 8.** (8a) the sets $T_1^{(A)}$ , $T_2^{(A)}$ , $T_3^{(A)}$ ; (8b) the sets $T_1^{(B)}$ , $T_2^{(B)}$ , $T_3^{(B)}$ ; (8c) the sets $T_i = T_i^A \cap T_i^B, i = 1,2,3$ for the mesh with step $h = 0.0019$ .

**Table 3.** Optimal parameters with a given error for the meshes with steps $h = 0.0038$ and $h = 0.0019$ .

| Percent of error | $\delta$ | $\nu$ | $\nu^*$ | Percent of error | $\delta$ | $\nu$ | $\nu^*$ |
|---|---|---|---|---|---|---|---|
| +5% | 0.00387 | 2.3..3.2 | 0.0 | +5% | 0.00193 | 2.3..2.8 | 0.0 |
| | 0.00773 | 1.7..2.5 | 0.0 | | 0.00387 | 1.6..2.6 | 0.0 |
| | 0.0116 | 1.7..2.3 | 0.0 | | 0.00387 | 2.9..2.9 | 0.0 |
| | 0.01547 | 1.9..2.6 | 0.0 | | 0.0058 | 1.7..2.5 | 0.0 |
| | 0.01934-0.02707 | 1.8..2.2 | 0.0 | | 0.00773 | 1.8..2.9 | 0.0 |
| | 0.03094- 0.3 | 1.9..2.1 | 0.0 | | 0.00967- 0.3 | 1.8..2.1 | 0.0 |
| +10% | 0.00387 | 2.0..3.2 | 0.0 | +10% | 0.00193 | 1.9..2.8 | 0.0 |
| | 0.00773 | 1.4..2.7 | 0.0 | | 0.00387 | 1.4..2.9 | 0.0 |
| | 0.00773 | 3.3..3.3 | 0.0 | | 0.00387 | 2.4..2.8 | 0.1 |
| | 0.00773 | 2.5..3.1 | 0.1 | | 0.0058 | 1.4..2.7 | 0.0 |
| | 0.0116 | 1.4..2.6 | 0.0 | | 0.0058 | 2.9..2.9 | 0.0 |
| | 0.0116 | 3.3..3.3 | 0.0 | | 0.0058 | 2.5..2.7 | 0.1 |
| | 0.0116 | 3.0..3.1 | 0.1 | | 0.00773 | 1.6..2.9 | 0.0 |
| | 0.01547 | 1.6..2.9 | 0.0 | | 0.00967 | 1.5..2.7 | 0.0 |
| | 0.01547 | 3.3..3.3 | 0.0 | | 0.00967 | 2.9..2.9 | 0.0 |
| | 0.01934- 0.3 | 1.5..2.4 | 0.0 | | 0.0116- 0.3 | 1.5..2.5 | 0.0 |

### 4.2 Numerical Results

Figures 3, 4, 5, 6, 7, 8 show the parameters $\nu$ , $\nu^*$ , $\delta$ at which the errors differ from the best error by no more than 5% (green), 10% (yellow), 15% (red) at $h = 0.062, 0.031, 0.015, 0.0077, 0.0038, 0.0019$ respectively. We present the results for tasks A and B in Figs 3a - 8a and Figs 3b - 8b, respectively. Figures 3c - 8c depict the sets $T_i$ , $i = 1,2,3$ . In Tables 1, 2 and 3 we indicated the intervals of the parameters $\nu$ , $\nu^*$ , $\delta$ of the BOP at which the relative error in the norm of the weight space deviates from the best error by no more than the indicated values for $h = 0.062$ and $h = 0.031$ , $h = 0.015$ and $h = 0.0077$ , $h = 0.0038$ and $h = 0.0019$ respectively.

We present the values of the parameters at which the deviation of the relative error from the best error does not exceed 5% , 5.5% , and 6% for problem A on the mesh with a step $h = 0.0015$ (Fig. 9) and $h = 0.0038$ (Fig. 10).

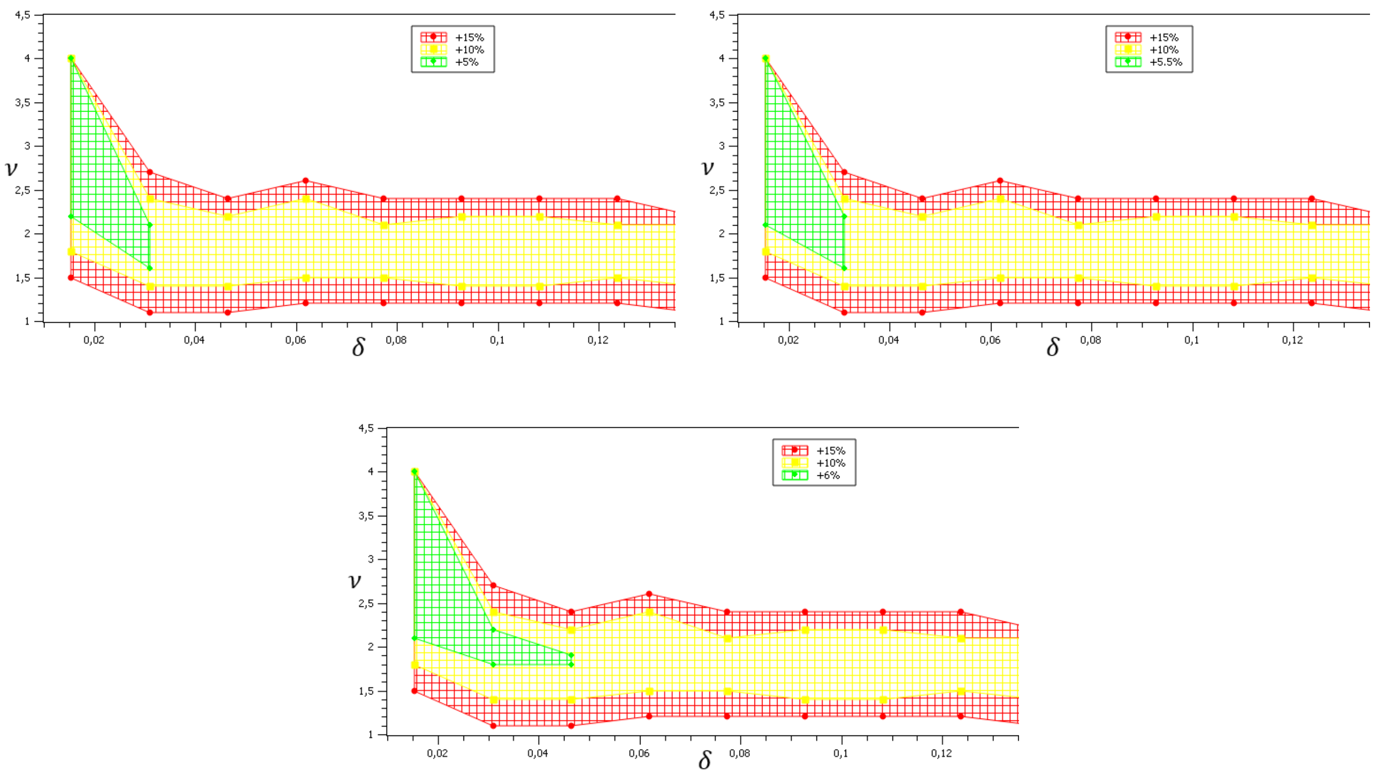


**Fig. 9.** Parameter values at which the deviation of the relative error from the best error does not exceed 5% , 5.5% , and 6% for problem A on the mesh with a step $h = 0.015$ .

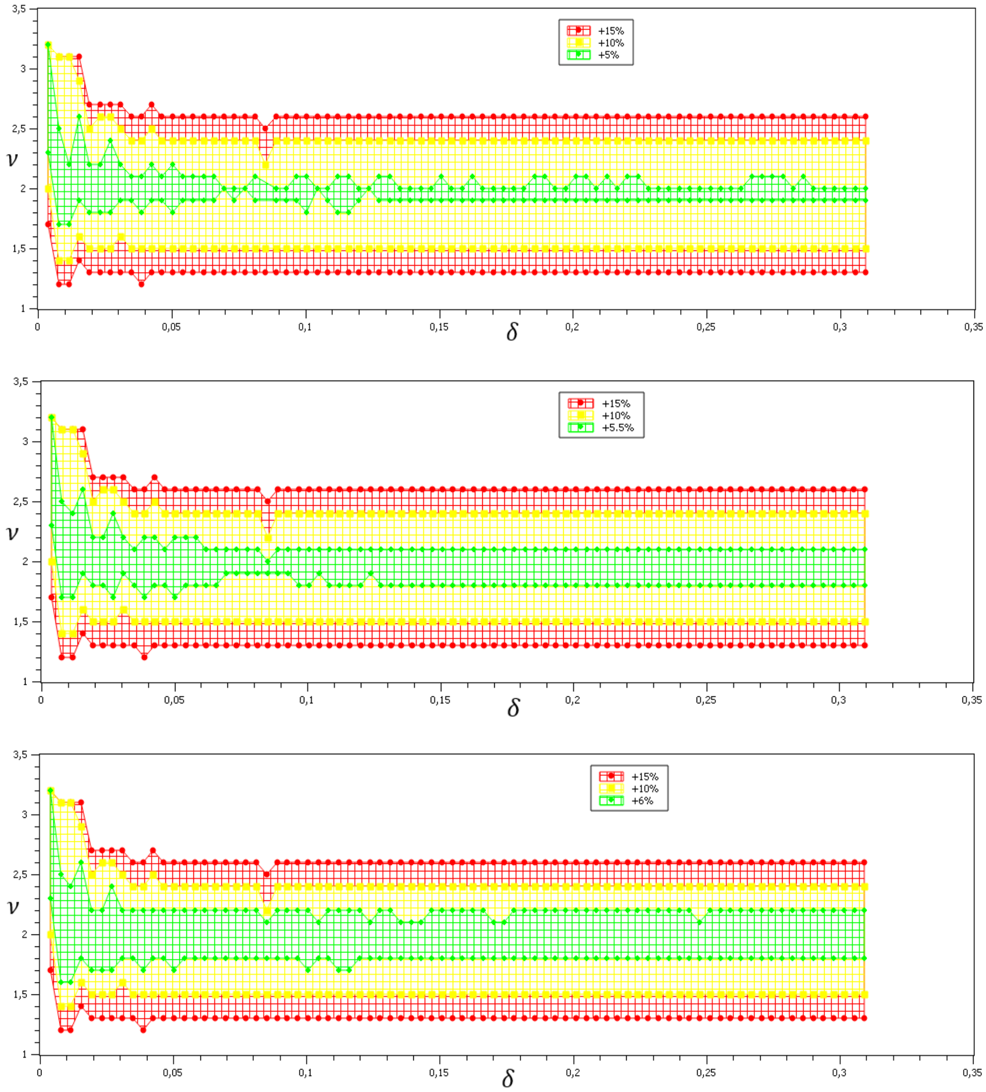


**Fig. 10.** Parameter values at which the deviation of the relative error from the best error does not exceed $5\%$, $5.5\%$, and $6\%$ for problem A on the mesh with a step $h = 0.0038$.

## 5. Conclusion

In this paper we defined the body of optimal parameters in the weighted finite element method to find an approximate solution to the crack problem with high accuracy. Finding the BOP is based on a series of numerical experiments. We used the knowledge about the asymptotic behavior of the solution in the neighborhood of the singularity point and the conditions on the input data $\nu$, $\delta$ of the existence and uniqueness theorem for the $R_\nu$-generalized solution. The results of the experiments led to the following conclusions:

1. The proposed approach allows us to determine the BOP for the weighted finite element method (Figs. 3, 4, 5, 6, 7, 8, Table 1, 2, 3).
2. BOP depends on the dimension of the mesh (mesh step).
3. With a small deviation in the choice of parameters from the best parameters in WFEM, the relative error in the norm of the Sobolev weight space grows slightly (see Fig. 9, Fig. 10). This indicates the stability of the process, i.e., a small change in the control parameters corresponds to a small increase in the relative error of the approximate solution.
4. If we choose the parameters $\delta = 0.062$, $\nu = 2.0$, $\nu^* = 0$ then the error does not exceed $6.75\%$ of the best value the error on all meshes simultaneously. In our opinion this fact is not important. The optimal parameters for carrying out calculations should be chosen depending on the dimension of the mesh (mesh step).
5. The proposed method with the found BOP can be used for calculating engineering problems.

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### Conflict of Interest

The author declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

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## Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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## ORCID iD

Viktor A. Rukavishnikov https://orcid.org/0000-0002-3702-1126



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