J. Appl. Comput. Mech., 8(3) (2022) 1054-1064
DOI: 10.22055/jacm.2022.39327.3388

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran
University of Ahvaz

Journal of
Applied and Computational Mechanics

Research Paper

Numerical Investigations of the Coupled Nonlinear Non-homogeneous Partial Differential Equations

Z.M. Alaofi[1,2], T.A. El-Danaf[3], Faisal E.I. Abd Alaal[4], S.S. Dragomir[2]

[1] Department of Mathematics, College of Science and Arts, King Khalid University, Muhayil Asir, Saudi Arabia, Email: zaleawfi@kku.edu.sa

[2] Mathematics, College of Engineering & Science, Victoria University, Melbourne, Australia, Email: Zaki.alaofi@live.vu.edu.au (ZMA), sever.dragomir@vu.edu.au (SSD)

[3] Department of Mathematics, Faculty of Sciences and Arts, Taibah University, Medina, Saudi Arabia, Email: tdanaf@taibahu.edu.sa

[4] Department of Mathematics, Faculty of Science, Damanhour University, Damanhour, Egypt, Email: faisal_ezz@sci.dmu.edu.eg

Received November 29 2021; Revised January 13 2022; Accepted for publication February 12 2022.
Corresponding author: Z.M. Alaofi (Zaki.alaofi@live.vu.edu.au)

Abstract. The mathematical description of various processes such as the nonlinear Klein-Gordon equation occurring in mathematical physics leads to a nonlinear partial differential equation. The mathematical model is only the first step, however, towards finding the solution of the problem under consideration. It has become possible to develop realistic mathematical models with the currently available computing power for complicated problems in science and engineering. To the best of our knowledge, systematically using the collocation method to acquire the numerical solution has not been previously used for the Klein-Gordon equation. The main aim of this paper is to systematically use the collocation method to acquire the numerical solution of the two coupled nonlinear non-homogeneous Klein-Gordon partial differential equations. We examine and analyze their stability, in detail. To this aim, we use the Von Neumann stability method to show that the proposed method is conditionally stable. A numerical example is introduced to demonstrate the performance and the efficiency of the proposed method for solving the coupled nonlinear non-homogeneous Klein-Gordon partial differential equations. The numerical results demonstrated that the proposed algorithm is efficient, accurate, and compares favorably with the analytical solutions.

Keywords: Nonlinear coupled hyperbolic Klein-Gordon equations; Nonlinear phenomena; Jacobi collocation method; Stability analysis.

1. Introduction

Nonlinear phenomena modelled by partial differential equations appear in many areas of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics. The Klein-Gordon equation is one of the most important mathematical models in quantum field theory. The nonlinear Klein-Gordon equation (NKGE) is used to model many nonlinear phenomena. NKGE appears in theoretical physics, particularly in the area of relativistic quantum mechanics. This equation is a relativistic version of the Schrödinger equation, which describes scalar spineless particles [1-2]. Many researchers have used various numerical methods to solve the nonlinear Klein-Gordon equation. In 2002, Masmoudi and Nakanishi showed that the solutions for the nonlinear Klein-Gordon equation can be described by using a system of two coupled nonlinear Schrödinger equations as the speed of light tends to infinity, in the strong topology of the energy space [3]. Later, John (2004) argued in favour of a numerical study of a particular form of the nonlinear Klein-Gordon (nlKG) equation. It is based on resonant structures within the nonlinear Klein-Gordon equations, the nlKG equation was solved numerically using finite-difference methods in one spatial dimension, with asymmetric double well potential as its nonlinear term [4]. In 2007, Khusnutdinova asserted that a system of coupled Klein-Gordon equations is a model for one-dimensional nonlinear wave processes in two-component media, e.g. long longitudinal waves in elastic bi-layers, where nonlinearity comes only from the bonding material. He proposed general properties for the model (lie group classification, conservation laws, invariant solutions) and special solutions exhibiting an energy exchange between the two physical components of the system [5]. A year later, Dehghan and Shokri proposed a numerical scheme to solve the one-dimensional nonlinear Klein-Gordon equation with quadratic and cubic nonlinearity. Their scheme used the collocation points and approximates the solution using Thin Plate Splines (TPS) radial basis functions (RBF) [6]. In 2010, Sassaman studied the coupled Klein-Gordon equations in (1+1) and (1+2) dimensions [7]. Li, in 2011, proposed a one-dimensional (1D) lattice Boltzmann scheme with an amending function for the nonlinear Klein-Gordon equation. With the Taylor and Chapman-Enskog expansion, the nonlinear Klein-Gordon equation was recovered correctly from the lattice Boltzmann equation [8]. Wu and Ge studied the Klein-Gordon equation coupled with the Maxwell equation in the rotationally symmetric bounded domains when a non-homogeneous term breaks the symmetry of the

Published online: February 20 2022

associated functional. Under some suitable assumption on nonlinear perturbation, they obtained infinite radially symmetric solutions to the non-homogeneous Klein-Gordon-Maxwell system [9]. In 2013, Krämer's diploma thesis demonstrated the derivation of an approximate solution to the nonlinear Klein-Gordon equation via the method of multiple scales. This method follows the concept of expanding the solution into a perturbation series, including multiple temporal and spatial scales [10]. In the same year, Chen and Li proved the existence of multiple solutions for the non-homogeneous Klein–Gordon equation coupled with Born–Infeld theory [11]. Guo et al. (2015) presented a numerical analysis of the one-dimensional Klein–Gordon equation with quadratic and cubic nonlinearity, using the element-free reproducing kernel particle Ritz method (kp-Ritz method) [12]. Also in 2015, Sarboland and Aminataei provided a numerical scheme to approximate solutions of the nonlinear Klein-Gordon equation by applying the multiquadric quasi-interpolation scheme and the integrated radial basis function network scheme [13]. A year later, Raza et al. presented a scheme for numerical approximation of solutions of the one-dimensional nonlinear Klein-Gordon equation (KGE). They used a common approach to find a solution of a nonlinear system by first linearizing the equations through successive substitution, or the Newton iteration method, and then solving a linear least squares problem [14]. Rashidinia and Jokar used polynomial wavelets to find the numerical solution of nonlinear Klein-Gordon equation in 2016 [15]. In 2018, Shi and Chen studied the multiplicity of positive solutions for a class of non-homogeneous Klein-Gordon-Maxwell equations. They proved the existence of two positive solutions through use of Ekeland's variational principle and the Mountain Pass Theorem [16]. Recently, in 2021, Ghazi and Tawfiq considered a new approach to solve a type of partial differential equation by using coupled Laplace transformation with decomposition method to find the exact solution for nonlinear non-homogenous equation with initial conditions [17]. We consider the nonlinear Klein-Gordon partial differential equation in the following form [1]:

$$\frac{\partial^2 u}{\partial x^2}(x,t)+\frac{\partial^2 u}{\partial t^2}(x,t)+v(x,t)u(x,t)=f(x,t), \tag{1}$$

$$\frac{\partial^2 v}{\partial x^2}(x,t)+\frac{\partial^2 v}{\partial t^2}(x,t)+u(x,t)v(x,t)=g(x,t), \tag{2}$$

for $a\le x\le b$ and $t\ge 0$, subject to the conditions:

$$u(a,t)=\varepsilon_1(t),\ u(b,t)=\varepsilon_2(t), \tag{3}$$

$$v(a,t)=\rho_1(t),\ v(b,t)=\rho_2(t), \tag{4}$$

$$u(x,0)=\tau_1(x),\ \frac{\partial u}{\partial t}(x,0)=\tau_2(x), \tag{5}$$

$$v(x,0)=\sigma_1(x),\ \frac{\partial v}{\partial t}(x,0)=\sigma_2(x). \tag{6}$$

2. The Numerical Method

To approximate $u(x,t)$ and $v(x,t)$ through collocation using cubic B spline, let the region $R=[a,b]\times[0,\infty]$ be discretized by a set of points R_{ij}, which are the vertices of a grid of points (x_i,t_j), where $x_i=a+ih, h=\Delta x$ for $i=0,1,\ldots,n$ and $t_j=jk, k=\Delta t$ for $j=0,1,\ldots$ and let $\phi_i(x)$ be cubic B splines with knots at $x_{-2}<x_{-1}<\ldots<x_{n+1}<x_{n+2}$ where:

$$\phi_i(x)=\begin{cases}(x-x_{i-2})^3 & x\in[x_{i-2},x_{i-1}]\\ h^3+3h^2(x-x_{i-1})+3h(x-x_{i-1})^2-3(x-x_{i-1})^3 & x\in[x_{i-1},x_i]\\ h^3+3h^2(x_{i+1}-x)+3h(x_{i+1}-x)^2-3(x_{i+1}-x)^3 & x\in[x_i,x_{i+1}]\\ (x_{i+2}-x)^3 & x\in[x_{i+1},x_{i+2}]\\ 0 & \text{otherwise}\end{cases}$$

Table 1 presents values of $\phi_i(x)$ and its derivatives at the knots. Since $\phi_i(x)$ and its first and second derivatives vanish outside the interval (x_{i-2},x_{i+2}) there is no need to tabulate ϕ_i for other values of x.

The collocation method for approximation solving Eqs. (1) and (2) consists in seeking approximations U and V of u and v from the finite dimensional subspace of $C^2[a,b]$, which is spanned by the linearly independent set of cubic B splines $\{\phi_{-1},\phi_0,\phi_1,\ldots,\phi_n,\phi_{n+1}\}$ having the forms

$$U(x,t)=\beta_{-1}(t)\phi_{-1}(x)+\beta_0(t)\phi_0(x)+\ldots+\beta_n(t)\phi_n(x)+\beta_{n+1}(t)\phi_{n+1}(x)\,, \tag{7}$$

$$V(x,t)=\alpha_{-1}(t)\phi_{-1}(x)+\alpha_0(t)\phi_0(x)+\ldots+\alpha_n(t)\phi_n(x)+\alpha_{n+1}(t)\phi_{n+1}(x)\,, \tag{8}$$

Table 1. The values of $\phi_i(x)$ and their derivative within the interval $[x_{i-2},x_{i+2}]$.

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$\phi_i(x)$	0	1	4	1	0
$\phi_i'(x)$	0	$3/h$	0	$-3/h$	0
$\phi_i''(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

such that

$$\frac{\partial^2 U}{\partial x^2}(x_i,t_j)+\frac{\partial^2 U}{\partial t^2}(x_i,t_j)+V(x_i,t_j)U(x_i,t_j)=f(x_i,t_j)\,, \tag{9}$$

$$\frac{\partial^2 V}{\partial x^2}(x_i,t_j)+\frac{\partial^2 V}{\partial t^2}(x_i,t_j)+U(x_i,t_j)V(x_i,t_j)=g(x_i,t_j)\,, \tag{10}$$

for $a\le x\le b$ and $t\ge 0$, subject to the conditions

$$U(a,t_j)=\varepsilon_1(t_j),\ U(b,t_j)=\varepsilon_2(t_j), \tag{11}$$

$$V(a,t_j)=\rho_1(t_j),\ V(b,t_j)=\rho_2(t_j), \tag{12}$$

$$U(x_i,0)=\tau_1(x\ ),\ \frac{\partial U}{\partial t}(x_i,0)=\tau_2(x_i), \tag{13}$$

$$V(x_i,0)=\sigma_1(x_i),\ \frac{\partial V}{\partial t}(x_i,0)=\sigma_2(x_i), \tag{14}$$

where $i=0,1,\ldots,n$ and $j=0,1,\ldots$. Substituting Eqs. (7) and (8) into Eqs. (9) and (10) gives:

$$\sum_{m=-1}^{n+1}\beta_m(t_j)\phi_m''(x_i)+\sum_{m=-1}^{n+1}\frac{d^2\beta_m(t_j)}{dt^2}\phi_m(x_i)+V_{i,j}\sum_{m=-1}^{n+1}\beta_m(t_j)\phi_m(x_i)=f_{i,j}, \tag{15}$$

$$\sum_{m=-1}^{n+1}\alpha_m(t_j)\phi_m''(x_i)+\sum_{m=-1}^{n+1}\frac{d^2\alpha_m(t_j)}{dt^2}\phi_m(x_i)+U_{i,j}\sum_{m=-1}^{n+1}\alpha_m(t_j)\phi_m(x_i)=g_{i,j}, \tag{16}$$

where

$$U_{i,j}=\sum_{m=-1}^{n+1}\beta_m(t_j)\phi_m(x_i)\ \text{and}\ V_{i,j}=\sum_{m=-1}^{n+1}\alpha_m(t_j)\phi_m(x_i)\,.$$

Using the values in Table 1 in Eqs. (15) and (16) gives:

$$\frac{6}{h^2}\left(\beta_{i-1,j}-2\beta_{i,j}+\beta_{i+1,j}\right)+\left(\frac{d^2\beta_{i-1,j}}{dt^2}+4\frac{d^2\beta_{i,j}}{dt^2}+\frac{d^2\beta_{i+1,j}}{dt^2}\right)+V_{i,j}\left(\beta_{i-1,j}+4\beta_{i,j}+\beta_{i+1,j}\right)=f_{i,j}\,, \tag{17}$$

$$\frac{6}{h^2}\left(\alpha_{i-1,j}-2\alpha_{i,j}+\alpha_{i+1,j}\right)+\left(\frac{d^2\alpha_{i-1,j}}{dt^2}+4\frac{d^2\alpha_{i,j}}{dt^2}+\frac{d^2\alpha_{i+1,j}}{dt^2}\right)+U_{i,j}\left(\alpha_{i-1,j}+4\alpha_{i,j}+\alpha_{i+1,j}\right)=g_{i,j}\,, \tag{18}$$

where

$$U_{i,j}=\left(\beta_{i-1,j}+4\beta_{i,j}+\beta_{i+1,j}\right),\quad V_{i,j}=\left(\alpha_{i-1,j}+4\alpha_{i,j}+\alpha_{i+1,j}\right),\beta_{i,j}=\beta_i(t_j)\ \text{and}\ \alpha_{i,j}=\alpha_i(t_j)\,.$$

The central finite difference approximations:

$$\beta_{i,j}\cong\frac{\beta_{i,j-1}+\beta_{i,j+1}}{2},\ \frac{d^2\beta_{i,j}}{dt^2}\cong\frac{\beta_{i,j-1}-2\beta_{i,j}+\beta_{i,j+1}}{k^2}, \tag{19}$$

$$\alpha_{i,j}\cong\frac{\alpha_{i,j-1}+\alpha_{i,j+1}}{2},\ \frac{d^2\alpha_{i,j}}{dt^2}\cong\frac{\alpha_{i,j-1}-2\alpha_{i,j}+\alpha_{i,j+1}}{k^2}, \tag{20}$$

can be substituted into Eqs. (17) and (18) to give the following systems:

$$A_{i,j}\beta_{i-1,j+1}+B_{i,j}\beta_{i,j+1}+A_{i,j}\beta_{i+1,j+1}=\frac{2}{k^2}\left(\beta_{i-1,j}+4\beta_{i,j}+\beta_{i+1,j}\right)-A_{i,j}\left(\beta_{i-1,j-1}+\beta_{i+1,j-1}\right)-B_{i,j}\beta_{i,j-1}+f_{i,j}\,, \tag{21}$$

and

$$C_{i,j}\alpha_{i-1,j+1}+D_{i,j}\alpha_{i,j+1}+C_{i,j}\alpha_{i+1,j+1}=\frac{2}{k^2}\left(\alpha_{i-1,j}+4\alpha_{i,j}+\alpha_{i+1,j}\right)-C_{i,j}\left(\alpha_{i-1,j-1}+\alpha_{i+1,j-1}\right)-D_{i,j}\alpha_{i,j-1}+g_{i,j}\,, \tag{22}$$

where

$$A_{i,j}=\frac{3}{h^2}+\frac{1}{k^2}+0.5V_{i,j},\ B_{i,j}=\frac{-6}{h^2}+\frac{4}{k^2}+2V_{i,j}\,,$$

$$C_{i,j}=\frac{3}{h^2}+\frac{1}{k^2}+0.5U_{i,j},\ D_{i,j}=\frac{-6}{h^2}+\frac{4}{k^2}+2U_{i,j}\,,$$

$$V_{i,j} = (\alpha_{i-1,j} + 4\alpha_{i,j} + \alpha_{i+1,j}),\ U_{i,j} = (\beta_{i-1,j} + 4\beta_{i,j} + \beta_{i+1,j}).$$

For each $i = 0,1,...,n$ and $j = 1,2,...$. Systems (21) and (22) have to be complemented by the boundary conditions:

$$U(a,t_j) = U(x_0,t_j) = \varepsilon_1(t_j),\ \ U(b,t_j) = U(x_n,t_j) = \varepsilon_2(t_j),$$

$$V(a,t_j) = V(x_0,t_j) = \rho_1(t_j),\ \ V(b,t_j) = V(x_n,t_j) = \rho_2(t_j).$$

Using Eqs. (7) and (8) and Table 1 these conditions give:

$$\beta_{-1,j} + 4\beta_{0,j} + \beta_{1,j} = \varepsilon_1(t_j), \tag{23}$$

$$\beta_{n-1,j} + 4\beta_{n,j} + \beta_{n+1,j} = \varepsilon_2(t_j), \tag{24}$$

$$\alpha_{-1,j} + 4\alpha_{0,j} + \alpha_{1,j} = \rho_1(t_j), \tag{25}$$

$$\alpha_{n-1,j} + 4\alpha_{n,j} + \alpha_{n+1,j} = \rho_2(t_j), \tag{26}$$

for each $j = 0,1,2,...$. Eliminating $\beta_{-1,j}$ and $\alpha_{-1,j}$ from the first equation of (21)

$$A_{0,j}\beta_{-1,j+1} + B_{0,j}\beta_{0,j+1} + A_{0,j}\beta_{1,j+1} = \frac{2}{k^2}(\beta_{-1,j} + 4\beta_{0,j} + \beta_{1,j}) - A_{0,j}(\beta_{-1,j-1} + \beta_{1,j-1}) - B_{0,j}\beta_{0,j-1} + f_{0,j}$$

and Eqs. (23) and (25)

$$\beta_{-1,j} + 4\beta_{0,j} + \beta_{1,j} = \varepsilon_1(t_j),\ \ j = 0,1,2,\dots\ .$$

$$\alpha_{-1,j} + 4\alpha_{0,j} + \alpha_{1,j} = \rho_1(t_j),\ \ j = 0,1,2,\dots\ .$$

where

$$A_{0,j} = \frac{3}{h^2} + \frac{1}{k^2} + 0.5V_{0,j},\ B_{0,j} = \frac{-6}{h^2} + \frac{4}{k^2} + 2V_{0,j}\ \text{ and }\ V_{0,j} = (\alpha_{-1,j} + 4\alpha_{0,j} + \alpha_{1,j}),$$

for each $i = 0,1,...,n$ and $j = 1,2,...$. We find

$$\frac{-18}{h^2}\beta_{0,j+1} = \frac{18}{h^2}\beta_{0,j-1} + \frac{2}{k^2}\varepsilon_1(t_j) - \left(\frac{3}{h^2} + \frac{1}{k^2} + \frac{1}{2}\rho_1(t_j)\right)(\varepsilon_1(t_{j+1}) + \varepsilon_1(t_{j-1})) + f_{0,j}. \tag{27}$$

Similarly, eliminating $\beta_{n+1,j}$ and $\alpha_{n+1,j}$ from the last of Eq. (21) and Eqs. (24) and Eq. (26), we find

$$\frac{-18}{h^2}\beta_{n,j+1} = \frac{18}{h^2}\beta_{n,j-1} + \frac{2}{k^2}\varepsilon_2(t_j) - \left(\frac{3}{h^2} + \frac{1}{k^2} + \frac{1}{2}\rho_2(t_j)\right)(\varepsilon_2(t_{j+1}) + \varepsilon_2(t_{j-1})) + f_{n,j}. \tag{28}$$

Equations (27), (28), and Eq. (21) for $i = 1,2,...,n-1$ can be written in matrix form

$$A\beta = d, \tag{29}$$

where

$$A = \begin{bmatrix} L & 0 & 0 & \dots & & & & 0 \\ A_{1,j} & B_{1,j} & A_{1,j} & 0 & \dots & & & 0 \\ 0 & A_{2,j} & B_{2,j} & A_{2,j} & 0 & \dots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & A_{n-2,j} & B_{n-2,j} & A_{n-2,j} & 0 \\ 0 & & \cdots & & 0 & A_{n-1,j} & B_{n-1,j} & A_{n-1,j} \\ 0 & \cdots & & & & 0 & 0 & L \end{bmatrix},$$

and,

$$L = \frac{-18}{h^2}, A_{i,j} = \frac{3}{h^2} + \frac{1}{k^2} + 0.5V_{i,j},\ B_{i,j} = \frac{-6}{h^2} + \frac{4}{k^2} + 2V_{i,j},$$

$$V_{i,j} = (\alpha_{i-1,j} + 4\alpha_{i,j} + \alpha_{i+1,j}),\ \beta = (\beta_{0,j+1}, \beta_{1,j+1}, ..., \beta_{n,j+1})^t,$$

$$d = (d_0, d_1, ..., d_n)^t,$$

$$d_0=\frac{18}{h^2}\beta_{0,j-1}+\frac{2}{k^2}\varepsilon_1(t_j)-\left(\frac{3}{h^2}+\frac{1}{k^2}+\frac{1}{2}\rho_1(t_j)\right)\left(\varepsilon_1(t_{j+1})+\varepsilon_1(t_{j-1})\right)+f_{0,j},$$

$$d_n=\frac{18}{h^2}\beta_{n,j-1}+\frac{2}{k^2}\varepsilon_2(t_j)-\left(\frac{3}{h^2}+\frac{1}{k^2}+\frac{1}{2}\rho_2(t_j)\right)\left(\varepsilon_2(t_{j+1})+\varepsilon_2(t_{j-1})\right)+f_{n,j},$$

$$d_i=\frac{2}{k^2}\left(\beta_{i-1,j}+4\beta_{i,j}+\beta_{i+1,j}\right)-A_{i,j}\left(\beta_{i-1,j-1}+\beta_{i+1,j-1}\right)-B_{i,j}\beta_{i,j-1}+f_{i,j}.$$

For each $i=1,2,\dots,n-1$ and $j=1,2,\dots$. A similar system can be developed by eliminating $\alpha_{-1,j},\beta_{-1,j},\alpha_{n+1,j}$ and $\beta_{n+1,j}$ from the first and the last equations of (21). The result is the following system

$$C\alpha=w, \tag{30}$$

where

$$C=\begin{bmatrix} L & 0 & 0 & \dots & & & 0\\ C_{1,j} & D_{1,j} & C_{1,j} & 0 & \dots & & 0\\ 0 & C_{2,j} & D_{2,j} & C_{2,j} & 0 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots\\ 0 & \cdots & \cdots & 0 & C_{n-2,j} & D_{n-2,j} & C_{n-2,j} & 0\\ 0 & & \cdots & & 0 & C_{n-1,j} & D_{n-1,j} & C_{n-1,j}\\ 0 & \cdots & & & & 0 & 0 & L \end{bmatrix},$$

and,

$$L=\frac{-18}{h^2}, C_{i,j}=\frac{3}{h^2}+\frac{1}{k^2}+0.5U_{i,j},\ D_{i,j}=\frac{-6}{h^2}+\frac{4}{k^2}+2U_{i,j},$$

$$U_{i,j}=\left(\beta_{i-1,j}+4\beta_{i,j}+\beta_{i+1,j}\right),\ \alpha=\left(\alpha_{0,j+1},\alpha_{1,j+1},\dots,\alpha_{n,j+1}\right)^t,$$

$$w=\left(w_0,w_1,\dots,w_n\right)^t,$$

$$w_0=\frac{18}{h^2}\alpha_{0,j-1}+\frac{2}{k^2}\rho_1(t_j)-\left(\frac{3}{h^2}+\frac{1}{k^2}+\frac{1}{2}\varepsilon_1(t_j)\right)\left(\rho_1(t_{j+1})+\rho_1(t_{j-1})\right)+g_{0,j},$$

$$w_n=\frac{18}{h^2}\alpha_{n,j-1}+\frac{2}{k^2}\rho_2(t_j)-\left(\frac{3}{h^2}+\frac{1}{k^2}+\frac{1}{2}\varepsilon_2(t_j)\right)\left(\rho_2(t_{j+1})+\rho_2(t_{j-1})\right)+g_{n,j},$$

$$w_i=\frac{2}{k^2}\left(\alpha_{i-1,j}+4\alpha_{i,j}+\alpha_{i+1,j}\right)-C_{i,j}\left(\alpha_{i-1,j-1}+\alpha_{i+1,j-1}\right)-D_{i,j}\alpha_{i,j-1}+g_{i,j},$$

for each $i=1,2,\dots,n-1$ and $j=1,2,\dots$. Since $A_{i,j}$ and $B_{i,j}$ are given by $A_{i,j}=3/h^2+1/k^2+0.5V_{i,j}$ and $B_{i,j}=-6/h^2+4/k^2+2V_{i,j}$, it is easy to see that $B_{i,j}=2\,(1/k^2+V_{i,j}/2-6/h^2)+2A_{i,j}$. Taking absolute values with sufficiently small k, we have

$$\left|B_{i,j}\right|=\left|2(\frac{1}{k^2}+\frac{1}{2}V_{i,j}-\frac{6}{h^2})+2A_{i,j}\right|=2(\frac{1}{k^2}+\frac{1}{2}V_{i,j}-\frac{6}{h^2})+2A_{i,j},$$

therefore, $|B_{i,j}|>2A_{i,j}=|A_{i,j}|+|A_{i,j}|$. From this we observe that A is diagonally dominant, hence nonsingular by Gershgorin's theorem. Since A is nonsingular, system (2) has a unique solution. Similarly, we can prove that system (30) has a unique solution if k is chosen to be small enough.

Systems (29) and (30) imply that $(j+1)st$ time step requires values from $(j)st$ and $(j-1)st$ time steps. This produces a minor starting problem since values for $j=0$ are given by the first parts of initial conditions (13) and (14),

$$\begin{gathered}\frac{\partial U}{\partial x}(x_0,0)=\frac{d\tau_1(x_0)}{dx},\\ U(x_i,0)=\tau_1(x_i),\ i=0,1,\dots,n,\\ \frac{\partial U}{\partial x}(x_n,0)=\frac{d\tau_1(x_n)}{dx},\end{gathered} \tag{31}$$

$$\begin{gathered}\frac{\partial V}{\partial x}(x_0,0)=\frac{d\sigma_1(x_0)}{dx},\\ V(x_i,0)=\sigma_1(x_i),\ i=0,1,\dots,n,\end{gathered} \tag{32}$$

$$\frac{\partial V}{\partial x}(x_n,0)=\frac{d\sigma_1(x_n)}{dx},$$

which can be rewritten by using Eqs. (7) and (8) and Table 1 in the matrix forms

$$A^*\beta^*=d^*, \tag{33}$$

where

$$A^*=\begin{bmatrix} \frac{-3}{h} & 0 & \frac{3}{h} & \dots & & & & 0 \\ 1 & 4 & 1 & 0 & \dots & & & 0 \\ 0 & 1 & 4 & 1 & 0 & \dots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & \cdots & & & 0 & 1 & 4 & 1 \\ 0 & \cdots & & & & \frac{-3}{h} & 0 & \frac{3}{h} \end{bmatrix},$$

$$\beta^*=\left(\beta_{-1,0},\beta_{0,0},\dots,\beta_{n+1,0}\right)^t \text{ and } d^*=\left(\frac{d\tau_1(x_0)}{dx},\tau_1(x_0),\tau_1(x_1),\dots,\tau_1(x_n),\frac{d\tau_1(x_n)}{dx}\right)^t,$$

and,

$$C^*\alpha=w^*, \tag{34}$$

where

$$C^*=\begin{bmatrix} \frac{-3}{h} & 0 & \frac{3}{h} & \dots & & & & 0 \\ 1 & 4 & 1 & 0 & \dots & & & 0 \\ 0 & 1 & 4 & 1 & 0 & \dots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & \cdots & & & 0 & 1 & 4 & 1 \\ 0 & \cdots & & & & \frac{-3}{h} & 0 & \frac{3}{h} \end{bmatrix},$$

$$\alpha^*=\left(\alpha_{-1,0},\alpha_{0,0},\dots,\alpha_{n+1,0}\right)^t \text{ and } w^*=\left(\frac{d\sigma_1(x_0)}{dx},\sigma_1(x_0),\sigma_1(x_1),\dots,\sigma_1(x_n),\frac{d\sigma_1(x_n)}{dx}\right)^t.$$

But the values for j = 1 which are needed in (29) and (30) to compute $(\beta_{-1,2},\beta_{0,2},\dots,\beta_{n+1,2})^t$ and $(\alpha_{-1,2},\alpha_{0,2},\dots,\alpha_{n+1,2})^t$ must be obtained from the second parts of (13) and (14),

$$\frac{\partial U}{\partial t}(x_i,0)=\tau_2(x_i), i=0,1,2,\dots,n,$$

$$\frac{\partial V}{\partial t}(x_i,0)=\sigma_2(x_i),\ i=0,1,2,\dots,n,$$

Using a second Maclaurin polynomial in t for U

$$U(x_i,t_1)\cong U(x_i,0)+k\frac{\partial U(x_i,0)}{\partial t}+\frac{k^2}{2}\frac{\partial^2 U(x_i,0)}{\partial t^2}=\tau_1(x_i)+k\tau_2(x_i)+\frac{k^2}{2}\frac{\partial^2 U(x_i,0)}{\partial t^2},$$

and Eq. (9) calculated at t=0 ; that is,

$$\frac{\partial^2 U(x_i,0)}{\partial t^2}=-\frac{\partial^2 U(x_i,0)}{\partial x^2}-V(x_i,0)U(x_i,0)+f(x_i,0)=-\frac{d^2\tau_1(x_i)}{dx^2}-\sigma_1(x_i)\tau_1(x_i)+f_{i,0},$$

we obtain the following system

$$U(x_i,t_1)=\tau_1(x_i)+k\tau_2(x_i)+\frac{k^2}{2}\left(-\frac{d^2\tau_1(x_i)}{dx^2}-\sigma_1(x_i)\tau_1(x_i)+f_{i,0}\right),$$

for each i = 0,1,…,n. After using the values in Table 1 this equation gives the system

$$\beta_{i-1,1}+4\beta_{i,1}+\beta_{i+1,1}=\tau_1(x_i)+k\tau_2(x_i)+\frac{k^2}{2}\left(-\frac{d^2\tau_1(x_i)}{dx^2}-\sigma_1(x_i)\tau_1(x_i)+f_{i,0}\right), \tag{35}$$

for each i = 0,1,…,n. System (35) consists of n+1 equations and the n+3 unknowns $(\beta_{-1,1},\beta_{0,1},\dots,\beta_{n+1,1})^t$. To solve the last system, we need two additional equations. These equations are

$$\frac{\partial^2 U}{\partial x \partial t}(x_0,0)=\frac{d\tau_2(x_0)}{dx},$$

$$\frac{\partial^2 U}{\partial x \partial t}(x_n,0)=\frac{d\tau_2(x_n)}{dx}.$$

Table 1 enables us to rewrite the last two equations in the forms

$$\frac{-3}{h}\frac{d\beta_{-1}}{dt}(0)+\frac{3}{h}\frac{d\beta_1}{dt}(0)=\frac{d\tau_2(x_0)}{dx},$$

$$\frac{-3}{h}\frac{d\beta_{n-1}}{dt}(0)+\frac{3}{h}\frac{d\beta_{n+1}}{dt}(0)=\frac{d\tau_2(x_n)}{dx}.$$

The use of the forward-difference formula

$$\frac{d\beta_i}{dt}(0)=\frac{\beta_{i,1}-\beta_{i,0}}{k},$$

in the last two equations results in the equations

$$\frac{-3}{h}\beta_{-1,1}+\frac{3}{h}\beta_{1,1}=\frac{-3}{h}\beta_{-1,0}+\frac{3}{h}\beta_{1,0}+k\frac{d\tau_2(x_0)}{dx}, \tag{36}$$

$$\frac{-3}{h}\beta_{n-1,1}+\frac{3}{h}\beta_{n+1,1}=\frac{-3}{h}\beta_{n-1,0}+\frac{3}{h}\beta_{n+1,0}+k\frac{d\tau_2(x_n)}{dx}. \tag{37}$$

Eqs. (36), (37), and (35) can be written in matrix form

$$A^{**}\beta^{**}=d^{**}, \tag{38}$$

where

$$A^{**}=\begin{bmatrix} \frac{-3}{h} & 0 & \frac{3}{h} & \cdots & & & & 0 \\ 1 & 4 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & \cdots & & & 0 & 1 & 4 & 1 \\ 0 & \cdots & & & & \frac{-3}{h} & 0 & \frac{3}{h} \end{bmatrix},$$

$$\beta^{**}=\left(\beta_{-1,1},\beta_{0,1},\ldots,\beta_{n+1,1}\right)^t \text{ and } d^{**}=\left(d_{-1},d_0,\ldots,d_n,d_{n+1}\right)^t,$$

$$d_i^{**}=\tau_1(x_i)+k\tau_2(x_i)+\frac{k^2}{2}\left(-\frac{d^2\tau_1(x_i)}{dx^2}-\sigma_1(x_i)\tau_1(x_i)+f_{i,0}\right), \text{ for } i=0,1,2,\ldots,n,$$

$$d_{-1}^{**}=\frac{-3}{h}\beta_{-1,0}+\frac{3}{h}\beta_{1,0}+k\frac{d\tau_2(x_0)}{dx},\; d_{n+1}^{**}=\frac{-3}{h}\beta_{n-1,0}+\frac{3}{h}\beta_{n+1,0}+k\frac{d\tau_2(x_n)}{dx},$$

Similar system can be developed for computing $(\alpha_{-1,1},\alpha_{0,1},\ldots,\alpha_{n+1,1})^t$

$$C^{**}\alpha^{**}=w^{**}, \tag{39}$$

where

$$C^{**}=\begin{bmatrix} \frac{-3}{h} & 0 & \frac{3}{h} & \cdots & & & & 0 \\ 1 & 4 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & \cdots & & & 0 & 1 & 4 & 1 \\ 0 & \cdots & & & & \frac{-3}{h} & 0 & \frac{3}{h} \end{bmatrix},$$

$$\alpha^{**} = (\alpha_{-1,1}, \alpha_{0,1}, \ldots, \alpha_{n+1,1})^t \text{ and } w^{**} = (w_{-1}, w_0, \ldots, w_n, w_{n+1})^t ,$$

$$w_i^{**} = \sigma_1(x_i) + k\sigma_2(x_i) + \frac{k^2}{2}\left(-\frac{d^2\sigma_1(x_i)}{dx^2} - \sigma_1(x_i)\tau_1(x_i) + g_{i,0}\right), \text{ for } i = 0,1,2,\ldots,n ,$$

$$w_{-1}^{**} = \frac{-3}{h}\alpha_{-1,0} + \frac{3}{h}\alpha_{1,0} + k\frac{d\sigma_2(x_0)}{dx}, \ w_{n+1}^{**} = \frac{-3}{h}\alpha_{n-1,0} + \frac{3}{h}\alpha_{n+1,0} + k\frac{d\sigma_2(x_n)}{dx} .$$

Since the matrices A^*, C^*, A^{**} and C^{**} are strictly diagonally dominant, it follows from Gershgorin's theorem that A^*, C^*, A^{**} and C^{**} are nonsingular. Hence, systems (33), (34), (38), and (39) have unique solutions.

3. Stability Analysis

For stability analysis, we use the Von Neumann technique. To do this, we must linearize the nonlinear term $v(x,t)u(x,t)$ of Eq. (1) by making $v(x,t)$ locally constant, which is equivalent to assuming that the corresponding value $V_{i,j}$ is equal to a local constant V^* in system (21). According to the Von Neumann technique we have:

$$\beta_{i,j} = \xi^j e^{(q\varphi hi)} , \tag{40}$$

where $q^2 = -1$, φ is the mode number, h is the element size, and ξ is the amplification factor. Inserting the last expression for $\beta_{i,j}$ into system (21) to obtain the characteristic equation:

$$\xi^{j+1}\left\{A_{i,j}\left(e^{q\varphi h(i-1)} + e^{q\varphi h(i+1)}\right) + B_{i,j}e^{q\varphi h(i)}\right\} = \frac{2}{k^2}\xi^j\left\{\left(e^{q\varphi h(i-1)} + e^{q\varphi h(i+1)}\right) + 4e^{q\varphi hi}\right\} - \xi^{j-1}\left\{A_{i,j}\left(e^{q\varphi h(i-1)} + e^{q\varphi h(i+1)}\right) + B_{i,j}e^{q\varphi hi}\right\} ,$$

where

$$A_{i,j} = \frac{3}{h^2} + \frac{1}{k^2} + 0.5V^*, \ B_{i,j} = \frac{-6}{h^2} + \frac{4}{k^2} + 2V^* ,$$

Dividing by $\xi^{j-1}e^{q\varphi hi}$, this equation becomes:

$$\xi^2\left\{A_{i,j}\left(e^{-q\varphi h} + e^{q\varphi h}\right) + B_{i,j}\right\} = \frac{2}{k^2}\xi\left\{\left(e^{-q\varphi h} + e^{q\varphi h}\right) + 4\right\} - \left\{A_{i,j}\left(e^{-q\varphi h} + e^{q\varphi h}\right) + B_{i,j}\right\} ,$$

Simple calculations enable us to write:

$$\xi^2 + 2\mu\xi + 1 = 0 , \tag{41}$$

where

$$\mu = \frac{-(4 + 2\cos\varphi)}{k^2\left(B_{i,j} + 2A_{i,j}\cos\varphi\right)} .$$

The two roots of the characteristic equation (41) are:

$$\xi_\pm = -\mu \pm \sqrt{\mu^2 - 1} .$$

The necessary and sufficient condition for stability is $|\xi_\pm| \le 1$. If $\mu^2 - 1 \le 0$, then it is easy to verify that $|\xi_\pm| \le 1$, and if $\mu^2 - 1 > 0$ then one of ξ_- and ξ_+ does not satisfy the condition $|\xi_\pm| \le 1$. Hence, the necessary and sufficient condition for stability is $\mu^2 - 1 \le 0$. In other words, we must have $-1 \le \mu \le 1$, which implies that:

$$-1 \le \frac{(4 + 2\cos\varphi)}{(4 + 2\cos\varphi) + k^2\left(\left(\frac{-6}{h^2} + 4V^*\right) + 2\left(\frac{-6}{h^2} + 2V^*\right)\cos\varphi\right)} \le 1 .$$

when k is sufficiently small, such that $k^2 \to 0$, this inequality is satisfied. Thus, stability analysis of system (22) can be developed.

4. Numerical Results

Consider the coupled nonlinear partial differential equations:

$$\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\partial^2 u}{\partial t^2}(x,t) + v(x,t)u(x,t) = -(\pi^2 + 1)\sin\pi x\cos t + \sin^2\pi x\cos^2 t ,$$

$$\frac{\partial^2 v}{\partial x^2}(x,t) + \frac{\partial^2 v}{\partial t^2}(x,t) + u(x,t)v(x,t) = -(\pi^2 + 1)\sin\pi x\cos t + \sin^2\pi x\cos^2 t , \tag{42}$$

for $0 \le x \le 1$ and $t \ge 0$ subject to the conditions:

$$u(0,t) = 0, \ u(1,t) = 0, \ v(0,t) = 0, \ v(1,t) = 0 , \tag{43}$$

$$u(x,0)=\sin \pi x, \frac{\partial u}{\partial t}(x,0)=0,\ v(x,0)=\sin \pi x, \frac{\partial v}{\partial t}(x,0)=0\ . \tag{44}$$

The exact solution of the system (42)-(44) is

$$\begin{aligned} u(x,t) &= \sin \pi x \cos t, \\ v(x,t) &= \sin \pi x \cos t, \end{aligned} \tag{45}$$

the results are presented in Tables (2)-(7). Tables (2)-(5) list the maximum absolute errors with $k=0.0005$ and $h=0.1$, with $k=0.00005$ and $h=0.01$. Tables (6)-(7) list the numerical solutions $V_{i,j}=(\alpha_{i-1,j}+4\alpha_{i,j}+\alpha_{i+1,j})$, and $U_{i,j}=(\beta_{i-1,j}+4\beta_{i,j}+\beta_{i+1,j})$ and exact solutions $u_{i,j}$ and $v_{i,j}$ with $k=0.0005$ and $h=0.1$.

5. Conclusion

In this paper, systematical use of the collocation analysis method has been successfully employed in identifying the approximate solutions for coupled nonlinear non-homogeneous Klein-Gordon partial differential equations. We applied the Von Neumann stability method and found that the proposed method is conditionally stable. The basis of this approach can be widely utilized to solve other strongly nonlinear evolution problems. We provided a numerical example to examine the accuracy and efficiency of the proposed method. It was evident from the example that the approximate solution is very close to the exact solution. The observed errors were summarized in tables to verify the stability of our presented scheme. Its accuracy has been demonstrated by calculating L_∞ error norms. The obtained numerical results showed that the present method is a remarkably efficient numerical technique for solving coupled nonlinear non-homogeneous Klein-Gordon equations, which makes it useful for a wide range of applications.

Table 2. Max. Absolute error $k=\Delta t=0.0005$ and $h=\Delta x=0.1$.

Time	0.1	0.3	0.4	0.5
Max $\left\|U_{i,j}-u_{i,j}\right\|$	4.0749×10^{-4}	3.8535×10^{-3}	7.1709×10^{-3}	1.3249×10^{-2}

Table 3. Max. Absolute error $k=\Delta t=0.0005$ and $h=\Delta x=0.1$.

Time	0.1	0.3	0.4	0.5
Max $\left\|V_{i,j}-v_{i,j}\right\|$	4.0749×10^{-4}	3.8535×10^{-3}	7.1709×10^{-3}	1.3249×10^{-2}

Table 4. Max. Absolute error $k=\Delta t=0.00005$ and $h=\Delta x=0.01$.

Time	0.0005	0.005	0.054
Max $\left\|U_{i,j}-u_{i,j}\right\|$	9.1326×10^{-11}	1.0047×10^{-8}	1.4709×10^{-6}

Table 5. Max. Absolute error $k=\Delta t=0.00005$ and $h=\Delta x=0.01$.

Time	0.0005	0.005	0.054
Max $\left\|V_{i,j}-v_{i,j}\right\|$	9.1326×10^{-11}	1.0047×10^{-8}	1.4709×10^{-6}

Table 6. $u_{i,j}$ and $U_{i,j}$ with $k=\Delta t=0.00005$ and $t=0.3, h=\Delta x=0.01$.

x	$u_{i,j}$	$U_{i,j}$
0.2	0.5615326992848998	0.563810889237005
0.4	0.90857899323744	0.9122450638802360
0.6	0.90857899323744	0.9122450638156732
0.8	0.5615326992848999	0.5638108891169986

Table 7. $v_{i,j}$ and $V_{i,j}$ with $k=\Delta t=0.00005$ and $t=0.3, h=\Delta x=0.1$.

x	$v_{i,j}$	$V_{i,j}$
0.2	0.5615326992848998	0.563810889237005
0.4	0.90857899323744	0.9122450638802360
0.6	0.90857899323744	0.9122450638156732
0.8	0.5615326992848999	0.5638108891169986

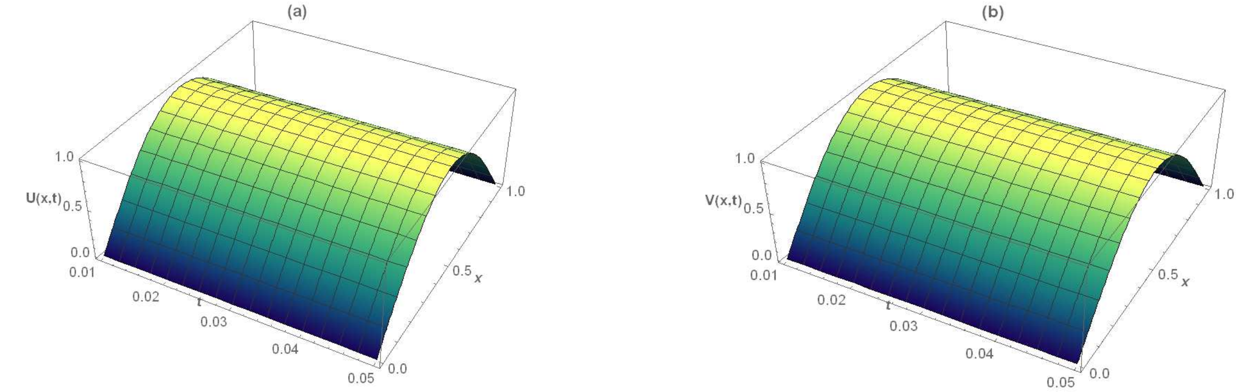

Fig. 1. Graphs of approximate solutions at k = 0.00005 and h = 0.01 for $U(x,t)$ part (a) and $V(x,t)$ part (b).

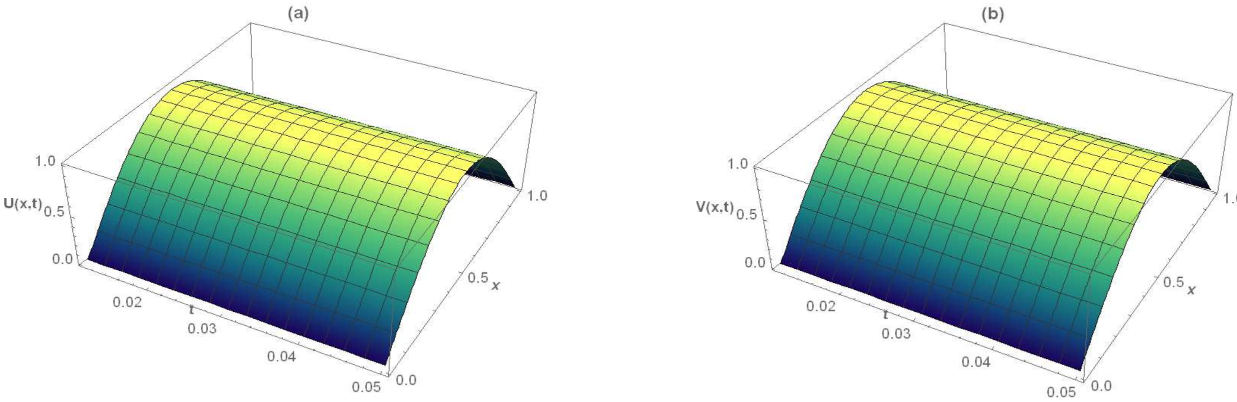

Fig. 2. Graphs of approximate solutions at Δt = 0.00005 and h = 0.01 for $U(x,t)$ part (a) and $V(x,t)$ part (b).

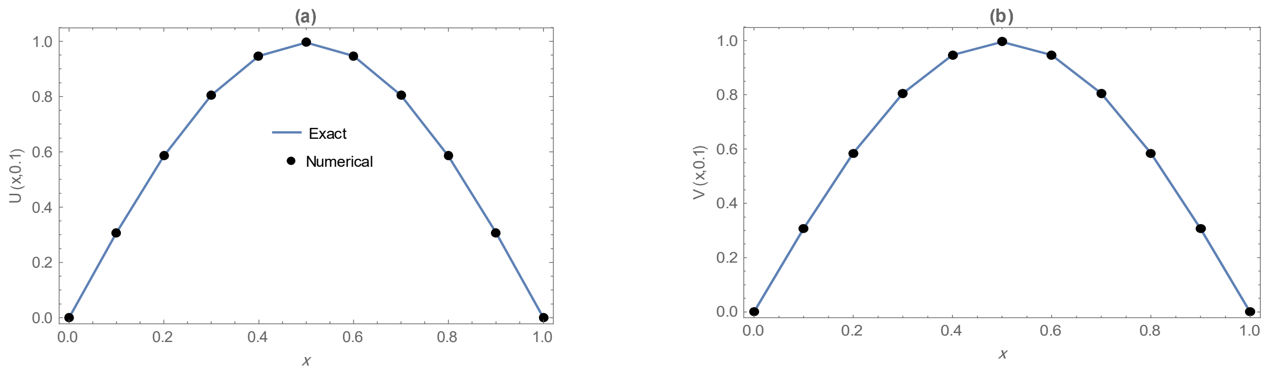

Fig. 3. Graphs of approximate solutions at Δt = 0.0005 and h = 0.1 for $U(x,t)$ part (a) and $V(x,t)$ part (b).

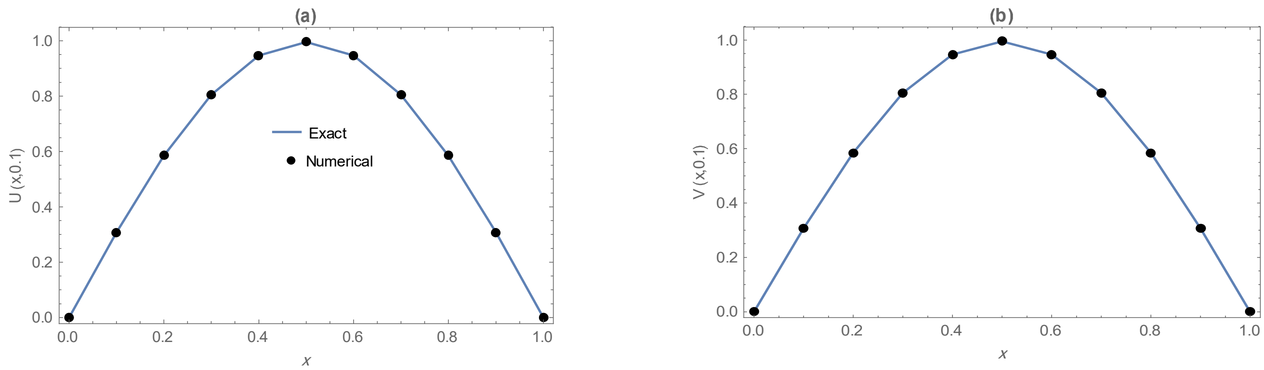

Fig. 4. Graphs of approximate solutions at Δt = 0.00005 and h = 0.1 for $U(x,t)$ part (a) and $V(x,t)$ part (b).

Author Contributions

Z.M. Alaofi planned the scheme, initiated the project, and suggested the experiments; T.A. El-Danaf conducted the experiments and analyzed the empirical results; F.E.I. Abd Alaal developed the mathematical modeling and examined the theory validation and made the mathematica programming. S.S. Dragomir developed the mathematical modeling and examined the theory validation. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Acknowledgments

Not applicable.

Conflict of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

Funding

The authors received no financial support for the research, authorship, and publication of this article.

Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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ORCID ID

Z.M. Alaofi https://orcid.org/0000-0002-6556-999X
T.A. El-Danaf https://orcid.org/0000-0001-7139-7031
Faisal E.I. Abd Alaal https://orcid.org/0000-0002-4769-5191
S.S. Dragomir https://orcid.org/0000-0003-2902-6805

How to cite this article: Alaofi Z.M., El-Danaf T.A., Abd Alaal F.E.I., Dragomir S.S. Numerical Investigations of the Coupled Nonlinear Non-homogeneous Partial Differential Equations, *J. Appl. Comput. Mech.*, 8(3), 2022, 1054–1064.
https://doi.org/10.22055/jacm.2022.39327.3388