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Shahid Chamran
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Rotations of a Rigid Body Close to the Lagrange Case under the Action of Nonstationary Perturbation Torque

Dmytro Leshchenko[1], Sergey Ershkov[2], Tetiana Kozachenko[3]

[1] Department of Theoretical Mechanics, Odessa State Academy of Civil Engineering and Architecture, 4 Didrikhson st., Odessa, 65029, Ukraine, Email: leshchenkodmytro@gmail.com

[2] Department of Scientific Researches, Plekhanov Russian University of Economics,
Scopus number 60030998, 36 Stremyanny lane, Moscow, 117997, Russia, Email: sergej-ershkov@yandex.ru

[3] Department of Theoretical Mechanics, Odessa State Academy of Civil Engineering and Architecture, 4 Didrikhson st., Odessa, 65029, Ukraine, Email: kushpil.t.a@gmail.com

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Corresponding author: D. Leshchenko (leshchenkodmytro@gmail.com)

Abstract. Perturbed motions of a rigid body, close to the Lagrange case, under the action of restoring and perturbation torques of forces are investigated in the paper. The following problem is formulated: investigating solutions behavior of system of equations of motion for nonzero values of small parameter on a sufficiently long time interval. To analyze a nonlinear system of equations of motion, the averaging method is used. The problem can be decomposed into slowly and quickly changing variables. Conditions for the possibility of averaging the equations of motion with respect to the phase of nutation angle are presented and averaging procedure for slow variables of a perturbed motion of a rigid body in the first approximation is described. As an example of the developed procedure, we investigate a perturbed motion, close to Lagrange case, taking into account constant dissipative and small torque, and dissipative torques depending on slow time. A new class of rotational motions of a dynamically symmetric rigid body about a fixed point has been investigated with restoring and perturbation torques of forces being taken into account.

Keywords: Perturbed motion, Averaging method, Torque, Lagrange's case, Rigid body.

1. Introduction

The motion of a rigid body about a fixed point is one of the important problems in mechanics. In dynamics of a rigid body with a fixed point there exists vast bibliography on the perturbed motions, close to Lagrange top, and on the applications in the problems of flying vehicle entry into the atmosphere [1, 2], rotating projectile motion [3] and gyroscopy [4-8].

In the theoretical aspect the problems attract the attention of specialists in the field of theoretical mechanics. They can be quite rigorously formulated within the framework of dynamic rigid body models in Lagrange's case, which is the unperturbed one. The refinement of the models under study is carried out by taking into account the perturbation torques of various physical nature, both internal and external.

The mathematical description of the symmetrical top motion in the field of gravity is one of the solved problems of rigid body dynamics. Many advanced treatise of classical mechanics include this problem [1-9]. The investigations of the dynamics of rotating bodies is important for some applications in astronautics [10].

In V.S. Aslanov's monograph [2] the motion of the rigid body in the atmosphere under the action of biharmonic air dynamic torque and small perturbations was studied.

In [5, 11], an averaging procedure for slow variables of a perturbed motion of a rigid body, close to Lagrange's case, in the first approximation was described. A perturbed motion of Lagrange's top, taking into account the torques acting on a rigid body from the external medium was studied.

The evolution of the motion of a rigid body, close to the Lagrange gyroscope, under the action of an unsteady perturbation torque was investigated by Akulenko et al. [12]. The generalization of this problem is considered in [13], when the restoring and perturbation torques are slowly varying in time.

The perturbed fast rotations of a rigid body, close to regular precession in the Lagrange case, were considered by Akulenko et al. [14]. Leshchenko [15] studied the motion of a rigid body, close to regular Lagrangian precessions, under the action of perturbation torque and a restoring torque, depending on the nutation angle. Akulenko et al. [16-18] studied the evolution of the rigid body rotations, close to regular precession, under the action of a restoring torque, depending on slow time and nutation angle, as well as a perturbation torque slowly varying in time. The perturbed motions of a rigid body, similar to Lagrange top, have been considered in a number of works such as [1, 2, 5, 11-21].

In [5 (Sections 4.8.2, 11.3), 14-18, 21] the perturbed fast rotational motions of a rigid body, close to regular precession in Lagrange's case, were studied for different orders of smallness of the projections of the perturbation torque vector. In [5 (Section

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4.8.3)], the perturbation torques are small compared to the restoring one. In contrast to [5 (Sections 4.8.1, 11.1, 11.2), 11-13, 20], studies [14-18, 21] considered the case of a rigid body that rotates rapidly about the axis of dynamic symmetry, and therefore the unperturbed solution was not the trajectory of motion in Lagrange case, but rather some simpler solution.

The motion of a symmetric Lagrange's gyroscope under the action of perturbation torques, Newtonian force field and gyro moment vector was considered by Amer, W.S. [20]. The procedure of averaging proposed in [5, 14] for studying the fast rotation of Lagrange's top was applied by Abady and Amer, T.S. [21] for investigation of the rotation a rigid body in the presence of a Newtonian field of force, gyro and perturbation torques.

Dissipation is an important factor of determination of heavy symmetric top's motion. Tanriverdi [22] estimated dragging with simple models; it is investigated as torque in Euler equations to be solved mathematically. The motion of a heavy symmetric rigid body with a fixed point under the action of forces caused by the surrounding dissipative medium has been considered in [23]. Some qualitative and quantitative results on motion of a slightly asymmetric heavy top subject to small viscous damping were investigated in [24].

A problem of stationary motions of a dynamically symmetric heavy rigid body under the action of dissipative torque and the constant torque was considered by Karapetyan [25]. Kononov and Vasylenko [26] considered rotation about a fixed point of a heavy dynamically symmetric rigid body with arbitrary asymmetric cavity completely filled with ideal fluid in a resisting medium. The conditions of asymptotic stability of the uniform rotations of an asymmetric rigid body in a resisting medium was obtained by Kononov [27]. The rotation of a rigid body is maintained by a constant torque that is directed along the third principal axes. The motion of a rigid body with an arbitrary cavity containing a heavy multilayer ideal fluid was studied using a linear problem statement in [28]. Ivashchenko [29] studied heavy symmetrical top's motion with a cavity filled with viscous fluid, when the axis of the top is diverged from the vertical.

Scarpello and Ritelli [30] computed in the Lagrange case the Euler angles of precession ψ and proper rotation ϕ in actual form through hypergeometric functions. The motion of symmetrical rigid body without weight under viscous dissipation was studied.

The author in [31] analyzes lower-order resonances in the motion of the Lagrange top with a small mass asymmetry. The secondary resonance effects in the spherical motion of a heavy asymmetrical rigid body with moving masses were investigated by Lyubimov [32], in the case, close to the Lagrange top.

The asymptotic stabilization of a top to a steady rotation about axes of symmetry was investigated by Wan et al. [33]. A rigid body forced by a nonstationary perturbation torque with zero mean value was studied by Aleksandrov and Tikhonov [34]. The control strategy for attitude stabilization of the body was based on the usage of dissipative and restoring torques. The motion of a heavy Lagrange's gyroscope with inequality of the equatorial moments of inertia was studied by Holmes and Marsden [35].

Interest to rigid body rotation about a fixed point attracts a wide circle of specialists, and not only in rigid body dynamics, but also in control theory [36], hydrodynamics [37], physics [38] and elasticity theory [39].

When axisymmetric magnetized body moves in constant field, close to regular precession, the following equations coincide: motion of the satellite to motion of the Lagrange gyroscope. It is known that a dynamically symmetric satellite moves the same way as a heavy rigid body in the Lagrange case, once the satellite possesses a magnetic torque moved along dynamic symmetry axis [40].

The resemblance of the problem of Lagrange's top motion in case of potential perturbations to the problem of satellite's rotation can be observed. The latter's mass center repositions in the equatorial plane's circular orbit, being affected by the Earth's magnetic field [41-43].

The Lagrange case occupies a special place in the range of problems in rigid body dynamics. This is explained by its proximity to a wide range of practical problems of gyroscopic technology and is associated with the possibilities of its theoretical analysis by classical methods of theoretical mechanics and vibration theory. The initial conditions are the nature of the body's motion, without resorting to integrating the equations of motion.

The mathematical model of a symmetric top represents an entire range of physical systems, including a child's spinning top toy, a variety of navigational instruments, the spinning earth, etc. These are all examples of gyroscopic systems, so called because they all exhibit peculiar behavior characteristic of the spinning gyroscope. We shall choose the symmetric top to develop the mathematical theory explaining the phenomenon of gyroscopic motion [44].

Besides the toy, there are many industrial applications of the spinning top such as navigation of the closely related gyroscope [45]. Because the earth had an initial spin on its polar axis when formed and because it is an oblate spheroid (slightly flattened at the poles), it acts like a top. The torque is due to gravitational attraction, primarily by the sun and moon, and would be zero if the earth were spherical. This torque is extremely weak and gives a processional period of 26000 years; in 80 years the spin axis processes 1° [46].

Author of book [47] considers an analytical solution for the dynamics of axially symmetric rotating objects. This work provides the gyroscopic effects theory, elaborating on their physics and utilizing mathematical models of Euler's form for the motion of non-fixed spinning objects.

In dynamics of a rigid body with fixed point there is vast bibliography on the theoretical researches of the perturbed motions that are close to Lagrange case, and on the applications to dynamics of space vehicle and flying machines, of gyrosystems and other engineering objects [2, 5, 7, 13, 22, 44-49]. A series of books and papers are dedicated to dynamics of a rigid body in a resistant medium (see, for example, works [2, 5, 7, 8, 11-24, 25, 26, 29, 48, 50, 51]).

In the first approximation, we provide a description of an averaging procedure for slow variables of a rigid body's perturbed motion similar to the Lagrange top. A series of applied problems permit averaging over the phase of the nutation angle θ. We analyze a perturbed motion similar to Lagrange's case, taking into consideration the torques influencing the rigid body from external medium. In contrast to the procedure of averaging with respect to the Euler-Poinsot motion, averaging with respect to the Lagrange motion permits us to examine the motion with external force torques, large in absolute value, as the unperturbed motion.

A non-standard approach to the selection of evolutionary variables is proposed to consider the roots of the Lagrange polynomial. Its results may be of interest to specialists in the field of rigid body dynamics, gyroscopy, and applications of asymptotic methods.

In this paper we present a new approach for the investigation of perturbed motions of Lagrange top for perturbations which assumes averaging with respect to the phase of the nutation angle. Nonlinear equations of motions are simplified and solved explicitly, so that the description of motion is obtained.

Asymptotic approach permits to obtain some qualitative results and to describe evolution of rigid body motion using simplified averaged equations. Thus it is possible to avoid numerical integration. We present a unified approach to the dynamics of angular motions of rigid bodies subject to perturbation torques of different physical nature.

In our paper the perturbed fast rotations of a rigid body are considered. An averaged system of the motion equations is obtained and investigated. We consider another possible variant of application of the averaging method for the perturbed motion close to Lagrange's top, this variant being different from the known ones (see, for example [5]).

In many applications, the angular velocity of the proper rotation of a rigid body significantly exceeds other components of the angular velocity of the body. The practical value of the results of this work is that it gives a qualitative and quantitative analysis of the motion of a rigid body under the influence of a number of perturbations that occur in the dynamics of satellites and gyroscopes. The obtained results can be used in studying the problems of stabilization of motions of mechanical systems.

The importance of this study is due to its different applications in many fields, for example in physics, celestial mechanics and engineering.

We can see from this survey that there is an extensive literature on the dynamics of a rigid body under the action of perturbation torques of various physical nature. The researches in this area is in connection with the problems of motion of flying vehicles, gyroscopes, and other objects of modern technology.

The plan of paper is as follows. In Sect. 1 the original equations are derived and the assumptions are formulated. In Sect. 2 we make the assumptions which means that the direction of the angular velocity of the body is close to the axis of dynamic symmetry; the angular velocity of the body is sufficiently large. In Sect. 3 we find expressions for real roots of the cubic polynomial. This is a distinctive feature of our problem as opposed to analysis carried out in [5, 11-20]. The perturbed motions of a rigid body motion, close to Lagrange top, were investigated in general case with the help of the averaging method in the works [5, 11-13]. The fast variable θ for unperturbed motion of expressed in terms of elliptic sine. In our paper θ is elementary function of sine. This is an element of novelty in our investigation. Conditions for the possibility of averaging the equations of motion over the phase of the nutation angle are presented and averaged system of equations is obtained. In Sect. 3, an example, corresponding to the body's motion in a medium with linear dissipation, is considered. In Sect. 4, we study the rigid body motion under the action of dissipative torques depending on slow time.

The averaged system for the projection of the angular momentum vector on the vertical G_z and total energy H was integrated numerically for various initial conditions and parameters of the problem.

2. Equations of Motion

Consider the perturbed motion about a fixed point of a dynamically symmetrical heavy rigid body in the case of perturbations of arbitrary nature. The equations of motion have the form:

$$\begin{aligned}
&A\frac{dp}{dt}+(C-A)\,qr=\mu\sin\theta\cos\varphi+\varepsilon M_1, && \frac{d\psi}{dt}=(p\sin\varphi+q\cos\varphi)\operatorname{cosec}\theta,\\
&A\frac{dq}{dt}+(A-C)\,pr=-\mu\sin\theta\sin\varphi+\varepsilon M_2, && \frac{d\theta}{dt}=p\cos\varphi-q\sin\varphi,\\
&C\frac{dr}{dt}=\varepsilon M_3,\quad M_i=M_i(p,q,r,\psi,\theta,\varphi,\tau),\quad i=1,2,3, && \frac{d\varphi}{dt}=r-(p\sin\varphi+q\cos\varphi)\operatorname{ctg}\theta.
\end{aligned} \tag{1}$$

Dynamic Eq. (1) are written in projections on the principal axes of inertia of the body, passing through point O. Here p, q, r are the projections of the angular velocity vector of the body on these axes; εM_i, $i=1,2,3$ are the projections of the vector of the perturbation torques on the same axes; ψ, φ, θ are the Euler angles; ε is a small parameter characterizing the magnitude of the perturbations; A and C are the body's respectively, equatorial and axial moments of inertia relative the fixed point O, $A\neq C$.

In particular, when $\varepsilon=0$ the system Eq. (1) describes motion in the Lagrange case [3, 5-9]. Figure 1 shows the Lagrange top, in which, the ellipsoid of inertia relative to a fixed point O is an ellipsoid of revolution. $Oxyz$ - stationary coordinate system; $Ox_1y_1z_1$ - moving coordinate system associated with the ellipsoid of revolution; axis Oz_1 is the axis of dynamic symmetry of the body. The center of gravity of the body lies on the axis Oz_1, the distance from the stationary point O of the body to the center of gravity C_1 is l. Force of gravity of the spinning top $\mathbf{P}=m\mathbf{g}$ and is directed vertically downward, m is the mass of the body, g is the acceleration due to gravity. It is assumed that at the initial moment of time the body performs a fast rotation, θ the angle of deviation of the axis Oz_1 of dynamic symmetry from the vertical. In the case of a heavy top, in the first three equations (1), the restoring moment $\mu=mgl$ and $\varepsilon=0$. The study of the system of equations (1) is related to the problems of motion of gyroscopes, flying vehicles, and other devices of modern technology.

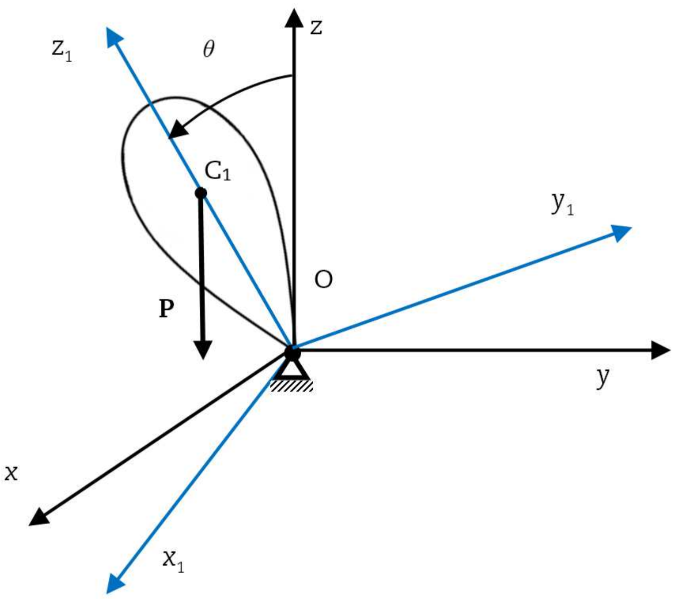

Fig. 1. Lagrange's top.

The problem that we formulate is that of investigating the asymptotic behavior of the solution of system Eq. (1) for small ε. This will be done by employing the averaging method [52] on a time interval of order ε^{-1}. We present the basic information about the unperturbed motion in the general case.

The first integrals of the equations of motion for the unperturbed system Eq. (1) are [3, 5-7]:

$$G_z = A\sin\theta(p\sin\varphi + q\cos\varphi) + Cr\cos\theta = c_1,\quad H = \frac{1}{2}[A(p^2+q^2) + Cr^2] + \mu\cos\theta = c_2,\quad r = c_3, \tag{2}$$

where G_z is the projection of the angular momentum vector on the vertical Oz, H is the total energy of the body, r is the projection of the angular velocity vector on the axis of dynamic symmetry, c_i, $i=1,2,3$ are arbitrary constants $(c_2 \geq -\mu)$.

The expression for the nutation angle θ in the unperturbed motion for the general case as a function of time t, of the motion integrals Eq. (2) and of arbitrary phase constant β is known [3, 5-7]:

$$\begin{aligned} &u = \cos\theta = u_1 + (u_2 - u_1)\mathrm{sn}^2(\alpha t + \beta, k),\ -1 \leq u_1 \leq u_2 \leq 1 < u_3 < +\infty,\\ &\alpha = [\mu(u_3 - u_1)/(2A)]^{1/2},\quad \mathrm{sn}(\alpha t + \beta, k) = \sin\mathrm{am}(\alpha t + \beta, k), \end{aligned} \tag{3}$$

$$k^2 = (u_2 - u_1)(u_3 - u_1)^{-1},\qquad 0 \leq k^2 < 1. \tag{4}$$

Here u is a periodic function of time with the period $K(k)/\alpha$, where $K(k)$ is the complete elliptic integral of the first kind; $\mathrm{sn}\,u$, $\mathrm{am}\,u$ are respectively called elliptic sine and the delta amplitude [53], k is the modulus of the ellipticity of the functions, u_1, u_2 and u_3 are real roots of the cubic polynomial:

$$Q(u) = A^{-2}[(2H - Cr^2 - 2\mu u)(1-u^2)A - (G_z - Cru)^2]. \tag{5}$$

Relations between the roots of the polynomial $Q(u)$ of Eq. (5) and first integrals Eq. (2) can be written in the following way:

$$\begin{aligned} &u_1 + u_2 + u_3 = \frac{H}{\mu} - \frac{Cr^2}{2\mu} + \frac{C^2r^2}{2A\mu} \equiv F_1,\\ &u_1u_2 + u_1u_3 + u_2u_3 = \frac{G_zCr}{A\mu} - 1 \equiv F_2,\\ &u_1u_2u_3 = -\frac{H}{\mu} + \frac{Cr^2}{2\mu} + \frac{G_z^2}{2A\mu} \equiv F_3. \end{aligned} \tag{6}$$

Formulas Eq. (2), (3), (6) describe the solution of system Eq. (1) when $\varepsilon = 0$.

3. The Averaging Procedure

Let us make the following basic assumptions:

$$p^2 + q^2 << r^2,\qquad Cr^2 >> \mu, \tag{7}$$

which means that the direction of the angular velocity of the body is close to the axis of dynamic symmetry; the angular velocity is sufficiently large.

If the body performs fast rotation about the axis of symmetry, then the potential energy of the body is small in comparison with the kinetic energy T, and we obtain the following in the first approximation:

$$G_z \approx Cr,\quad H \approx T \approx \frac{1}{2}Cr^2. \tag{8}$$

If the angular velocity r of the body is large, the square modulus of ellipticity of the functions can be presented in the form:

$$k^2 = (u_2 - u_1)(u_3 - u_1)^{-1} << 1, \tag{9}$$

after realization of the second initial assumption Eq. (7) we obtain:

$$u_1 << u_3,\quad u_2 << u_3,\quad u_1 + u_2 << u_3. \tag{10}$$

Then from relations Eq. (6) we obtain:

$$u_3 = F_1,\qquad u_1u_2 = \frac{F_3}{F_1}. \tag{11}$$

After a number of transformations with regard to Eq. (6) we find expressions for real roots of the cubic polynomial Eq. (5) in the form:

$$u_1 = \frac{1}{2F_1}[F_2 - \sqrt{F_2^2 - 4F_1F_3}],\quad u_2 = \frac{1}{2F_1}[F_2 + \sqrt{F_2^2 - 4F_1F_3}],\quad u_3 = F_1. \tag{12}$$

Then, there is no need to solve cubic Eq. (5) with respect to u_i. This is the main distinguishing feature of our problem.

Let us reduce the equations of perturbed motion Eq. (1) to the form allowing application of the averaging method. To do this, we identify slow and fast variables. In the considered problem, the first integrals Eq. (2) will be slow variables for the perturbed motion Eq. (1). In the case of fast rotation of a body about the axis of symmetry the angle of precession ψ is also slow variable [7].

We reduce the first three equations in Eq. (1) after several transformations to the form [5, 11]:

$$\begin{aligned}
&\frac{dG_z}{dt} = \varepsilon[(M_1\sin\varphi + M_2\cos\varphi)\sin\theta + M_3\cos\theta],\\
&\frac{dH}{dt} = \varepsilon(M_1 p + M_2 q + M_3 r),\\
&\frac{dr}{dt} = \varepsilon C^{-1}M_3, \quad M_i = M_i(p,q,r,\psi,\theta,\varphi),\ i=1,2,3.
\end{aligned} \tag{13}$$

Here and in the last three equations in Eq. (1) it is assumed that variables $p,\ q,\ r$ are expressed as functions of $G_z,\ H,\ r,\ \psi,\ \theta,\ \varphi$ and substituted into Eq. (1) and Eq. (7). Here ε is a small parameter.

The right-hand sides of Eq. (13) contain two fast variables, which creates difficulties for the application of the averaging method connected with the possibility of appearance of resonances. To eliminate this difficulty we require that the right-hand sides of Eq. (13) for slow variables depend just on a single fast variable, the angle of nutation θ, and are periodic functions with respect to phase of nutation angle θ with the period 2π. We also require that the right-hands sides of Eq. (13) have the following structural properties of perturbation torque (see Eq. (2)):

$$M_1\sin\varphi + M_2\cos\varphi = M_1^*(G_z,H,r,\theta), \quad M_1 p + M_2 q = M_2^*(G_z,H,r,\theta), \quad M_3 = M_3^*(G_z,H,r,\theta). \tag{14}$$

We consider for determination the case when the perturbation torques satisfy the conditions:

$$M_1 = pf, \quad M_2 = qf, \quad M_3 = M_3^*, \quad f = f(G_z,H,r,\theta). \tag{15}$$

We assume satisfied the necessary and sufficient conditions Eq. (14) or, in particular, the sufficient conditions Eq. (15), which ensure that relations Eq. (14) are held. Then the system Eq. (13) of equations of the perturbed motion of a rigid body, close to Lagrange case, can be represented in the form:

$$\begin{aligned}
&\frac{dG_z}{dt} = \varepsilon U_1(G_z,H,r,\theta), \quad U_1 = M_1^*\sin\theta + M_3^*\cos\theta,\\
&\frac{dH}{dt} = \varepsilon U_2(G_z,H,r,\theta), \quad U_2 = M_2^* + M_3^* r,\\
&\frac{dr}{dt} = \varepsilon U_3(G_z,H,r,\theta), \quad U_3 = C^{-1}M_3^*.
\end{aligned} \tag{16}$$

Here U_1, U_2, U_3 are 2π – periodic functions of phase of nutation angle θ.

For fast spinning rigid body if $k^2 << 1$ from Eq. (3) for u we obtain approximate formula:

$$u = \cos\theta \approx u_1 + (u_2 - u_1)\sin^2(\alpha t + \beta). \tag{17}$$

The procedure of averaging for equations Eq. (16) for slow variables $G_z,\ H,\ r$ of the first approximation consists in the following. We substitute into the right-hand sides of system Eq. (16) the fast variable θ from expression Eq. (17) for the unperturbed motion:

$$\theta \approx \arccos[u_1 + (u_2 - u_1)\sin^2(\alpha t + \beta)]. \tag{18}$$

Averaging the right-hand sides of the resulting system with respect to t, we obtain with regard to Eq. (3), (6) the averaged system of the first approximation:

$$\begin{aligned}
&\frac{dG_z}{dt} = \varepsilon V_1(G_z,H,r), \quad \frac{dH}{dt} = \varepsilon V_2(G_z,H,r), \quad \frac{dr}{dt} = \varepsilon V_3(G_z,H,r),\\
&V_i(G_z,H,r) = \frac{\alpha}{2\pi}\int_0^{2\pi/\alpha} U_i(G_z,H,r,\theta(t))dt, \quad i=1,2,3,
\end{aligned} \tag{19}$$

while expression Eq. (18) is inserted into Eq. (19) instead of $\theta = \theta(t)$.

Thus, according to the proposed method, the study of perturbed Lagrange motion is carried out in the following way. Let perturbation torques εM_i satisfy conditions Eq. (14) or, in particular Eq. (15). We calculate functions $M_i^*,\ U_i,\ i=1,2,3,$ with the help of relations Eq. (14) - (16). Then, according to Eq. (19), we average functions U_i using expressions Eq. (17), (18), and from the averaged system Eq. (19). System Eq. (19) is much simpler that the initial system Eq. (1), since it has a lower order (three instead of six), is autonomous and does not contain fast oscillations.

The question about proximity of solutions of system Eq. (16) and averaged system Eq. (19) is considered in [54] in the case of solution for θ of the form Eq. (17) having error $O(\delta)$, where $\delta << 1$ is a small parameter. Estimate of proximity of solutions of system Eq. (16), (19) on time interval $\sim\varepsilon^{-1}$ consists of the sum of estimate of approximation of unperturbed solution δ and small parameter ε, which characterizes the value of perturbations [54].

After investigating and solution of system Eq. (14) for $G_z,\ H,\ r$ slow variables $u_i,\ i=1,2,3$ is determined according to formulas Eq. (12).

4. Motion of a Rigid Body under the Action of Dissipative Torque and a Small Torque that is Constant in the Attached Axes

As an example of the technique, let us consider perturbed Lagrange motion with allowance for the torques acting on our rigid body from the surrounding medium and under the action of a torque that is constant in the attached axes and is applied along the axis of symmetry. We take the perturbation torque $\varepsilon M_i,\ i=1,2,3$ in the form [4, 5, 7, 11]:

$$M_1 = -ap, \quad M_2 = -aq, \quad M_3 = -br - \eta, \quad a,\ b > 0. \tag{20}$$

Here a and b are certain constant proportionality coefficients depending on the medium's properties and the body's shape, $\eta = \text{const}$.

Torques Eq. (20) satisfy the conditions Eq. (14), (15) for the possibility of averaging with respect to phase of the nutation angle θ. System Eq. (13) can be written as follows:

$$\begin{aligned} \frac{dG_z}{dt} &= -\varepsilon[a(p\sin\varphi + q\cos\varphi)\sin\theta + br\cos\theta] - \varepsilon\eta\cos\theta, \\ \frac{dH}{dt} &= -\varepsilon[a(p^2+q^2) + br^2] - \varepsilon\eta r, \\ \frac{dr}{dt} &= -\varepsilon C^{-1}br - \varepsilon\eta C^{-1}. \end{aligned} \tag{21}$$

Integrating the third equation in Eq. (21), we obtain (r_0 is the arbitrary initial value of the axial rotation velocity, $\tau = \varepsilon t$ is the slow time):

$$r = (r_0 + \eta b^{-1})\exp(-bC^{-1}\tau) - \eta b^{-1}. \tag{22}$$

Substituting Eq. (22) for r in the first two Eq. (21), we average according to Eq. (19). We note, that with regard to Eq. (8), (11):

$$\frac{1}{2}(u_1 + u_2) = \frac{F_2}{2F_1} \sim 1. \tag{23}$$

After some transformations averaged system of the first approximation has the form:

$$\begin{aligned} &\frac{dG_z}{d\tau} + aA^{-1}G_z = (aA^{-1}C - b)(r_0 + \eta b^{-1})\exp(-bC^{-1}\tau) - \eta b^{-1}aA^{-1}C, \\ &\frac{dH}{d\tau} + 2aA^{-1}H = (aA^{-1}C - b)(r_0 + \eta b^{-1})^2\exp(-2bC^{-1}\tau) - \eta(r_0 + \eta b^{-1})(2b^{-1}aA^{-1}C - 1)\exp(-bC^{-1}\tau) + \eta^2 b^{-2}aA^{-1}C + 2\mu aA^{-1}. \end{aligned} \tag{24}$$

Solution of the system Eq. (24) is described as follows:

$$\begin{aligned} &G_z = (G_{z0} - Cr_0)\exp(-aA^{-1}\tau) + C(r_0 + \eta b^{-1})\exp(-bC^{-1}\tau) - \eta C b^{-1}, \\ &H = (H_0 - \frac{1}{2}Cr_0^2 - \mu)\exp(-2aA^{-1}\tau) + \frac{1}{2}C(r_0 + \eta b^{-1})^2\exp(-2bC^{-1}\tau) - \eta C b^{-1}(r_0 + \eta b^{-1})\exp(-bC^{-1}\tau) + \frac{1}{2}\eta^2 C b^{-2} + \mu. \end{aligned} \tag{25}$$

Here $G_{z0,}$ H_0 are arbitrary initial conditions of the projection of the angular momentum vector on the vertical Oz and of the body's total energy.

Let us point out some qualitative features of motion in the case in question. The modulus of the axial rotational velocity r and the projection of the angular momentum vector on the vertical Oz G_z asymptotically approach to values $r = -\eta b^{-1}$, $G_z = -\eta C b^{-1}$. Total energy H is changed asymptotically and approaches to value $H = 0.5\eta^2 C b^{-2} + \mu$.

5. The Motion of Rigid Body under the Action of Dissipative Torques Depending on Slow Time

We investigate the perturbed Lagrange motion with torques applied to the body from the surrounding medium. This is the case, for example, for a medium the viscous properties of which change due to varies in the density, temperature of which is linearly dissipative and has the form [12, 13, 17, 18]:

$$M_1 = -a(\tau)p, \quad M_2 = -a(\tau)q, \quad M_3 = -b(\tau)r, \quad a(\tau), b(\tau) > 0, \ \tau = \varepsilon t. \tag{26}$$

Here $a(\tau)$ and $b(\tau)$ are positive integrable functions depending on the medium's properties and the body's shape.

Torques Eq. (26) satisfy the conditions Eq. (15) for the possibility of averaging with respect by the phase of nutation angle θ. System Eq. (13) can be written as follows:

$$\begin{aligned} \frac{dG_z}{dt} &= -\varepsilon[(a(\tau)p\sin\varphi + a(\tau)q\cos\varphi)\sin\theta + b(\tau)r\cos\theta], \\ \frac{dH}{dt} &= -\varepsilon[a(\tau)(p^2+q^2) + b(\tau)r^2], \\ \frac{dr}{dt} &= -\varepsilon C^{-1}b(\tau)r. \end{aligned} \tag{27}$$

Integrating the third equation in Eq. (27), we obtain (r_0 is the arbitrary initial value of the axial rotation velocity):

$$r = r_0\exp(-\varepsilon C^{-1}\int_0^t b(\varepsilon t))dt. \tag{28}$$

Consider a case where $a(\tau)$, $b(\tau)$ have the form:

$$a(\tau) = a_0 + a_1\tau, \ \ b(\tau) = b_0 + b_1\tau, \ \ a_0, \ a_1, \ b_0, \ b_1 - \text{const.}. \tag{29}$$

Integrating the equation Eq. (28), we obtain:

$$r(\tau) = r_0\exp(-C^{-1}b_0\tau). \tag{30}$$

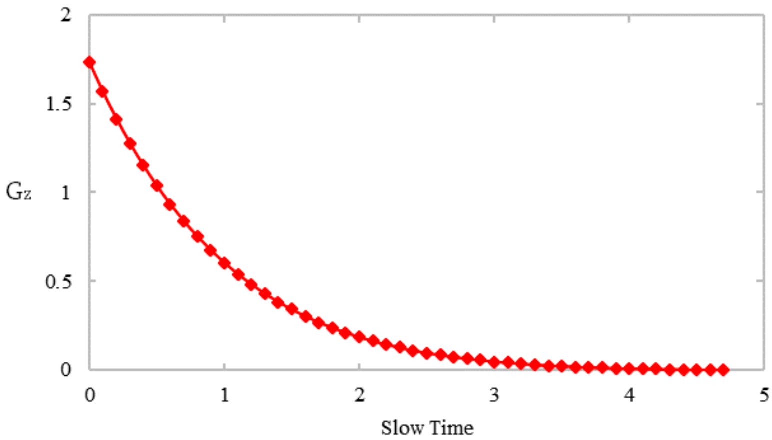

Fig. 2. Graph of changes of projection of the angular momentum vector on the vertical Oz.

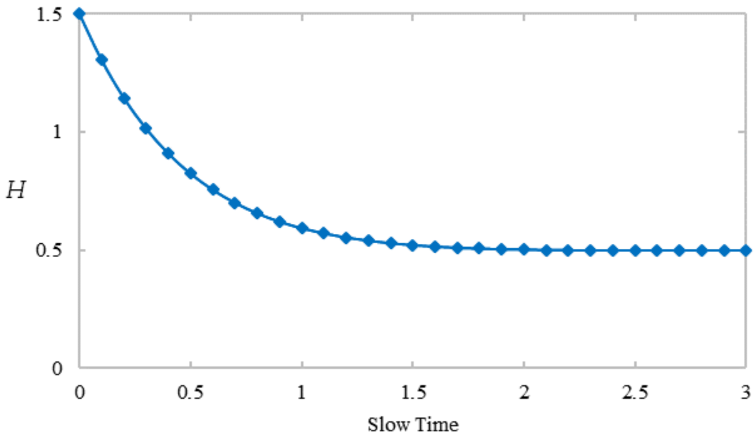

Fig. 3. Graph of changes of total energy.

First two equations of system Eq. (27) after sequence of transformations and averaging by the phase of nutation angle assume the form:

$$\begin{aligned}\frac{dG_z}{d\tau} &= -A^{-1}G_z a(\tau) + [A^{-1}Ca(\tau) - b(\tau)]r(\tau),\\ \frac{dH}{d\tau} &= -2A^{-1}Ha(\tau) + [A^{-1}Ca(\tau) - b(\tau)]r^2(\tau) + 2A^{-1}\mu a(\tau).\end{aligned} \tag{31}$$

In Figures 2, 3, the graphs of solutions of system Eq. (31) are presented for the following parameter values $A = 1.5,\ C = 1,\ \mu = 0.5,\ a_0 = 1.25,\ b_0 = 1,\ a_1 = b_1 = 0.1$. At the initial moment, the body received the angular velocity of rotation about the axis of dynamic symmetry, equal $r_0 = \sqrt{3}$, based on the assumptions Eq. (8) to the initial values $G_{z0} = 1.73$ and $H_0 = 1.5$. As can be seen from the Figs. 2 and 3, the projection of the angular momentum vector the vertical Oz tends to zero. The total energy H decreases monotonically, approaching the value $H = \mu$.

Approximate solution of the first Eq. (31) for projection of the angular momentum vector on the vertical Oz has a form:

$$\begin{aligned}&G_z = [G_{z0} - Cr_0(1-\lambda)]\exp(-A^{-1}a_0\tau) + [1 - \lambda + \xi\tau]Cr(\tau),\\ &\lambda = \frac{A^{-1}a_1 - C^{-1}b_1}{\left(A^{-1}a_0 - C^{-1}b_0\right)^2}, \quad \xi = \frac{A^{-1}a_1 - C^{-1}b_1}{A^{-1}a_0 - C^{-1}b_0}.\end{aligned} \tag{32}$$

6. Conclusion

We presented some new qualitative and quantitative results of fast motion of a heavy top subject to small perturbation torques. The averaging method and its methodological treatment were presented and applied to the nonlinear equations of motion. We suggested a new procedure of the averaging method, different from works [5 (Sections 4.8.2, 11.3.1, 11.3.2), 14]. The main goal of this article was to extend the results of previous investigations for problem of the fast motion of a dynamically symmetric rigid body under the action of perturbation torques independent or dependent on the slow time. The numerical solution was gained and plotted in some graphs taking into consideration the case of dissipative torques. The paper presented a unified approach to the dynamics of rigid bodies subjected to perturbation torques of different physical nature. Our article contained both the foundations of rigid body dynamics and the application of the asymptotic method of averaging. Nonlinear equations of motion were simplified and often solved explicitly, so the description of motion was obtained. The approach presented in the paper is suitable for attitude dynamics of gyroscopes, spacecraft and engineering applications.

Author Contributions

In this research, Dmytro Leshchenko is responsible for the general ansatz and solving procedure, as well as survey in literature on the problem under consideration. Sergey Ershkov is responsible for the theoretical investigations and analysis of obtained results. Tetiana Kozachenko is responsible for receiving the averaged system of equations of motion and for the plots and numerical solutions for test example. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

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Data Availability Statements

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Nomenclature

a, b	Constant coefficients	m	Body's mass
am	Amplitude	$p,\ q,\ r$	Projections of the angular velocity onto the principal axes of the body
$a(\tau)$, $b(\tau)$	Positive integrable functions	sn	Elliptic sine
A	Equatorial moment of inertia	$u_1,\ u_2,\ u_3$	The real roots of the cubic polynomial $Q(u)$
C	Axial moment of inertia	$u=\cos\theta$	
g	Acceleration due to gravity	$\varepsilon<<1$	Small parameter
G_z	Projection the vector of angular momentum onto the vertical Oz	εM_i	Projections of the vector of the perturbation torque on the principal axes of inertia of the body
H	Body's total energy	η	Constant
k	Modulus of ellipticy of the functions	$\theta,\ \varphi,\ \psi,$	Euler angles
$K(k)$, $E(k)$	Complete elliptic integrals of the first and second kind	μ	Restoring torque
l	Distance from the fixed point to the body's center of gravity	$\tau=\varepsilon t$	Slow time

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ORCID iD

Dmytro Leshchenko https://orcid.org/0000-0003-2436-221X
Sergey Ershkov https://orcid.org/0000-0002-6826-1691
Tetiana Kozachenko https://orcid.org/0000-0001-9034-3776

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