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Research Paper

Finite Deformations of Fibre-reinforced Elastic Solids with Fibre Bending Stiffness: A Spectral Approach

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Abstract. In this paper, we propose a spectral approach to model finite deformations of fibre-reinforced elastic solids with fibre bending stiffness. The constructed constitutive equations depend on spectral invariants, where each one has a clear physical meaning and hence are attractive for use in experiment and analysis. With the use of spectral invariants, we easily obtain the number of independent invariants and the number of invariants in the corresponding minimal integrity or irreducible basis. The proposed finite strain energy prototypes are consistent with their infinitesimal strain energy function counterparts. Some results for pure bending of a slab, and the extension and torsion of solid cylinder, that could be useful for experiments and numerical validations, are given. The proposed model could be used to obtain numerical results via modification of some computational methods found in the literature.

Keywords: Finite elasticity; Bending stiffness; Fibre-reinforced solids; Bending; Torsion; Spectral invariants.

1. Introduction

Fibre-reinforced composites have widely been applied in automotive, aerospace, civil engineering, sports, wind energy and so on across the world. The (nonlinear) combination of a matrix and a reinforcement equips composite materials with much better mechanical performance than the matrix and reinforcement material alone [14]. However, in most previous theoretical models, the influence of bending stiffness is not considered. Modelling fibre-reinforced solids with bending stiffness (see, for example, reference [30]) gives significantly different results from modelling fibre-reinforced solids if we assume that the fibres are perfectly flexible [5, 6, 14]; in particular bending stiffness models produce couple-stress and non-symmetric stress and is accordingly classified as a second-gradient theory [2, 4, 7, 10, 11].

However, in the past, bending stiffness models used traced-based classical invariants [29] (or their variants) to describe their constitutive equations [5, 6]. Since the 1940s trace-based classical invariants have played an important role in the development of constitutive models in continuum mechanics. Rivlin and others developed trace based invariants, because they are convenient and easy to evaluate. However, in many theoretical works, where such invariants are used, there is no interest about fitting with experimental data, the issue of propagation of error, nor being consistent with physics and the infinitesimal theory. Problems arise because most of the classical invariants do not have an immediate physical meaning and, hence, they are not attractive in seeking to design a rational program of experiments. For example, it is not straightforward to design an experiment [8, 19] (denoted by R-experiment), where in order to rigorously construct a specific functional form of the strain energy, requires to capture the mechanical behavior of the material in terms of a single classical invariant, while keeping the remaining (classical) invariants fixed. We note that an R-experiment requires the number of independent invariants in the corresponding minimal integrity or irreducible basis. It is shown in references [21, 25, 27] that the number of independent invariants is generally less than the number of invariants in the corresponding minimal integrity or irreducible basis, and is far less if the number of classical invariants in a minimal integrity or irreducible basis is large. Because of the unclear physical meaning of the classical invariants, it is not clear how to select the relevant independent classical invariants from the set of invariants in the corresponding minimal or irreducible basis. In addition to this, researchers are not sure which invariants are best needed for a given problem, and for simplicity a reduced number of invariants is commonly considered, which may create problems in order to capture the response of the material [13, 22]. However, it is shown by Shariff [19, 20] that spectral invariants, each one with a clear physical meaning, are easy to analyze and attractive for use in R-experiment. Furthermore, to evaluate the number of independent classical invariants in a minimal integrity basis is not straightforward due to the difficulty in constructing relations (syzygies) among classical invariants. However, relations among the spectral variables are easily constructed [21, 25, 27] and, hence, the number of independent spectral

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invariants can be easily obtained.

In view of the convenience of using spectral invariants, in this communication, following the authors' previous works (see, for example references [19, 20, 22, 23]), we develop spectral constitutive models to describe the mechanical behaviour of fibre-reinforced elastic solids with bending stiffness. Advantages of physically meaningful spectral invariants over classical invariants [29] (or their variants) are described in, for example, Shariff [23]; see also Appendix A.

Results for pure bending and, extension and torsion of a solid cylinder are given in this paper.

2. Preliminaries

Spectrally, the deformation gradient is given by

$$\boldsymbol{F}(\lambda_i,\boldsymbol{v}_i,\boldsymbol{u}_i)=\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}=\sum_{i=1}^{3}\lambda_i\boldsymbol{v}_i\otimes\boldsymbol{u}_i \tag{1}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ denote the position vectors in the current and reference configurations, respectively, of a particle in the solid body, λ_i is a principal stretch, $\boldsymbol{v}_i$ is an eigenvector of the left stretch tensor $\boldsymbol{V}=\boldsymbol{F}(\lambda_i,\boldsymbol{v}_i,\boldsymbol{v}_i)$ and $\boldsymbol{u}_i$ is an eigenvector of the right- stretch tensor $\boldsymbol{U}=\boldsymbol{F}(\lambda_i,\boldsymbol{u}_i,\boldsymbol{u}_i)$. Note that the right Cauchy-Green tensor $\boldsymbol{C}=\boldsymbol{F}(\lambda_i^2,\boldsymbol{u}_i,\boldsymbol{u}_i)$. Note that $=\boldsymbol{RU}$, where the rotation tensor $\boldsymbol{R}=\boldsymbol{F}(\lambda_i=1,\boldsymbol{v}_i,\boldsymbol{u}_i)$.

We consider a material with the preferred unit direction $a(x)$ in the reference configuration and this preferred direction becomes the vector $\boldsymbol{b}=\boldsymbol{Fa}$ in the current configuration.

2.1 Strain energy function

Following the work of Spencer and Soldatos [30], we assume the strain energy function (SE I)

$$W_{(e)}=W(\boldsymbol{U},\boldsymbol{\Lambda},\boldsymbol{a}) \tag{2}$$

where

$$\boldsymbol{\Lambda}=\boldsymbol{F}^T\boldsymbol{G}-\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{x}}\ ,\qquad \boldsymbol{G}=\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{x}}. \tag{3}$$

We call the tensor $\boldsymbol{\Lambda}$, the bending stiffness tensor. We note that Spencer and Soldatos [30] define $\boldsymbol{\Lambda}=\boldsymbol{F}^T\boldsymbol{G}$; we find that with this definition, $\boldsymbol{\Lambda}$ has the value of $\frac{\partial a}{\partial x}$ in rigid-body motion. In our definition $(3)_1$, $\boldsymbol{\Lambda}=\boldsymbol{0}$ in rigid-body motion.

It is plausible that in fibre composite solids the fibre curvature plays an important role. We model this by first considering the vector

$$\boldsymbol{b}=\boldsymbol{Fa}=\lambda\boldsymbol{f}\qquad \lambda=\sqrt{\boldsymbol{b}\cdot\boldsymbol{b}}=\sqrt{\boldsymbol{a}\cdot\ \boldsymbol{C}\,\boldsymbol{a}}\ , \tag{4}$$

in the current configuration, where $\boldsymbol{f}$ is the unit vector in the direction of $\boldsymbol{b}$.

We assume that the strain energy function $W_{(e)}$ depends on the fibre curvature

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{a}, \tag{5}$$

the deformation tensor $\boldsymbol{F}$ and the preferred direction $\boldsymbol{a}$. Hence, we can write

$$W_{(e)}=W_{(F)}\left(\boldsymbol{F},\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{a},\boldsymbol{a}\right). \tag{6}$$

For a strain energy function to be objective, we express the strain energy function (SE II) in the form

$$W_{(e)}=W_{(O)}(\boldsymbol{C},\boldsymbol{d},\boldsymbol{a}),\qquad \boldsymbol{d}=\boldsymbol{F}^T\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{a} \tag{7}$$

We note that Spencer and Soldatos [30] uses

$$\boldsymbol{d}=\boldsymbol{F}^T\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{x}}\boldsymbol{a} \tag{8}$$

in their strain energy function, instead of $(7)_2$. However, we show in the Appendix B that, with the definition (8), the value of $\boldsymbol{d}$ is not always zero when the fibres are not bend.

Note that, in view of $\frac{\partial a}{\partial x}\boldsymbol{a}=\boldsymbol{0}$, we have the useful form required in Section 3

$$\boldsymbol{d}(\boldsymbol{C},\boldsymbol{\Lambda},\boldsymbol{a})=\frac{1}{\lambda}\boldsymbol{\Lambda a}-\frac{1}{\lambda^3}(\boldsymbol{a}\cdot\boldsymbol{\Lambda a})\boldsymbol{Ca} \tag{9}$$

Note that $\boldsymbol{d}\cdot\boldsymbol{a}=\boldsymbol{0}$ as expected. In Section 4 we study both strain energy formulations (2) and (7).

3. Stress and Couple Stress

The inclusion of the tensor $\boldsymbol{G}$ in the strain energy function produces a non-symmetric stress tensor [30]

$$\boldsymbol{T}=\boldsymbol{T}_{(s)}+\boldsymbol{T}_{(a)}\,,\quad J\boldsymbol{T}_{(s)}=2\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{C}}\boldsymbol{F}^T+\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{\Lambda}}\boldsymbol{G}^T+\boldsymbol{G}\frac{\partial W}{\partial \boldsymbol{\Lambda}^T}\boldsymbol{F}^T,\qquad \boldsymbol{T}_{(a)}=-\frac{1}{2}\,\mathbb{E}\,\mathrm{div}\boldsymbol{M}, \tag{10}$$

where $\mathbb{E}$ is the three-dimensional alternating tensor, div is the divergence of a tensor in the current configuration, and $\boldsymbol{M}$ is the couple stress tensor. The deviatoric part of the couple stress tensor $\bar{\boldsymbol{M}}=\boldsymbol{M}-\frac{1}{3}\mathrm{tr}(\boldsymbol{M})\,\boldsymbol{I}$,where tr denotes the trace of a tensor and $\boldsymbol{I}$ is the identity tensor, is related to the deformation via [30] the relation

$$J\frac{3}{2}\bar{\boldsymbol{M}}=\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{\Lambda}}\boldsymbol{F}^T\mathbb{E}\,\boldsymbol{b}-\boldsymbol{b}\otimes\mathbb{E}\,\boldsymbol{F}\frac{\partial W}{\partial\,\boldsymbol{\Lambda}}\boldsymbol{F}^T. \tag{11}$$

The operation $\mathbb{E}(\boldsymbol{l}\otimes\boldsymbol{n})=\boldsymbol{n}\times\boldsymbol{l}$, where $\boldsymbol{l}$ and $\boldsymbol{n}$ are vectors and $\times$ is the cross-product, is used to obtained the expression in (11) . Let $\boldsymbol{\omega}$ be the spin vector, taking note of the constraint $\operatorname{tr}\frac{\partial\boldsymbol{\omega}}{\partial y}=0$. A couple stress power is given by $\operatorname{tr}\left(\boldsymbol{M}\frac{\partial\boldsymbol{\omega}}{\partial y}\right)$. Hence, we obtain the relations

$$\operatorname{tr}\left(\boldsymbol{M}\frac{\partial\boldsymbol{\omega}}{\partial y}\right)=\operatorname{tr}\left([\boldsymbol{M}+\bar{p}\,\boldsymbol{I}]\frac{\partial\boldsymbol{\omega}}{\partial y}\right)=\operatorname{tr}\left(\bar{\boldsymbol{M}}\frac{\partial\boldsymbol{\omega}}{\partial y}\right)=\frac{2}{3J}\operatorname{tr}\left(\left[\boldsymbol{F}\frac{\partial W}{\partial \Lambda}\boldsymbol{F}^T\mathbb{E}\,\boldsymbol{b}-\boldsymbol{b}\otimes\mathbb{E}\boldsymbol{F}\frac{\partial W}{\partial \Lambda}\boldsymbol{F}^T\right]\frac{\partial\boldsymbol{\omega}}{\partial y}\right),$$

where $\bar{p}$ can be considered as a Lagrange multiplier associated with the constraint $\operatorname{tr}\frac{\partial\boldsymbol{\omega}}{\partial y}=0$. Since $\frac{\partial\boldsymbol{\omega}}{\partial y}$ is arbitrary, we have,

$$\boldsymbol{M}=\frac{2}{3J}\left(\boldsymbol{F}\frac{\partial W}{\partial \Lambda}\boldsymbol{F}^T\mathbb{E}\,\boldsymbol{b}-\boldsymbol{b}\otimes\mathbb{E}\boldsymbol{F}\frac{\partial W}{\partial \Lambda}\boldsymbol{F}^T\right)-\bar{p}\,\boldsymbol{I}. \tag{12}$$

From (11), we have $\operatorname{tr}\bar{\boldsymbol{M}}=0$, hence in view of (12), we obtain the spherical part $\frac{1}{3}\operatorname{tr}\boldsymbol{M}=-\bar{p}$, which states that the spherical part is indeterminate (arbitrary) and is a Lagrange multiplier associated with the constraint $\operatorname{tr}\frac{\partial\boldsymbol{\omega}}{\partial y}=0$. It is clear that the indeterminate spherical part makes no contribution to the energy balance equation.

The equation of motion and the balance of rotational momentum for the couple stress, in the absence of body forces, take the forms

$$\operatorname{div}\boldsymbol{T}=\boldsymbol{0},\qquad \operatorname{div}\boldsymbol{M}+\mathbb{E}\,\boldsymbol{T}^T=0\,. \tag{13}$$

4. Spectral Formulations

The strain energy function SE I must satisfy the relation

$$W_{(e)}=W(\boldsymbol{U},\boldsymbol{\Lambda},\boldsymbol{a})=W(\boldsymbol{QUQ}^T,\boldsymbol{Q\Lambda Q}^T,\boldsymbol{Qa}) \tag{14}$$

for every rotation tensor $\boldsymbol{Q}$. Following, the work of Shariff [27], we can express the strain energy function $W_{(e)}$ in terms of the spectral invariants

$$\lambda_i,\quad a_i=\boldsymbol{u_i}\cdot\boldsymbol{a}\,,\quad \Lambda_{ij}=\boldsymbol{u_i}\cdot\boldsymbol{\Lambda}\,\boldsymbol{u}_j,\quad a_1^2+a_2^2+a_3^2=1\,. \tag{15}$$

Hence, only 14 of the 15 invariants in (15) are independent. For strain energy SE II, it must satisfy the invariant relation

$$W_{(O)}(\boldsymbol{C},\boldsymbol{d},\boldsymbol{a})=W_{(O)}\,(\boldsymbol{Q}\,\boldsymbol{C}\,\boldsymbol{Q}^T,\boldsymbol{Qd},\boldsymbol{Qa}) \tag{16}$$

Expressing

$$\boldsymbol{d}=\rho\,\boldsymbol{k}\,,\quad \rho=\sqrt{\boldsymbol{d}\cdot\boldsymbol{d}}\,, \tag{17}$$

where $\boldsymbol{k}$ is the unit vector in the $\boldsymbol{d}$ direction, and following the work of Shariff [27], the strain energy $\boldsymbol{W}_{(e)}$ can be expressed in terms of the spectral invariants

$$\lambda_i,\quad k_i=\boldsymbol{u_i}\cdot\boldsymbol{k}\,,\quad a_i=\boldsymbol{u_i}\cdot\ \boldsymbol{a}\,,\quad \rho \tag{18}$$

Since, $\sum_{i=1}^3 k_i^2=\sum_{i=1}^3 a_i^2=1$, only 9 of the 11 invariants in (18) are independent. Note that

$$\rho^2=\frac{1}{\lambda^2}\left(\boldsymbol{a}\cdot\boldsymbol{\Lambda}^T\boldsymbol{\Lambda}\,\boldsymbol{a}-\frac{2\kappa}{\lambda^2}\boldsymbol{a}\cdot\boldsymbol{C\Lambda}\,\boldsymbol{a}+\frac{\kappa^2}{\lambda^4}\boldsymbol{a}\cdot\boldsymbol{C}^2\boldsymbol{a}\right),\quad \kappa=\boldsymbol{a}\cdot\boldsymbol{\Lambda a}. \tag{19}$$

Hence, we can express

$$W_{(e)}=W_{(s)}(\lambda_i,a_i,k_i,\rho)\,. \tag{20}$$

It is important to note that non-uniqueness of the eigenvectors of $\boldsymbol{U}$ when two or more of the principal value have the same value, the strain energies described by the spectral invariants must satisfy the P-property as described in Shariff [24].

4.1 Spectral derivative components

The spectral formulation requires the Lagrangian spectral tensor components for $\frac{\partial W_{(e)}}{\partial \boldsymbol{C}}$ and $\frac{\partial W_{(e)}}{\partial \Lambda}$, i.e.,

$$\left(\frac{\partial W_{(e)}}{\partial \boldsymbol{C}}\right)_{ii}=\frac{1}{2\lambda_i}\frac{\partial W_{(e)}}{\partial\lambda_i}\qquad (i\quad \text{not summed}) \tag{21}$$

and for the shear components

$$\left(\frac{\partial W_{(e)}}{\partial \boldsymbol{C}}\right)_{ij}=\frac{\frac{\partial W_{(e)}}{\partial \boldsymbol{u}_i}\cdot\boldsymbol{u}_j-\frac{\partial W_{(e)}}{\partial \boldsymbol{u}_j}\cdot\boldsymbol{u}_i}{2\left(\lambda_i^2-\lambda_j^2\right)},\ i\neq j \tag{22}$$

$$\left(\frac{\partial W_{(e)}}{\partial \boldsymbol{\Lambda}}\right)_{ij}=\boldsymbol{u}_i\cdot\frac{\partial W_{(e)}}{\partial \boldsymbol{\Lambda}}\boldsymbol{u}_j. \tag{23}$$

Take note that

$$\frac{\partial W_{(e)}}{\partial \boldsymbol{C}}=\sum_{i,j=1}^3\left(\frac{\partial W_{(e)}}{\partial \boldsymbol{C}}\right)_{ij}\boldsymbol{u}_i\otimes\boldsymbol{u}_j,\ \frac{\partial W_{(e)}}{\partial \boldsymbol{\Lambda}}=\sum_{i,j=1}^3\left(\frac{\partial W_{(e)}}{\partial \boldsymbol{\Lambda}}\right)_{ij}\boldsymbol{u}_i\otimes\boldsymbol{u}_j \tag{24}$$

5. Strain Energy Prototypes

Due to the non-existent of experiment data for this class of materials, we are not able to rigorously construct a finite-strain constitutive equation for specific types for this class of materials. However, a finite strain spectral constitutive equation must be consistent with its infinitesimal strain counterpart and satisfies the P-property described in Shariff [24]. Based on this, and considering only incompressible materials, we propose the following strain energy functions.

Before we construct strain energy prototypes for finite strain deformation, we give a brief description on infinitesimal elasticity. When the gradient of the displacement field $\mathbf{u}$ is very small

$$\|\boldsymbol{F}-\boldsymbol{I}\| = \left\|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right\| = O(e)\,, \tag{25}$$

where ‖•‖ is an appropriate norm and the magnitude of e is much less than unity. In this case we have

$$\boldsymbol{\Lambda} \approx (\boldsymbol{I} + \boldsymbol{E} - \boldsymbol{W})\boldsymbol{G} - \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{x}}, \tag{26}$$

where $\boldsymbol{E}$ and $\boldsymbol{W}$ are the infinitesimal strain and rotational tensors, respectively. Since $\|\boldsymbol{E}\|$ and $\|\boldsymbol{W}\|$ are both $O(e)$, we have for $\|\boldsymbol{\Lambda}\|$ of $O(e)$ that

$$\boldsymbol{\Lambda} = \frac{\partial \boldsymbol{u}_a}{\partial x} + \left(\frac{\partial \boldsymbol{u}}{\partial x}\right)^T \frac{\partial \boldsymbol{a}}{\partial x}\,, \quad \boldsymbol{u}_a = \frac{\partial \boldsymbol{u}}{\partial x}\boldsymbol{a}\,, \quad \boldsymbol{U} - \boldsymbol{I} = \boldsymbol{E} \tag{27}$$

and $\lambda_i - 1 = e_i$ is $O(e)$, where e_i are the eigenvalues of the infinitesimal strain $\boldsymbol{E}$ and we do not distinguish the eigenvectors of $\boldsymbol{U}$ and $\boldsymbol{E}$.

Taking into consideration that $W_{(e)}$ should be independent of the sign of $\boldsymbol{a}$, and the fact that the invariant $\boldsymbol{a}\cdot\boldsymbol{\Lambda}\,\boldsymbol{a}$ does not contribute to the couple stress (11) [30], and in order that stress $\boldsymbol{T}$ and the couple stress $\bar{\boldsymbol{M}}$ to be of $O(e)$, the most general strain energy function takes the form

$$W_{(e)} = W_{(T)} + W_{(\Lambda)}\,, \tag{28}$$

where following the work of Shariff [23]

$$W_{(T)} = \mu_T \hat{I}_1 + 2(\mu_L - \mu_T)\hat{I}_2 + \frac{\beta}{2}\hat{I}_3, \tag{29}$$

$$\hat{I}_1 = \operatorname{tr}\boldsymbol{E}^2 = \sum_{i=1}^{3} e_i^2, \quad \hat{I}_2 = \boldsymbol{a}\cdot\boldsymbol{E}^2\boldsymbol{a} = \sum_{i=1}^{3}\alpha_i e_i^2, \quad \hat{I}_3 = (\boldsymbol{a}\cdot\boldsymbol{E}\,\boldsymbol{a})^2 = \left(\sum_{i=1}^{3}\alpha_i e_i\right)^2, \quad \alpha_i = a_i^2. \tag{30}$$

μ_T, μ_L and β are ground state constants [23].

For the Cauchy and the couple stresses to be of $O(e)$, taking into account that $W_{(\Lambda)}$ must be independent of the signs of $\boldsymbol{a}$ and $\boldsymbol{\Lambda}$, and in view the work of [30], we have,

$$W_{(\Lambda)} = \bar{b}_1\bar{J}_1 + \bar{b}_2\bar{J}_2 + \bar{b}_3\bar{J}_3 + b_4\bar{J}_4\,, \tag{31}$$

where

$$\begin{gathered}\bar{J}_1 = \operatorname{tr}(\boldsymbol{\Lambda}^2) = \sum_{i,j=1}^{3}\Lambda_{ij}\Lambda_{ij}, \quad \bar{J}_2 = \boldsymbol{a}\cdot\boldsymbol{\Lambda}^2\boldsymbol{a} = \sum_{i,j,k=1}^{3} a_i\Lambda_{ik}\Lambda_{kj}a_j\,, \\ \bar{J}_3 = \operatorname{tr}(\boldsymbol{\Lambda})(\boldsymbol{a}\cdot\boldsymbol{\Lambda}\,\boldsymbol{a}) = \left(\sum_{i,=1}^{3}\Lambda_{ii}\right)\left(\sum_{i,j=1}^{3}a_i\Lambda_{ij}a_j\right), \quad \bar{J}_4 = (\operatorname{tr}\boldsymbol{\Lambda})^2 = \left(\sum_{i,=1}^{3}\Lambda_{ii}\right)^2.\end{gathered} \tag{32}$$

In this paper, we impose constraints on the ground-state constants of $W_{(T)}$ based on the work of [23]; however, we will not give the conditions here. In the case of the ground-state constants of $W_{(\Lambda)}$, based on the satisfaction of positive semi-definiteness, following the work of Soldatos et al. [28], we necessarily have

$$\bar{b}_1 = \bar{b}_2 = 0\,, \quad b_4 \geq 0\,, \quad 0 \geq \frac{\bar{b}_3^2}{4b_4}. \tag{33}$$

Hence, we have the reduced strain energy

$$W_{(\Lambda)} = b_4\bar{J}_4\,. \tag{34}$$

For nonlinear strain energy SE I to be consistent with infinitesimal elasticity, we assume that strain energy function of the form

$$W_{(e)} = W_{(T)}(\boldsymbol{U},\boldsymbol{a}) + W_{(\Lambda)}(\boldsymbol{\Lambda},\boldsymbol{a}), \tag{35}$$

where [23]

$$W_{(T)} = \mu_T\bar{I}_1 + 2(\mu_L-\mu_T)\bar{I}_2 + \frac{\beta}{2}\bar{I}_3\,, \tag{36}$$

$\bar{I}_1 = \sum_{i=1}^{3} r_1(\lambda_i)$, $\bar{I}_2 = \sum_{i=1}^{3}\alpha_i r_2(\lambda_i)$, $\bar{I}_3 = (\sum_{i=1}^{3}\alpha_i r_3(\lambda_i))^2$, with conditions, [23] $r_1(1) = r_2(1) = r_3(1) = r'_1(1) = r'_2(1) = 0$ and $r''_1(1) = r''_2(1) = 2$.

Similarly, a strain energy $W_{(\Lambda)}$ for finite strain that is consistent with infinitesimal elasticity [28], can be easily constructed, i.e.,

$$W_{(\Lambda)} = b_1 f_1(\bar{J}_1) + b_2 f_2(\bar{J}_2) + b_3 f_{3a}(t_1) f_{3b}(t_2) + b_4 f_4(t_1) + b_5 f_5(t_2)\,, \tag{37}$$

$$t_1 = \text{tr}\, \boldsymbol{\Lambda} = \sum_{i=1}^{3} \Lambda_{ii}, \quad t_2 = \boldsymbol{a} \cdot \boldsymbol{\Lambda}\, \boldsymbol{a} = \sum_{i,j=1}^{3} a_i \Lambda_{ij} a_j, \tag{38}$$

with conditions $f_1(0) = f_2(0) = f_{3a}(0) = f_{3b}(0) = f_4(0) = f_5(0) = 0$, $f_1{}'(0) = f_2{}'(0) = f'_{3a}(0) = f'_{3b}(0) = f'_4(0) = f'_5(0) = 0$, $f''_1(0) = f''_2(0) = f''_{3a}(0) = f''_{3b}(0) = f''_4(0) = f''_5(0) = 2$.

In the case of strain energy function SE II, taking into consideration that $W_{(e)}$ should be independent of the sign of $\boldsymbol{a}$ and the result of the invariant $\boldsymbol{a} \cdot \boldsymbol{k} = 0$, we propose a general quadratic strain energy function for infinitesimal elasticity,

$$W_{(e)} = W_{(T)} + W_{(\Lambda)}, \tag{39}$$

where $W_{(T)}$ is given (29) and

$$W_{(\Lambda)} = c\rho^2, \tag{40}$$

where c is ground state material constant. A necessary and sufficient condition for $W_{(\Lambda)}$ to be non-negative is $c \geq 0$.

For the strain energy SE II, we propose a prototype finite strain energy function that is consistent with its infinitesimal counterpart

$$W_{(e)} = W_{(T)} + W_{(\Lambda)}\,, \tag{41}$$

where $W_{(T)}$, is given by (36) and

$$W_{(\Lambda)} = cs(\rho)\,. \tag{42}$$

Note that

$$\rho^2 = \frac{1}{\lambda^2}\left(\boldsymbol{a} \cdot \boldsymbol{\Lambda}^T \boldsymbol{\Lambda}\, \boldsymbol{a} - \frac{2\kappa}{\lambda^2}\, \boldsymbol{a} \cdot \boldsymbol{C}\boldsymbol{\Lambda}\, \boldsymbol{a} + \frac{\kappa}{\lambda^4} \boldsymbol{a} \cdot \boldsymbol{C}^2 \boldsymbol{a}\right), \quad \kappa = \boldsymbol{a} \cdot \boldsymbol{\Lambda}\, \boldsymbol{a}\,. \tag{43}$$

To be consistent with infinitesimal elasticity, we must have,

$$s(0) = s'(0) = 0\,, \quad s''(0) = 2\,. \tag{44}$$

6. Boundary Value Problems

In this Section, we consider two-dimensional boundary value problems, where it is reasonable to assume that the couple stress component $\boldsymbol{e}_r \cdot \boldsymbol{M}\, \boldsymbol{e}_r = 0$ (see for example, reference [30]); $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z\}$ is the polar basis for the current configuration. For illustration purposes, we only consider two boundary value problems (the pure bending of a slab, and the extension and torsion of a solid cylinder), where the deformations are prescribed; these boundary value problems could be important from the experimental and numerical point of view. In view of (12), the couple stress can be then easily obtained from the deformation and the anti-symmetric stress $\boldsymbol{T}_{(a)}$ can be readily obtained from the relation $(13)_2$.

For boundary value problems, where the deformations are not prescribed, numerical solutions for the proposed constitutive equations could be obtained via modifications of numerical procedures, such as those developed in references [1, 15, 17, 32]. It is beyond the scope of this paper to give such numerical solutions.

6.1 Pure Bending

Consider the problem of pure bending in plane strain, in which a rectangular slab of incompressible material is bent into a sector of a circular annulus defined by

$$r = r(x_1), \quad \theta = \theta(x_2), \quad z = x_3, \tag{45}$$

where (r, θ, z) is the cylindrical polar coordinate for the current configuration and (x_1, x_2, x_3) is the Cartesian referential coordinate with the basis $\{\boldsymbol{g_1},\ \boldsymbol{g_2},\ \boldsymbol{g_3} = \boldsymbol{e_z}\}$.

The deformation tensor has the form

$$\boldsymbol{F} = r'\boldsymbol{e_r} \otimes \boldsymbol{g_1} + r\theta'\boldsymbol{e_\theta} \otimes \boldsymbol{g_2} + \boldsymbol{e_z} \otimes \boldsymbol{g_3} \tag{46}$$

From the incompressibility condition $\det \boldsymbol{F} = 1$ and the boundary conditions $\theta(0) = 0$ and $r(A) = a$ we obtain

$$r^2 - a^2 = 2\chi x_1\,, \quad \theta = \frac{x_2}{\chi}\,, \quad \chi = \frac{b^2 - a^2}{2B} > 0\,, \tag{47}$$

where $A \leq x_1 \leq B$ and $r(B) = b$. Hence, in view of (1), (46) and (50), we have

$$\lambda_1 = \frac{\chi}{r}\,, \quad \lambda_2 = \frac{r}{\chi}\,, \quad \lambda_3 = 1 \tag{48}$$

and the spectral basis vectors are $\boldsymbol{u_i} = \boldsymbol{g_i}$, $\boldsymbol{v_1} = \boldsymbol{e}_r$, $\boldsymbol{v_2} = \boldsymbol{e_\theta}$ and $\boldsymbol{v_3} = \boldsymbol{e_z}$.

In this section we study the case $\boldsymbol{a} = \boldsymbol{g_2}$, and, we have

$$b = \frac{r}{\chi}\, \boldsymbol{e_\theta}, \quad a_1 = a_3 = 0\,, \quad a_2 = 1, \quad \kappa = 0\,. \tag{49}$$

In view of the relation $\frac{\mathrm{d}\,\boldsymbol{e_\theta}}{\mathrm{d}x_2} = -\frac{\boldsymbol{e}_r}{\chi}$, we obtain

$$\boldsymbol{G} = -\frac{r}{\chi^2}\, \boldsymbol{e_r} \otimes \boldsymbol{g_2} + r^{-1}\, \boldsymbol{e_\theta} \otimes \boldsymbol{g_1}\,, \qquad \boldsymbol{\Lambda} = \frac{1}{\chi}(-\boldsymbol{g_1} \otimes \boldsymbol{g_2} + \boldsymbol{g_2} \otimes \boldsymbol{g_1})\,. \tag{50}$$

6.1.1 Strain energy SE I

For this type of strain energy function we have $t_1 = t_2 = 0$, $\bar{J}_1 = -\frac{2}{\chi^2}$, $\bar{J}_2 = -\frac{1}{\chi^2}$ and hence, in view of the property of SE I given in Section 5, we obtain

$$W_\Lambda = b_1\, f_1(\bar{J}_1) + b_2\, f_2(\bar{J}_2)\,, \quad \frac{\partial W_\Lambda}{\partial \boldsymbol{\Lambda}} = 2b_1 f_1'(\bar{J}_1)\boldsymbol{\Lambda} + b_2 f'_2(\bar{J}_2)[\boldsymbol{\Lambda a} \otimes \boldsymbol{a} + \boldsymbol{a} \otimes \boldsymbol{\Lambda}^T \boldsymbol{a}] \tag{51}$$

and

$$\bar{\boldsymbol{M}} = -\frac{3}{2}\,(\bar{\gamma}_1\,\lambda_2\,\chi^{-1}\,r + \bar{\gamma}_1\,\lambda_2^2 + \bar{\gamma}_2\,)\,\boldsymbol{e_\theta} \otimes \boldsymbol{e}_z, \quad \bar{\gamma}_1 = \frac{2b_1 f_1'(\bar{J}_1) + b_2 f_2'(\bar{J}_2)}{r}\,, \quad \bar{\gamma}_2 = \frac{r^2}{\chi^2}\,\bar{\gamma}_1. \tag{52}$$

The above implies that the non-zero component of $\boldsymbol{M}$ is

$$m_{\theta z} = -\frac{3}{2}\,(\bar{\gamma}_1\,\lambda_2\,\chi^{-1}\,r + \bar{\gamma}_1\,\lambda_2^2 + \bar{\gamma}_2) \tag{53}$$

and it is non-negative if $\bar{\gamma}_1 \leq 0$. The symmetric stress take the form

$$\boldsymbol{T}_{(s)} = \boldsymbol{S} + \frac{4b_1 f_1'(\bar{J}_1) + 2b_2 f_2'(\bar{J}_2)}{\chi^2}\,[\,\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \,\boldsymbol{e_z} \otimes \boldsymbol{e_z}]\,, \tag{54}$$

where

$$\boldsymbol{S} = \mu_T\,\lambda_1\,r'_1(\lambda_1)\,\boldsymbol{e_r} \otimes \boldsymbol{e_r} + \lambda_2\,[\mu_T\,r'_1(\lambda_2) + 2(\mu_L - \mu_T)r'_2(\lambda_2) + \beta r_3\,(\lambda_2)r'_3\,(\lambda_2)]\,\boldsymbol{e_\theta} \otimes \boldsymbol{e_\theta} - p\boldsymbol{I}. \tag{55}$$

6.1.2 Strain energy SE II

For this type of strain energy function, we have, $\boldsymbol{d} = -\frac{1}{\lambda\chi}\,\boldsymbol{g_1}, \boldsymbol{k} = -\,\boldsymbol{g_1}\,, \rho = \frac{1}{\lambda\chi}, \boldsymbol{a}\cdot\boldsymbol{k} = 0$. The derivative of (42) then simplify to

$$\frac{\partial W_{(\Lambda)}}{\partial \boldsymbol{\Lambda}} = -\frac{cs'(\rho)}{\lambda}\,\boldsymbol{g_2} \otimes \boldsymbol{g_1}\,. \tag{56}$$

We then have,

$$\bar{I}_1 = r_1(\lambda_1) + r_1(\lambda_2)\,, \quad \bar{I}_2 = r_2(\lambda_2)\,, \quad \bar{I}_3 = {r_3}^2(\lambda_2)\,. \tag{57}$$

The symmetric part of the stress is simply

$$\boldsymbol{T}_{(s)} = \boldsymbol{S} - \frac{2cs\prime(\rho)}{\lambda\chi}\,\boldsymbol{e_\theta} \otimes \boldsymbol{e_\theta}\,. \tag{58}$$

Hence, we have

$$\sigma_{rr} = \mu_T\,\lambda_1\,r'_1(\lambda_1) - p\,, \quad \sigma_{zz} = -p \tag{59}$$

$$\sigma_{\theta\theta} = \lambda_2\left[-\frac{2c}{\lambda r}\,s'(\rho) + \mu_T\,\,r'_1(\lambda_2) + 2(\mu_L - \mu_T)r'_2(\lambda_2) + \beta r_3\,(\lambda_2)r'_3\,(\lambda_2)\right] - p\,. \tag{60}$$

We also have

$$\bar{\boldsymbol{M}} = m_{\theta z}\,\boldsymbol{e_\theta} \otimes \boldsymbol{e}_z\,, \quad m_{\theta z} = \frac{4}{3}\,\frac{crs'(\rho)}{\lambda\chi} \geq 0\,. \tag{61}$$

6.1.3 Remark

The remark here applies for both strain energy functions SE I and SE II. It is clear from (61), (52) and (11) that the cylindrical components of $\boldsymbol{M}$ and $\bar{\boldsymbol{M}}$ are

$$\bar{m}_{rr} = \bar{m}_{\theta\theta} = \bar{m}_{zz} = m_{r\theta} = m_{\theta r} = m_{zr} = m_{rz} = m_{z\theta} = 0. \tag{62}$$

From (11) and (62), we have $m_{rr} = m_{\theta\theta} = m_{zz}$. It is reasonable to assume that $m_{rr} = 0$ (see [30]) and hence, we have $m_{rr} = m_{\theta\theta} = m_{zz} = 0$.

$$\sigma_{r\theta} + \sigma_{\theta r} = \sigma_{rz} + \sigma_{zr} = \sigma_{z\theta} + \sigma_{\theta z} = 0\,. \tag{63}$$

Hence, in view of the equilibrium equation $(13)_2$ and (63), we have

$$\sigma_{r\theta} = \sigma_{\theta r} = \sigma_{rz} = \sigma_{zr} = \sigma_{z\theta} = \sigma_{\theta z} = 0 \tag{64}$$

and hence $\boldsymbol{T} = \boldsymbol{T}_{(s)}$. It is clear that σ_{rr} and $\sigma_{\theta\theta}$ depends only on r and hence we have the equilibrium equation

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0\,. \tag{65}$$

If we assume that $\sigma_{rr} = 0$ at $r = b$, we then have

$$\sigma_{rr} = -\int_r^b G(y)\,\mathrm{d}y, \tag{66}$$

where

$$rG(r) = \lambda_2\left[-\frac{2c_1}{r}\,s'_1(\rho) + \mu_T\,r'_1(\lambda_2) + 2(\mu_L - \mu_T)2(\lambda_2) + \beta r_3(\lambda_2)r'_3(\lambda\_2)\right] - \mu_T\,\lambda_1 r'_1(\lambda_1)\,. \tag{67}$$

The incompressible Lagrange multiplier takes the form

$$p = \mu_T \lambda_1 r'_1(\lambda_1) + \int_r^b G(y)\,\mathrm{d}y \tag{68}$$

and with the above expression for p we obtain the stress-strain relations for $\sigma_{\theta\theta}$ and σ_{zz}.

The bending moment $\mathcal{M}$, and the normal force $\mathcal{N}$, per unit length in the x_3 direction, and applied to a section of constant θ, are

$$\mathcal{M} = \int_a^b (r\sigma_{\theta\theta} + m_{\theta z})\mathrm{d}r\ , \quad \mathcal{N} = \int_a^b \sigma_{\theta\theta}\,\mathrm{d}r\,. \tag{69}$$

6.2 Extension and torsion of a solid cylinder

In this section we consider an incompressible thick-walled circular cylindrical annulus with the initial geometry

$$0 \le R \le A, \quad 0 \le \Theta \le 2\pi, \quad 0 \le Z \le L, \tag{70}$$

where R, Θ and Z are reference polar coordinates with the corresponding basis $B_R = \{\boldsymbol{E_R}, \boldsymbol{E_\Theta}, \boldsymbol{E_Z}\}$. The deformation is described by

$$r = \lambda_z^{-\frac{1}{2}} R, \quad \theta = \Theta + \lambda_z \tau Z, \quad z = \lambda_z Z, \tag{71}$$

where $\boldsymbol{\tau}$ is the amount of torsional twist per unit deformed length and λ_z is the axial stretch. In the above formulation, r, θ and z are cylindrical polar coordinates in the deformed configuration with the corresponding basis $B_C = \{\boldsymbol{e_r}, \boldsymbol{e_\theta}, \boldsymbol{e_z}\}$. Here, we have allowed $\boldsymbol{e_r} = \boldsymbol{E_R}, \boldsymbol{e_\theta} = \boldsymbol{E_\Theta}$ and $\boldsymbol{e_z} = \boldsymbol{E}_Z$. The deformation gradient is

$$\boldsymbol{F} = \lambda_z^{-\frac{1}{2}}\, \boldsymbol{e_r} \otimes \boldsymbol{E_R} + \lambda_z^{-\frac{1}{2}}\, \boldsymbol{e_\theta} \otimes \boldsymbol{E_\Theta} + \lambda_z\, \gamma\, \boldsymbol{e_\theta} \otimes \boldsymbol{E_Z} + \lambda_z\ \boldsymbol{e_z} \otimes \boldsymbol{E_Z}\ , \tag{72}$$

where $\boldsymbol{\gamma} = \boldsymbol{r\tau}$ and in this paper, we only consider $\boldsymbol{\lambda_z \ge 1}$. The Lagrangian principal directions are

$$\boldsymbol{u_1} = \boldsymbol{E}_R\ , \qquad \boldsymbol{u_2} = c\,\boldsymbol{E_\Theta} + s\,\boldsymbol{E_Z}\ , \qquad \boldsymbol{u_3} = -s\,\boldsymbol{E_\Theta} + c\,\boldsymbol{E_Z}\ , \tag{73}$$

where

$$c = \cos(\phi) = \frac{2}{\sqrt{2(\hat{\gamma}^2+4) + 2\hat{\gamma}\sqrt{\hat{\gamma}^2\ +4}}}\ , \quad s = \sin(\phi) = \frac{\hat{\gamma} + \sqrt{\hat{\gamma}^2\ +4}}{\sqrt{2(\hat{\gamma}^2+4) + 2\hat{\gamma}\sqrt{\hat{\gamma}^2\ +4}}}, \tag{74}$$

with

$$\frac{\pi}{4} \le \frac{\pi - \tan^{-1}\left(\frac{1}{\sqrt{{\lambda_z}^3 - 1}}\right)}{2} \le \phi < \frac{\pi}{2}, \quad \hat{\gamma} = \frac{{\lambda_z}^3\gamma^2 + {\lambda_z}^3 - 1}{{\lambda_z}^{\frac{3}{2}}\gamma} \ge 0\,, \quad c^2 - s^2 = -\hat{\gamma}cs\,. \tag{75}$$

In the case of pure torsion, $\lambda_z = 1$ and we have $\hat{\gamma} = \gamma$. The principal stretches for a combined extension and torsion deformation are

$$\lambda_1 = \frac{1}{\lambda_z^{\frac{1}{2}}}, \quad \lambda_2 = \sqrt{\frac{1}{\lambda_z} + \frac{s\gamma\sqrt{\lambda_z}}{c}}, \quad \lambda_3 = \sqrt{\frac{1}{\lambda_z} - \frac{c\gamma\sqrt{\lambda_z}}{s}}. \tag{76}$$

In this section we consider the case when $\boldsymbol{a} = \boldsymbol{E_z}$, hence $\boldsymbol{b} = \lambda_z\, \gamma\, \boldsymbol{e_\theta} + \lambda_z\, \boldsymbol{e_z}$, and using

$$\text{Grad}\,\boldsymbol{b} = \frac{\partial \boldsymbol{b}}{\partial R} \otimes \boldsymbol{E_R} + \frac{1}{R}\,\frac{\partial \boldsymbol{b}}{\partial \Theta} \otimes \boldsymbol{E_\Theta} + \frac{\partial \boldsymbol{b}}{\partial Z} \otimes \boldsymbol{E_Z}, \tag{77}$$

we obtain

$$\boldsymbol{\Lambda} = \tau(-\boldsymbol{E_R} \otimes \boldsymbol{E_\Theta} + \boldsymbol{E_\Theta} \otimes \boldsymbol{E_R}) + \lambda_z^{\frac{3}{2}}\tau\gamma(-\boldsymbol{E_R} \otimes \boldsymbol{E_Z} + \boldsymbol{E_Z} \otimes \boldsymbol{E_R})\,. \tag{78}$$

Consider the stress-like tensor

$$\boldsymbol{S} = 2\boldsymbol{F}\frac{\partial W_{(T)}}{\boldsymbol{\partial C}}\boldsymbol{F^T} - p\boldsymbol{I} \tag{79}$$

In view of $a \equiv [0,0,1]^T$, we have $a_1 = 0, a_2 = s$ and $a_3 = c$ and

$$\boldsymbol{S} = \sigma^s_{rr}\,\boldsymbol{e_r} \otimes \boldsymbol{e_r} + \sigma^s_{\theta\theta}\,\boldsymbol{e_\theta} \otimes \boldsymbol{e_\theta} + \sigma^s_{zz}\,\boldsymbol{e_z} \otimes \boldsymbol{e_z} + \sigma^s_{z\theta}\,(\boldsymbol{e_z} \otimes \boldsymbol{e_\theta} + \boldsymbol{e_\theta} \otimes \boldsymbol{e_z}) - p\,\boldsymbol{I}\ , \tag{80}$$

where

$$\begin{gathered}\sigma^s_{\theta\theta} = 2\left[\frac{l_2\,c^2 + l_3\,s^2 - 2l_4 cs}{\lambda_z} + 2\sqrt{\lambda_z}\ \gamma((l_2 - l_3)cs + l_4\,(c^2 - s^2)) + {\lambda_z}^2\,\gamma^2\,(l_2\,s^2 + l_3\,c^2 + 2l_4\,cs)\right], \\ \sigma^s_{z\theta} = 2\left[\sqrt{\lambda_z}((l_2 - l_3)cs + l_4\,(c^2 - s^2)) + {\lambda_z}^2\,\gamma\,(l_2\,s^2 + l_3\,c^2 + 2l_4\,cs)\right], \\ \sigma^s_{zz} = 2{\lambda_z}^2\,(l_2\,s^2 + l_3\,c^2 + 2l_4\,cs)\ , \quad \sigma^s_{rr} = \frac{2l_1}{\lambda_z}\ ,\end{gathered} \tag{81}$$

where

$$l_1 = \frac{1}{2\lambda_1}\,[\mu_T\, r'_1(\lambda_1)]\ , \quad l_2 = \frac{1}{2\lambda_2}\,[\mu_T\, r'_1(\lambda_2) + 2s^2\,\mu_1\, r'_2(\lambda_2) + \beta s^2[s^2 r_3(\lambda_2) + c^2 r_3(\lambda_3)] r'_3(\lambda_2)]\ ,$$

$$l_3 = \frac{1}{2\lambda_3}\,[\mu_T\, r'_1(\lambda_3) + 2c^2\,\mu_1 r'_2(\lambda_3) + c^2\beta[s^2 r_3(\lambda_2) + c^2 r_3(\lambda_3)] r'_3(\lambda_3)]\ . \tag{82}$$

$$l_4 = \frac{cs}{\lambda_2^2 - \lambda_3^2}\,(2\mu_1\,[r_2(\lambda_2) - r_2(\lambda_3)] + \beta[s^2 r_3(\lambda_2) + c^2\, r_3(\lambda_3)][r_3(\lambda_2) - r_3(\lambda_3)])\ ,$$

$$\mu_1 = \mu_L - \mu_T\,. \tag{83}$$

6.2.1 Strain energy SE I

For this type of strain energy function, we obtain,

$$t_1 = t_2 = 0 \qquad \bar{J}_1 = 2\tau^2 + 2\lambda_z^3\,\tau^2\,\gamma^2, \qquad \bar{J}_2 = \lambda_z^3\,\tau^2\,\gamma^2. \tag{84}$$

In view of the above and the property of strain energy SEI, we have,

$$W_{(\Lambda)} = b_1 f_1(\bar{J}_1) + b_2 f_2(\bar{J}_2), \tag{85}$$

$$\frac{\partial W_{(\Lambda)}}{\partial \boldsymbol{\Lambda}} = \bar{\gamma}_1 \boldsymbol{\Lambda}\ + \bar{\gamma}_2 \lambda_z^{\frac{3}{2}} \tau\gamma[-\boldsymbol{E_R} \otimes \boldsymbol{E_Z} + \boldsymbol{E_Z} \otimes \boldsymbol{E_R}]\,, \tag{86}$$

We then have

$$\bar{\boldsymbol{M}} = \bar{m}_{rr}\, \boldsymbol{e_r} \otimes \boldsymbol{e_r} + \bar{m}_{\theta\theta}\, \boldsymbol{e_\theta} \otimes \boldsymbol{e_\theta} + \bar{m}_{zz}\, \boldsymbol{e_z} \otimes \boldsymbol{e_z} + \bar{m}_{\theta z}\, \boldsymbol{e_\theta} \otimes \boldsymbol{e_z} + \bar{m}_{z\theta}\, \boldsymbol{e_z} \otimes \boldsymbol{e_\theta}\ , \tag{87}$$

where

$$\bar{m}_{rr} = \frac{2\lambda_z}{3}[\bar{\alpha}_3\gamma - \bar{\alpha}_1]\,, \quad \bar{m}_{\theta\theta} = \frac{2\lambda_z}{3}[\bar{\alpha}_2 - \gamma(\bar{\alpha}_3 - \bar{\alpha}_4)]\,, \quad \bar{m}_{zz} = \frac{2\lambda_z}{3}[-\gamma\bar{\alpha}_4 - (\bar{\alpha}_2 - \bar{\alpha}_1)]\,, \tag{88}$$

$$\bar{m}_{\theta z} = \frac{2\lambda_z\gamma}{3}[\bar{\alpha}_1 - 2\bar{\alpha}_2]\,, \quad \bar{m}_{z\theta} = \frac{2\lambda_z}{3}[2\bar{\alpha}_4 - \bar{\alpha}_3]\,, \tag{89}$$

$$\bar{\alpha}_1 = \frac{\beta_1}{\sqrt{\lambda_z}}, \quad \bar{\alpha}_2 = \frac{\beta_2}{\sqrt{\lambda_z}} + \beta_4\lambda_z\gamma, \quad \bar{\alpha}_3 = \frac{\beta_3}{\sqrt{\lambda_z}}, \quad \bar{\alpha}_4 = \beta_4\,\lambda_z, \tag{90}$$

$$\beta_1 = -\tau\left[\bar{\gamma}_1\left(\lambda_z^{-\frac{1}{2}} + \lambda_z^{\frac{5}{2}}\gamma^2\right) + \bar{\gamma}_2\,\lambda_z^{\frac{5}{2}}\gamma^2\right], \quad \beta_2 = \bar{\gamma}_1\,\tau\lambda_z^{-\frac{1}{2}}, \quad \beta_3 = -\lambda_z^{\frac{3}{2}}\beta_4, \quad \beta_4 = \tau\gamma\lambda_z\,(\bar{\gamma}_1 + \bar{\gamma}_2)\,. \tag{91}$$

Since, the deformation can be considered two-dimensional, it is quite physically reasonable to assume $m_{rr} = 0$ (see [30]) and from the equilibrium equation (13), we obtain the anti-symmetric stress

$$\boldsymbol{T}_{(a)} = \frac{\bar{m}_{\theta\theta} - \bar{m}_{rr}}{2r}[\,\boldsymbol{e_\theta} \otimes \boldsymbol{e_z} - \boldsymbol{e_z} \otimes \boldsymbol{e_\theta}]\,. \tag{92}$$

The symmetric stress is

$$\boldsymbol{T}_{(s)} = \boldsymbol{S}\ + 2\tau\big(\beta_2\sqrt{\lambda_z} + \beta_4\,\lambda_z^2\,\gamma\big)\,\boldsymbol{e_r} \otimes \boldsymbol{e_r} + 2\beta_1\sqrt{\lambda_z}\,\tau\boldsymbol{e_\theta} \otimes \boldsymbol{e_\theta} + \beta_3\sqrt{\lambda_z}\,\tau(\boldsymbol{e_\theta} \otimes \boldsymbol{e_z} + \boldsymbol{e_z} \otimes \boldsymbol{e_\theta})\,. \tag{93}$$

6.2.2 Strain energy SE II

$$\boldsymbol{d} = -\frac{\lambda_z^3\,\gamma\tau}{\lambda}\boldsymbol{E_R}, \qquad \boldsymbol{d}\cdot\boldsymbol{C}\,\boldsymbol{a}\ = \boldsymbol{a}\cdot\boldsymbol{\Lambda}\,\boldsymbol{a} = 0\,, \quad \lambda = \sqrt{\lambda_z^2\,(1+\gamma^2)}\ , \quad \rho = \frac{\lambda_z^3\,\gamma\tau}{\lambda}, \qquad \boldsymbol{k}\ = -\boldsymbol{E_R}. \tag{94}$$

We also have

$$\frac{\partial W_{(\Lambda)}}{\partial \boldsymbol{\Lambda}} = w_{zr}\boldsymbol{e_z} \otimes \boldsymbol{e_r}, \quad w_{zr} = -\frac{cs'(\rho)}{\lambda}. \tag{95}$$

$$\bar{\boldsymbol{M}} = \eta(\gamma\boldsymbol{e_\theta} \otimes \boldsymbol{e_\theta} - \gamma^2\boldsymbol{e_\theta} \otimes \boldsymbol{e_z} + \boldsymbol{e_z} \otimes \boldsymbol{e_\theta} - \gamma\boldsymbol{e_z} \otimes \boldsymbol{e_z})\,, \quad \eta = -\frac{4cs'(\rho)\lambda_z^{\frac{3}{2}}}{3\lambda}. \tag{96}$$

If we assume $\boldsymbol{e_r}\cdot\boldsymbol{M}\boldsymbol{e_r} = 0$, from the momentum balance equation $(13)_2$, we obtain the anti-symmetric stress

$$\boldsymbol{T}_{(a)} = \frac{1}{2r}\,\eta\gamma[\boldsymbol{e_\theta} \otimes \boldsymbol{e_z} - \boldsymbol{e_z} \otimes \boldsymbol{e_\theta}]\,. \tag{97}$$

Let

$$\boldsymbol{S}_{(b)} = 2\,\boldsymbol{F}\frac{\partial W_{(\Lambda)}}{\partial \boldsymbol{C}}\boldsymbol{F}^T + \boldsymbol{F}\frac{\partial W_{(\Lambda)}}{\partial \boldsymbol{\Lambda}}\boldsymbol{G}^T + \boldsymbol{G}\frac{\partial W_{(\Lambda)}}{\partial \boldsymbol{\Lambda}^T}\boldsymbol{F}^T\ , \tag{98}$$

we then have

$$\boldsymbol{S}_{(b)} = \sigma_{\theta\theta}^b\,\boldsymbol{e_\theta} \otimes \boldsymbol{e_\theta} + \sigma_{\theta z}^b\,(\,\boldsymbol{e_z} \otimes \boldsymbol{e_\theta} + \boldsymbol{e_\theta} \otimes \boldsymbol{e_z}) + \sigma_{zz}^b\,\boldsymbol{e_z} \otimes \boldsymbol{e_\theta}\ , \tag{99}$$

where

$$\sigma_{\theta\theta}^{b} = 2q\gamma^2\lambda_z^2 + 2w_{zr}\,\gamma\lambda_z^{\frac{3}{2}}\,\tau\,, \quad \sigma_{\theta z}^{b} = 2q\gamma\lambda_z^2 + w_{zr}\,\lambda_z^{\frac{3}{2}}\,\tau\,, \quad \sigma_{zz}^{b} = 2q\lambda_z^2, \quad q = \frac{cs'(\rho)\lambda_z^{\frac{3}{2}}\gamma\tau}{2\lambda^3}. \tag{100}$$

The symmetric stress

$$\boldsymbol{T}_{(s)} = \boldsymbol{S} + \boldsymbol{S}_{(b)}\,. \tag{101}$$

6.2.3 Remark

The remark here applies for both strain energy functions SE I and SE II. The normal force $\mathcal{N}$ per unit area and the torque $\mathcal{M}$ applied at the ends of the cylinder are as follows

$$\mathcal{N} = 2\pi\int_0^a \sigma_{zz}\,r\mathrm{d}r, \quad \mathcal{M} = 2\pi\int_0^a [\boldsymbol{e_z}\cdot\boldsymbol{Te_\theta}r + m_{zz}]\,r\,\mathrm{d}r\,, \tag{102}$$

where $a = \frac{A}{\sqrt{\lambda_z}}$, and

$$\sigma_{zz} = \boldsymbol{e_z}\cdot\boldsymbol{Te_z}, \quad m_{zz} = \boldsymbol{e_z}\cdot\boldsymbol{Me_z}. \tag{103}$$

To remove the hydrostatic pressure term in $(102)_1$, and, in view of

$$\boldsymbol{e_r}\cdot\boldsymbol{Te_\theta} = \boldsymbol{e_\theta}\cdot\boldsymbol{Te_r} = \boldsymbol{e_r}\cdot\boldsymbol{Te_z} = \boldsymbol{e_z}\cdot\boldsymbol{Te_r} = 0 \tag{104}$$

and the relation

$$\sigma_{rr} + \sigma_{\theta\theta} = \frac{1}{r}\frac{\mathrm{d}(r^2\sigma_{rr})}{\mathrm{d}r}\,, \quad \sigma_{rr} = \boldsymbol{e_r}\cdot\boldsymbol{Te_r}, \quad \sigma_{\theta\theta} = \boldsymbol{e_\theta}\cdot\boldsymbol{Te_\theta}. \tag{105}$$

we reformulate $(102)_1$ in the following form

$$\mathcal{N} = \pi\int_0^a (2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta})\,r\,\mathrm{d}r\,. \tag{106}$$

7. Conclusion

Modelling constitutive equations for fibre-reinforced elastic solids with bending stiffness, using classical invariants [29], have been done in the past [30]. However, most of the classical invariants do not have an immediate physical meaning and, hence, they are not attractive in seeking to design an experiment that requires to capture the mechanical behavior of the material in terms of a single invariant, while keeping the remaining invariants fixed, so that a specific functional form of the strain energy can be obtained. To overcome this unattractiveness, we use a set of spectral invariants, each with a clear physical meaning, to construct a specific form of constitutive equation; this indicates that spectral invariants have an experimental advantage over other types of invariants with no physical interpretation such as most of the standard trace based classical invariants [29]. The use of spectral invariants also permits us to obtain a simple but general expression for the stress and the couple-stress that are consistent with their infinitesimal counterparts. We note that, since trace based classical invariants (and most of their variants) can be constructed using the trace operator, they can be explicitly expressed in terms of spectral invariants (see for example reference [27]). Hence if a constitutive equation is initially written in terms of the classical invariants, the stress-strain equation can be easily expressed in terms of both classical and spectral invariants. However, in general, a spectral invariant cannot be explicitly expressed in terms of classical invariants; this indicates the generality of the spectral-invariant formulation. In addition to this, independent invariants in an irreducible basis, that is essential in designing a rational program of experiments [8, 19], can be easily obtained via spectral invariants, as shown by Shariff [27]. We also showed that the use of $(\partial\boldsymbol{f}/\partial\boldsymbol{x})\boldsymbol{a}$ to construct our fibre curvature constitutive equation is physically more meaningful than Spencer and Soldatos [30] fibre curvature constitutive model that depends on $(\partial\boldsymbol{b}/\partial\boldsymbol{x})\boldsymbol{a}$.

In future, we will consider two-preferred-direction elastic materials, which is interesting from the point of view of applications in biomechanics, and we also intend to extend our model to a more general model that deal with an initial distribution of the orientation of the fibers [3, 12]. The elastic consistent tangent modulus tensor formula for the proposed models, required for finite element software, can be easily derived via the work of Shariff [24, 26] and numerical results, for the proposed spectral model, could be obtained via modifications of some numerical methods found in the literature.

Author Contributions

Conceptualization, methodology, formal analysis, investigation, writing-original draft preparation and writing-review and editing, M.H.B.M. Shariff, J. Merodio and R. Bustamante. All authors have read and agreed to the published version of the manuscript.

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Appendix A

In the literature, a constitutive equation of an anisotropic solid generally contains invariants of tensors that depend explicitly on the right Cauchy-Green tensor $\boldsymbol{C}$ (or left Cauchy-Green tensor $\boldsymbol{B}$). For example, the strain energy of a fibre-reinforced solid with the preferred direction $\boldsymbol{a}$ in the reference configuration is a function of the independent classical invariants

$$I_1 = \mathrm{tr}\boldsymbol{C}, \quad I_2 = \frac{1}{2}\left((\mathrm{tr}\,\boldsymbol{C})^2 - \mathrm{tr}\,\boldsymbol{C}^2\right), \quad I_3 = \det \boldsymbol{C}, \quad I_4 = \boldsymbol{a}\cdot\boldsymbol{C}\boldsymbol{a}, \quad I_5 = \boldsymbol{a}\cdot\boldsymbol{C}^2\boldsymbol{a}. \tag{A1}$$

Any attempt for a modeler to construct a good constitutive equation using the invariants in (A1) is hampered due to the restriction that the invariants in (A1) depend explicitly on $\boldsymbol{C}$ and the strain energy function generally depends explicitly on these invariants: This does not allow the modeler to use a more general invariants (that cannot be explicitly expressed in terms of $\boldsymbol{C}$) which could facilitate the construction of a good constitutive equation. In addition to this, except for I_3 and I_4, the invariants in (A1) do not have a direct physical interpretation; hence they are not experimentally attractive. For example, there are fairly good functions for the strain energy $W_{(e)}$ of an incompressible isotropic elastic solids that used the invariants I_1 and I_2, however, in 1967, Valanis and Landel [31] suggested that an efficient function of the strain energy had not been found before because of difficulties inherent in its dependence on classical strain invariants; functions $\partial W_{(e)}/\partial I_1$ and $\partial W_{(e)}/\partial I_2$ might be very complex and it is not easy to design experiments in which I_1 and I_2 are not interrelated. They hence proposed the well-known Valanis and Landel strain energy function

$$W_{(e)} = \sum_{i=1}^{3} \bar{r}(\lambda_i)\,, \tag{A2}$$

where λ_i is the principal stretch. Ogden [16] and Shariff [18] have used (A2) to construct specific forms for $W_{(e)}$ that manage to successfully model nonlinear isotropic elasticity. One of the main reason that the form (A2) is attractive is that the $W_{(e)}$ depends explicitly on the spectral invariants (principal stretches) which have an immediate physical interpretation and in view of this an appropriate experiment was constructed [9].

As for the case of a transversely isotropic solid, since the invariants in (A1), except for I_3 and I_4, do not have a direct physical interpretation they are not suitable for a rigorous experiment that requires to vary one invariant while keeping the remaining invariants fixed (we called this R-experiment) [8]. However, using the spectral invariants, developed for transversely isotropic solids, Shariff [19] has shown that it is possible to construct an R-experiment. Hence, it is obvious that for an anisotropic model that requires a large amount of classical invariants, for example, 33 invariants in Spencer and Soldatos [30] model and most of them have no clear physical meaning, it is far more difficult to construct an R-experiment using classical invariants than using spectral invariants.

Appendix B

Consider a compressible triaxial deformation defined by (in Cartesian basis)

$$\boldsymbol{F} \equiv \begin{pmatrix} g(X) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B1}$$

where (X,Y,Z) is a Cartesian point in the reference configuration and $g(X) > 0$. Let $\boldsymbol{a} \equiv [1,0,0]^T$ and in this case

$$\boldsymbol{b} \equiv [g(X),0,0]^T \tag{B2}$$

and the fibre is stretched but not bent. It is clear that for **d** defined in [30]

$$\boldsymbol{d} = \boldsymbol{F}^T \frac{\partial \boldsymbol{b}}{\partial x}\, \boldsymbol{a} \equiv [g'(X)g(X),0,0]^T \neq \boldsymbol{0} \tag{B3}$$

if $g'(X) \neq 0$. However, if we used the definition $(7)_2$, we obtain

$$\boldsymbol{d} = \boldsymbol{F}^T \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\, \boldsymbol{a} = \boldsymbol{0}\ , \tag{B4}$$

which is the required value when bending is absent. It is worth noting that under a rigid body motion, where $\boldsymbol{F} = \boldsymbol{R}$ and $\boldsymbol{R}$ is independent of $\boldsymbol{x}$ both $\boldsymbol{d}$ defined in [30] and $(7)_2$ have the value $= \boldsymbol{0}$.

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