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Research Paper

An Analysis of Nonlinear Beam Vibrations with the Extended Rayleigh-Ritz Method

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Abstract. The nonlinear deformation and vibrations of beams are frequently encountered as a typical example of structural analysis as well as a mathematical problem. There have been many methods and techniques for the approximate and exact solutions of nonlinear differential equations arising from the nonlinear phenomena of elastic beam structures. One method is particularly more powerful and flexible is proposed recently as the extended Rayleigh-Ritz method (ERRM) by adding the temporal variable as another dimension of deformation formulation but eliminated through the integration over a period of vibrations. Such a procedure leads to a simple, elegant, and powerful method for the approximate solutions of nonlinear vibration and deformation problems in dynamics and structural analysis. By utilizing the usual displacement function of beams, the nonlinear vibration frequencies of Euler-Bernoulli and Timoshenko beams are obtained with the same accuracy as from other approximate solutions.

Keywords: Extended Rayleigh-Ritz Method (ERRM), nonlinear vibration, Euler-Bernoulli beam, Timoshenko beam.

1. Introduction

Nonlinear problems from many engineering fields are frequently encountered in mathematical analysis with important roles in design and production of structures and components. Through mathematical classification and categorization, nonlinear problems are more appealing and intriguing in the dynamic behavior and solution techniques. To cope with challenges from many scientific problems and engineering applications, there have been tremendous efforts and long history in solving the typical nonlinear vibration problems with various methods and procedures. There have been abundant literatures on nonlinear vibration problems and solution techniques with broad applications.

Nonlinear vibrations of beams are common in engineering applications and natural phenomena with typical examples such as slender beams and similar structures like trees and plants. It also referred to as vibrations with large amplitudes in analysis with Euler-Bernoulli beam theory [1-11]. The applications of nonlinear vibrations of beams are also widely found in structures with composite materials, and a further refinement is done with Euler-Bernoulli, Timoshenko, and other beam theories [12-14]. As for the analysis of nonlinear vibrations, there are many approaches and techniques such as the differential quadrature method (DQM) [15], variational iteration method (VIM) [16], homotopy analysis method (HAM) [17], harmonic balance method (HBM) [18], and the homotopy perturbation analysis (HPM) [19-21]. It is generally accepted that the more accurate solutions can be obtained from the full solutions of the HAM [17, 22]. In a comparison with other approximate techniques, the extension of Rayleigh-Ritz method through the inclusion of harmonic terms, or the extended Rayleigh-Ritz method (ERRM) [23], will be utilized here to demonstrate the efficiency and simplicity in obtaining the first-order approximate solutions of frequency and corresponding mode shapes.

For linear vibrations of common structural components such as beams and rods, in addition to the analytical solutions with the consideration of boundary conditions by various techniques [24-25], one most popular and effective method is the Rayleigh-Ritz method (RRM) [26-28]. The method starts with the assumption of displacements with undetermined amplitudes, then the kinetic and potential energies are obtained through the integration of known displacement functions. By minimizing the Lagrangian functional with the vanishing of derivatives to unknown amplitudes, the vibration frequency is obtained from the resulting eigenvalue problem. If displacements are exact, the frequency will also be exact. The critical condition for exact displacements is that the boundary conditions are satisfied. In practice, the accurate solutions of both frequency and mode shape are obtained through using the known displacement solutions or approximations from a family of popular series such as trigonometric functions or Chebyshev polynomials. Because of its simplicity and easy implementation, the Rayleigh-Ritz method has been widely used for problems which are hard to obtain accurate analytical solutions with many known techniques [29-31]. Regrettably, it is generally recognized that the Galerkin and Rayleigh-Ritz methods are only effective for linear problems, and

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common nonlinear problems of vibrations are seldom tried with these methods even for approximate solutions of the frequency and mode shapes.

Recently, in an expanded search for a simple and effective method to obtain approximate solutions of nonlinear vibrations, it is discovered that the Rayleigh-Ritz method can be extended or modified as a novel and equally powerful technique with usual procedure and assumptions [23]. The basic idea is that the approximate displacement function is assumed to be harmonic with time, and the Lagrangian functional will have products of trigonometric terms. In addition to the integration over the spatial domain of the elastic structure, the integration over one cycle of the harmonic vibrations is also performed to eliminate the temporal variable. Then, the same exact procedure of the Rayleigh-Ritz method on the minimization of the Lagrangian is performed, and the vibration frequency, consequently the mode shape, are obtained just like the standard Rayleigh-Ritz method for linear problems. Clearly, it is a simple and effective extension of the traditional Rayleigh-Ritz method, making it is available to both linear and nonlinear problems with just one simple procedure as the extended Rayleigh-Ritz method for the first-order approximation of nonlinear problems.

With the excellent approximation to typical problems of nonlinear vibrations, the extended Rayleigh-Ritz method [23] is utilized for the nonlinear vibrations of beams as a demonstration of the new technique for nonlinear vibration problems in general.

2. Methods for Nonlinear Vibration Analysis

There is a long and rich history on methods and techniques for the analysis and solution of nonlinear vibration problems. There are extensive literatures like monographs, textbooks, research papers, review papers, and proceedings covering this subject in all aspects. The brief review and introduction in this paper will focus on a few methods closely related to the particular methods and applications in this study.

2.1 The Rayleigh-Ritz method

As it is generally known from popular textbooks and monographs, the standard Rayleigh-Ritz method starts from the calculation of potential energy U and kinetic energy T of an elastic structure with volume V for the Lagrangian functional

$$L = \int_V (U - T)\mathrm{d}V \tag{1}$$

which can also be obtained from the equation of motion through integration [23, 32-33]. However, it should be stressed that the energy terms in Eq. (1) are functions of spatial variables only, or they are represented by their maximum values. Then with solutions of displacements with undetermined amplitude A, frequency ω, and time t through

$$u = A\cos\omega t \tag{2}$$

and the vibration frequency and displacement amplitude satisfy

$$\frac{\partial L}{\partial A} = \frac{\partial U}{\partial A} - \frac{\partial T}{\partial A} = 0 \tag{3}$$

This will present a typical generalized eigenvalue problem of vibrations without appearance of temporal variable. Or, it should be emphasized that time is not in the analysis as a variable.

2.2 The variational approach and applications

As one recent innovative development of analytical methods for nonlinear vibration problems, the variational approach for nonlinear vibration solutions was first proposed and validated by He [34]. The integration of Lagrangian functional over a quarter of the vibration period was examined and validated with success in solving some typical problems based on a variational approach, or with Eq. (1) modified to

$$L = \int_0^{\frac{T}{4}} \int_V (U - T)\mathrm{d}V\mathrm{d}t \tag{4}$$

as the variational approach for nonlinear oscillations by He [34]. It is proved that such an integration can solve the nonlinear vibration problems with the same result as from other methods. Lately, the method was also used by Anderson et al. for nonlinear vibration problems as further validation [35]. To be more consistent and reflective with harmonic vibrations, an integration over one full period of fundamental vibration mode was proposed and demonstrated by Wang with the same approximate results [23]. The late approach, different in the upper limit of integration over time, was referred to as the extended Rayleigh-Ritz method, and a similar extension has also been made to the Galerkin method [36]. If the integrals of functions are proportional to the vibration period, the results of the variational approach and integration scheme with the extended Rayleigh-Ritz method are the same. In case there are situations that the integrals are also dependent on values of functions at limits of the integration, there will be differences to be examined through analytical and numerical procedures. Generally speaking, the integration over a full period can include the properties and behaviors of functions better, implying possibly improved approximation, to be explored in future studies.

2.3 The extended Rayleigh-Ritz method and Galerkin method

In a recent analysis of nonlinear vibration problems, it is believed that an extension to the Rayleigh-Ritz method by adding time as a variable and integrating over one period of vibrations will provide another approach for solving both linear and nonlinear vibration problems by using the Lagrangian functional in the form of [23]

$$L = \int_0^{T} \int_V (U - T)\mathrm{d}V\mathrm{d}t \tag{5}$$

Noting the change of upper limit of the integration above is the full period T now in comparison with Eq. (4), the variational approach by He [34]. For linear vibrations, the results with the above extension of the standard Rayleigh-Ritz method are exactly the same with the known solutions with and without integration of time. For typical nonlinear vibration problems such as nonlinear vibrations of pendulum and Duffing equation, earlier studies have been shown that the same results from other methods such as the harmonic balance method are obtained [23, 36]. Of course, the results of fundamental mode are also exactly the same with the variational approach [34-35]. In fact, the validity and accuracy of the variational approach [34] and the extended Rayleigh-Ritz method [23] can be traced to the general principle and technique of the averaged Lagrangian method [37-39], which was not adopted and utilized for elastic structural problems for a long time. It can be considered that Eq. (5) is a modification of Eq. (4) from the variational approach by He [34], but the integration over one full period of vibration with the Galerkin method was not practiced before [36], and these procedures are actually related to the averaged Lagrangian method by Miles [38-39]. With successful applications of the extended Rayleigh-Ritz method and the equivalent extended Galerkin method (EGM) [36], the objective of this study is to demonstrate that the ERRM can be easily implemented in the analysis of nonlinear vibrations of beams of different types, manifesting the potential for a unified formulation and solution of both linear and nonlinear vibrations with the popular Rayleigh-Ritz and Galekin methods.

3. The extended Rayleigh-Ritz Analysis of Nonlinear Vibrations of Euler-Bernoulli Beams

Linear vibrations of beams are important problems for the demonstration of analytical methods and procedures as part of the essential foundation of solid mechanics and continuous systems [40]. Without losing generality, the simple nonlinear vibration equation of an Euler-Bernoulli beam subjected to an axial loading in the flexural motion is taken as an example as shown in Fig. 1 [41].

$$EI\frac{\partial^4 w}{\partial x^4}+F\frac{\partial^2 w}{\partial x^2}+\frac{EA}{2L}\frac{\partial^2 w}{\partial x^2}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x+\rho A\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}=0 \tag{6}$$

where w is the transverse displacements, x is the axial coordinate of the beam, E is the modulus of elasticity, I is the moment of inertia, F is the axial force, A is the cross-sectional area, L is the length of the beam, ρ is the density, and t is time.

The fourth-order nonlinear differential equation in Eq. (1) can be solved with analytical method for the approximate solutions of frequency and mode shapes as shown in popular references [42-43]. More importantly, it can be transformed to a nonlinear vibration problem of the well-known Duffing equation [41-43]

$$\ddot{q}+\alpha q+\beta q^3=0 \tag{7}$$

The approximate frequency from many solution techniques is [34-35, 41-43]

$$\omega^2=\alpha+\frac{3}{4}\beta B^2,\quad \omega=\omega_0+\frac{3\beta}{8\omega_0}B^2,\quad \Delta\omega=\frac{3\beta}{8\omega_0}B^2 \tag{8}$$

with $\omega_0=\sqrt{\alpha}$ as the linear vibration frequency, β is coefficient of the third-order nonlinearity and B as the vibration amplitude.

To apply the ERRM for approximate solutions of the problem, it starts from the Lagrangian of the equation of motion in Eq. (1)

$$\bar{L}=\frac{1}{2}EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2-F\left(\frac{\partial w}{\partial x}\right)^2+\frac{EA}{4L}\left(\frac{\partial w}{\partial x}\right)^2\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x+\frac{1}{2}\rho A\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 \tag{9}$$

As with the Rayleigh-Ritz method, the flexure of the beam is assumed to be a harmonic function in the form of

$$w=Wf(x)\cos\omega t \tag{10}$$

with W as the amplitude of the flexure and $f(x)$ is the mode shape function of the beam. Now the Lagrangian with the flexural displacement is

$$\bar{L}=\frac{1}{2}EIW^2\left(\frac{\partial^2 f}{\partial x^2}\right)^2\cos^2\omega t-FW^2\left(\frac{\partial f}{\partial x}\right)^2\cos^2\omega t+\frac{EA}{4L}W^4\left(\frac{\partial f}{\partial x}\right)^2\int_0^L\left(\frac{\partial f}{\partial x}\right)^2\mathrm{d}x\cos^4\omega t-\frac{1}{2}\rho AW^2\omega^2\int_0^L f^2\,\mathrm{d}x\sin^2\omega t \tag{11}$$

which is exactly the function used in the traditional RRM if the nonlinear and harmonic terms are dropped.

As the extension of the traditional RRM, the Lagrangian in Eq. (11) is now integrated over the spatial domain of the elastic beam and time domain of one cycle of the vibration period $T=2\pi/\omega$, which results in

$$\begin{aligned}L&=\int_0^T\int_0^L\bar{L}\,\mathrm{d}x\,\mathrm{d}t\\&=\frac{\pi}{2\omega}EIW^2\int_0^L\left(\frac{\partial^2 f}{\partial x^2}\right)^2\mathrm{d}x-\frac{\pi}{\omega}FW^2\int_0^L\left(\frac{\partial f}{\partial x}\right)^2\mathrm{d}x\\&\quad+\frac{3\pi EA}{32\omega L}W^4\left(\int_0^L\left(\frac{\partial f}{\partial x}\right)^2\mathrm{d}x\right)^2-\frac{\pi}{2\omega}\rho AW^2\int_0^L f^2\,\mathrm{d}x\,\omega^2\end{aligned} \tag{12}$$

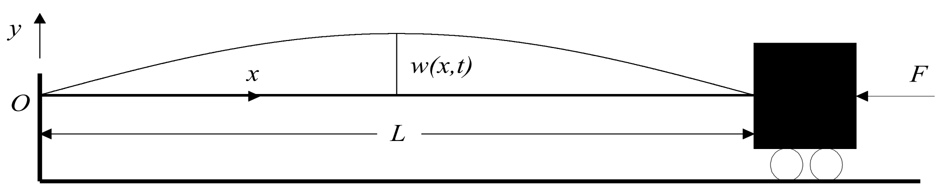

Fig. 1. An Euler-Bernoulli beam subjected to an axial load

in which the temporal variable t is now eliminated through the integration with identities

$$\int_0^T \cos^2 \omega t\, dt = \int_0^T \sin^2 \omega t\, dt = \frac{T}{2}, \quad \int_0^T \cos^4 \omega t\, dt = \frac{3T}{8} \tag{13}$$

and some coefficients associated to the integrations of mode shapes are

$$C_1 = \int_0^L \left(\frac{\partial^2 f}{\partial x^2}\right)^2 dx, \quad C_2 = \int_0^L \left(\frac{\partial f}{\partial x}\right)^2 dx, \quad C_3 = \int_0^L f^2\, dx \tag{14}$$

Then the Lagrangian is

$$L = \frac{\pi}{2\omega} EIW^2 C_1 - \frac{\pi}{\omega} FW^2 C_2 + \frac{3\pi EA}{32\omega L} W^4 C_2^2 - \frac{\pi}{2\omega} \rho A W^2 C_3 \omega^2 \tag{15}$$

Now the standard RRM can be applied for the solution of the nonlinear vibration problem with a new definition in Eq. (12). Apparently, for a linear problem, the integration over time provides the Lagrangian functional consistent with the expression without harmonic terms. Or, the elimination of time from Eq. (12) keeps the procedure valid for both linear and nonlinear vibration problems. This is certainly a useful extension which can expand the applications of the Rayleigh-Ritz method without restricting to linear cases.

If the nonlinear terms of the vibration amplitude are neglected, it is exactly the linear vibration problem solvable by the RRM. Next, the RRM requires that

$$\frac{\partial L}{\partial W} = \frac{\pi}{\omega} EIWC_1 - \frac{2\pi}{\omega} FWC_2 + \frac{3\pi EA}{8\omega L} W^3 C_2^2 - \frac{\pi}{\omega} \rho A W C_3 \omega^2 = 0 \tag{16}$$

Consequently, the vibration frequency is now

$$\omega^2 = \frac{EIC_1 - 2FC_2 + \frac{3EA}{8L} W^2 C_2^2}{\rho A C_3}, \quad \omega_0^2 = \frac{EIC_1 - 2FC_2}{\rho A C_3} \tag{17}$$

with ω_0 as the linear vibration frequency. This result can be compared with results from other methods [33, 42]. If there is no axial force F the resonant frequency is

$$\omega^2 = \frac{EIC_1 + \frac{3EA}{8L} W^2 C_2^2}{\rho A C_3} = \omega_0^2 + \beta W^2, \quad \omega_0^2 = \frac{EIC_1}{\rho A C_3}, \quad \beta = \frac{3E}{8\rho L} \frac{C_2^2}{C_3} \tag{18}$$

There are results published like this one from many earlier studies [7-11].

Clearly, the ERRM yields a very good first-order approximation of nonlinear vibrations of the beam. It is the extension of RRM with the inclusion of time and integration over one period, as demonstrated. It means the vibration solution is periodic as a harmonic motion, implying the approximation of the first-order.

Additionally, even for the simple solutions given in Eq. (2), the ERRM here is much simpler than the techniques shown or demonstrated by other methods [15-22]. This is the advantage of the ERRM for nonlinear vibration problems. To obtain the vibration frequency and mode shape, the displacement function $f(x)$ should be chosen carefully. If the displacement is the same, the same solutions from other techniques can be obtained due to the same effect on the elimination of higher-order harmonic terms. As known in nonlinear vibration problems, the frequency and mode shape are amplitude dependent, and the evaluation part will be the same.

For further demonstration of the applicability, now the ERRM is also applied to the nonlinear vibrations of Timoshenko beams.

4. Analysis of Nonlinear Vibrations of Timoshenko Beams

As before, only the isotropic and uniform beams are considered here for the nonlinear vibration analysis of the Timoshenko beams. With large deformation of the extension u and flexure w, it is known [44]

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \kappa_x = -\frac{\partial^2 w}{\partial x^2} \tag{19}$$

where ε_x and κ_x are the normal strain and curvature of the flexure with x as the length coordinate of the beam.

With the consideration of shear deformation, the rotation of the cross-section is now

$$\frac{\partial w}{\partial x} = \psi + \gamma \tag{20}$$

where ψ and γ are shear deformation and rotation, respectively. As a result, the shear force is

$$V = -kSG\gamma \tag{21}$$

with S, G and k as cross-sectional area, shear modulus, and shear factor, respectively.

For the Timoshenko beam, the strain and kinetic energies are

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \left[\rho S \left(\frac{\partial w}{\partial t}\right)^2 + \rho I \left(\frac{\partial \psi}{\partial t}\right)^2\right] dx \\ U &= \frac{1}{2}\int_0^L \left[EI\left(\frac{\partial \psi}{\partial x}\right)^2 + kSG\gamma^2 + ES\varepsilon_x^2\right] dx \end{aligned} \tag{22}$$

with L, I, E, ρ and t as length, inertia momentum, Young's modulus, density, and time, respectively.

With the nonlinear strain in Eq. (19), the axial force is [44]

$$N(x,t) = N_0 + ES\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \tag{23}$$

where N_0 is the initial axial force.

Since the two-ends of the beam are fixed, implying extension $u = 0$, and N is independent of x, so [17-23]

$$N(x,t) = N(t) = N_0 + \frac{ES}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \tag{24}$$

As a result, the strain energy of the beam will be

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial \psi}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L kSG\left(\frac{\partial w}{\partial x} - \psi\right)^2 dx + \frac{ES}{8L}\left(\frac{2N_0 L}{ES} + \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right)^2 \tag{25}$$

With the Rayleigh-Ritz method, the Lagrangian is

$$\overline{\Pi} = U - T \tag{26}$$

And the expressions with independent deformation variables f and w are

$$\begin{aligned}\overline{\Pi} = &\frac{1}{2}\int_0^L EI\left(\frac{\partial \psi}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L kSG\left(\frac{\partial w}{\partial x} - \psi\right)^2 dx \\ &+ \frac{ES}{8L}\left(\frac{2N_0 L}{ES} + \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right)^2 \\ &- \frac{1}{2}\int_0^L \left[\rho S\left(\frac{\partial w}{\partial t}\right)^2 + \rho I\left(\frac{\partial \psi}{\partial t}\right)^2\right] dx\end{aligned} \tag{27}$$

Now the nondimensional variables as

$$\hat{x} = \frac{x}{L},\ \hat{w} = \frac{w}{L},\ \hat{\psi} = \frac{\psi}{L} \tag{28}$$

are introduced. The Lagrangian has a new expression as

$$\begin{aligned}\overline{\Pi} = &\frac{1}{2}L\int_0^1 EI\left(\frac{\partial \hat{\psi}}{\partial \hat{x}}\right)^2 d\hat{x} + \frac{1}{2}L\int_0^1 kSG\left(\frac{\partial \hat{w}}{\partial \hat{x}} - L\hat{\psi}\right)^2 d\hat{x} \\ &+ \frac{ES}{8L}\left(\frac{2N_0 L}{ES} + L\int_0^1 \left(\frac{\partial \hat{w}}{\partial \hat{x}}\right)^2 d\hat{x}\right)^2 \\ &- \frac{1}{2}L^3\int_0^1 \left[\rho S\left(\frac{\partial \hat{w}}{\partial t}\right)^2 + \rho I\left(\frac{\partial \hat{\psi}}{\partial t}\right)^2\right] d\hat{x}\end{aligned} \tag{29}$$

Now the deformation of beam is assumed harmonic in the forms of [24-25]

$$\begin{aligned} w(x,t) &= AW(x)\cos\omega t \\ \psi(x,t) &= B\Psi(x)\cos\omega t \end{aligned} \tag{30}$$

where $W(x)$ and $\Psi(x)$ are the mode shape functions of a simply-supported beam. By inserting Eq. (30) in to Eq. (29), it yields

$$\begin{aligned}\overline{\Pi} = &\left[\frac{L}{2}\int_0^1 EI\left(B\frac{d\Psi(x)}{dx}\right)^2 dx + \frac{L}{2}\int_0^1 kSG\left(A\frac{dW(x)}{dx} - LB\Psi(x)\right)^2 dx\right]\cos^2\omega t \\ &+ \frac{N_0^2 L}{2ES} + \frac{N_0}{2}L\int_0^1 \left(A\frac{dW(x)}{dx}\right)^2 dx \cos^2\omega t \\ &+ \frac{ESL}{8}\left(\int_0^1 \left(A\frac{dW(x)}{dx}\right)^2 dx\right)^2 \cos^4\omega t \\ &- \omega^2\left\{\frac{L^3}{2}\int_0^1 \left[\rho SA^2W^2(x) + \rho IB^2\Psi^2(x)\right] dx\right\}\sin^2\omega t\end{aligned} \tag{31}$$

As a simplification, the following constants are defined

$$\begin{aligned} &c_1 = \int_0^1 \left(\frac{dW(x)}{dx}\right)^2 dx,\ c_2 = \int_0^1 W^2(x)dx, \\ &c_3 = \int_0^1 \left(\frac{d\Psi(x)}{dx}\right)^2 dx,\ c_4 = \int_0^1 \Psi^2(x)dx,\ c_5 = \int_0^1 \frac{dW(x)}{dx}\Psi(x)dx \end{aligned} \tag{32}$$

Then

$$\begin{aligned}\overline{\Pi} = &\left[\frac{L}{2}EIB^2c_3 + \frac{L}{2}kSG(A^2c_1 - 2LABc_5 + L^2B^2c_4)\right]\cos^2\omega t \\ &+ \frac{N_0^2L}{2ES} + \frac{N_0}{2}LA^2c_1\cos^2\omega t + \frac{ESL}{8}A^4c_1^2\cos^4\omega t \\ &- \omega^2\frac{L^3\rho}{2}(SA^2c_2 + IB^2c_4)\sin^2\omega t\end{aligned} \tag{33}$$

It should be noted that the Lagrangian now has time in trigonometric terms. The appearances of such terms will not change the results of the Rayleigh-Ritz method for linear vibrations if these terms are combined, or the maximum values are considered. In the treatment of the trigonometric terms of Eq. (33), it is found they can be eliminated without causing any difficulties in the analysis by integrating the functional over one cycle of vibration. Then, the normal procedure of the Rayleigh-Ritz method can be applied with the integration of the Lagrangian over time as

$$\Pi = \int_0^T \overline{\Pi}dt \tag{34}$$

where $T = 2\pi/\omega$. As a result, the Lagrangian functional now is

$$\begin{aligned}\Pi = &\left[\frac{L}{2}EIB^2c_3 + \frac{L}{2}kSG(A^2c_1 - 2LABc_5 + L^2B^2c_4)\right]\frac{T}{2} \\ &+ \frac{N_0^2L}{2ES} + \frac{N_0}{4}LA^2c_1T + \frac{3ESL}{64}A^4c_1^2T \\ &- \frac{L^3\rho}{4}(SA^2c_2 + IB^2c_4)\omega^2T\end{aligned} \tag{35}$$

The Rayleigh-Ritz method requires

$$\frac{\partial\widehat{\Pi}}{\partial A} = 0,\ \frac{\partial\widehat{\Pi}}{\partial B} = 0 \tag{36}$$

or

$$\begin{aligned}\frac{\partial\widehat{\Pi}}{\partial A} = &\frac{1}{2}kSGTL(Ac_1 - LBc_5) + \frac{1}{2}N_0TLAc_1 \\ &+ \frac{3ESTL}{16}A^3c_1^2 - \frac{L^3}{2}\rho STAc_2\omega^2\end{aligned} \tag{37}$$

and

$$\frac{\partial\widehat{\Pi}}{\partial B} = \frac{TL}{2}\left[EIBc_3 + kSG(L^2Bc_4 - LAc_5)\right] - \frac{L^3}{2}\rho ITBc_4\omega^2 \tag{38}$$

Further simplification of above expressions yields

$$kSG(Ac_1 - LBc_5) + N_0Ac_1 + \frac{3ES}{8}A^3c_1^2 - \omega^2L^2\rho SAc_2 = 0 \tag{39}$$

and

$$EIBc_3 + kSG(L^2Bc_4 - LAc_5) - \omega^2L^2\rho IBc_4 = 0 \tag{40}$$

In Eq. (39), if the shear deformation B is assumed to be zero, the resonant frequency of the Euler-Bernoulli beam is

$$\omega_B^2 = \frac{N_0c_1 + \frac{3ESA^2}{8}c_1^2}{\rho SL^2c_2} \tag{41}$$

If the momentum of inertia is neglected, the resonant frequency is

$$\omega_S^2 = \frac{kSGc_1 - \frac{kSGLB}{A}c_5 + N_0c_1 + \frac{3ESA^2}{8}c_1^2}{\rho SL^2c_2} \tag{42}$$

These are the results by Ramezani and Alasty [44] and Foda [45] with the method of multiple scales. Clearly, the ERRM obtained these results with a relatively simple procedure.

Furthermore, from Eqs. (39) and (40),

$$\frac{\left(EIc_3 + kSGL^2c_4 - \omega^2L^2\rho Ic_4\right)}{kSGLc_5}B = A \tag{43}$$

and

$$\omega^2 = \frac{\left(EIc_3 + kSGL^2c_4\right)B^2 - (kSGc_1 + N_0c_1 + \frac{3ES}{8}A^2c_1^2)A^2}{L^2\rho(Ic_4B^2 - Sc_2A^2)} \tag{44}$$

This is the resonant frequency of the nonlinear Timoshenko beam. By selecting the deformation and vibration modes, the frequencies of pure shear and bending modes of a beam shown in Eq. (41) and Eq. (42) are

$$\omega^2 = \frac{EIc_3 + kSGL^2c_4}{L^2\rho Ic_4} \tag{45}$$

and

$$\omega^2 = \frac{kSGc_1 + N_0c_1 + \frac{3ES}{8}A^2c_1^2}{L^2\rho Sc_2} \tag{46}$$

These are the known frequency solutions of vibrations of a beam with both nonlinear Euler-Bernoulli and Timoshenko equations. These solutions have been obtained by other methods before, making them ideal for comparison and validation of the ERRM presented as a new procedure. The advantage of the ERRM, as shown through the examples, is the simple procedure through the integration of the compounded harmonics which can be better handled and eliminated with integration. Thus the ERRM is capable and preferable for the first-order approximation of nonlinear equations related to vibrations and other structural problems. Further improvements for higher-order solutions should be explored along the same path for more accurate solutions which are important in applications in our next phase of research.

5. Conclusions

By utilizing the newly proposed extended Rayleigh-Ritz method and the equivalent extended Galerkin method, the approximate results of resonant frequency and mode shape of beams, including both popular beam theories of Euler-Bernoulli and Timoshenko were obtained with a simple procedure. The results were compared with other techniques for validation and consistency. More importantly, the ERRM had a much simpler procedure and calculation. It will be an important and effective method for the analysis of nonlinear vibrations for the approximate solutions of the fundamental mode. It is also important to extend the standard RRM to nonlinear problems, demonstrating the power of the RRM and universality for wide range of problems in structural analysis. Further studies of the ERRM can help better and more accurate solutions of nonlinear vibration problems which are widely encountered in mathematics and engineering applications with a simple procedure. Besides, the procedure is also applicable to similar methods like the Galerkin method for the treatment of nonlinear problems which are more common in structural analysis and applied mathematics. In addition, it is believed that further improvement of the extended Rayleigh-Ritz method will be able to provide higher-order approximations in a similarly simple and efficient procedure which is different from existing techniques. This, of course, is an important development in the solution techniques and methods for nonlinear differential equations in structural vibrations and more general mathematical problems and engineering applications.

Author Contributions

The method and applications were the suggestion of JW. XLG and RXW worked on the problems with formulation, derivation, calculation, and drafting of the paper in discussion with HMJ. The revision, polish and submission were done by HMJ, RXW, BH, and JW.

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Conflict of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

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Data Availability Statements

The data used and generated in this research will be publicly available from the publication's website and author's website.

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