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Research Paper

Bernoulli-Euler Beam Unsteady Bending Model with Consideration of Heat and Mass Transfer

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Abstract. The article describes the problem of unsteady vibrations of a Bernoulli-Euler beam taking into account the relaxation of temperature and diffusion processes. The initial mathematical model includes a system of equations for unsteady bending vibrations of the beam with consideration of heat and mass transfer. This model is obtained from the general model of thermomechanodiffusion for continuum using the D'Alembert's variational principle. The solution of the problem is obtained in the integral form. The kernels of the integral representations are Green's functions. For finding of Green's functions the expansion into trigonometric Fourier series and Laplace transform in time are used. The calculation example is investigated for a freely supported three-component beam made of zinc, copper and aluminum alloy under the action of unsteady bending moments, including the interaction of mechanical, temperature and diffusion fields.

Keywords: Thermoelastic diffusion, Laplace transform, Green's function, Bernoulli-Euler beam.

1. Introduction

An interaction of fields of different physical nature occur in the form of mechanodiffusion, thermomechanodiffusion, etc., and is frequently observed in the technical systems. These effects can cause negative impact on the stress-strain state of structures and their individual elements. It is important to note that, the vast majority of thin-walled structures (aviation and rocket and space equipment, shipbuilding etc.) are shell-and-rod systems, consisting of elements as thin plates and shells, interconnecting or reinforcing with rods. Modern high-intensity technological processes (such as explosive bonding, SOI producing, electrospark and laser material treatment, film producing etc.) are connected not only with large thermal flows and heat rates. When performing the most part of such processes, near-surface material or its specific component indention or transfer take place with high speed as well as associated variation of stress-strain state. For this reason, it is a very important task to build mathematical models capable of sufficiently describing these phenomena in various mechanical systems both scientifically and practical points of view.

Various models and methods for solving problems of thermomechanodiffusion (with possible consideration of other physical fields) in recent decades have been considered in the studies of international scientific groups [1-6]. It indicates the relevance of research in this area. The influence of diffusion and thermal processes on the bearing capacity of rods, plates, and shells is demonstrated in studies [7-13]. The engineering application of the beam with consideration of heat and mass transfer maybe include the works introduced in [14, 15].

The studies [7-9] investigate steady-state thermomechanodiffusion processes, which is useful for calculating steady-state modes of technical systems operation. It is necessary to use unsteady models for the analysis of short-term pulsed effects, which should also describe the relaxation thermodiffusion effects [11, 18-20]. Relaxation determines the finite speed of thermal/diffusion flows and is accounted by the additional inertial terms in the heat and mass transfer equations. All this aspects are represented in the problem formulation.

It is well known that inertia was first put into heat transfer equations by Maxwell, and in 1948 Cattaneo proposed a variant of the Fourier law with a relaxation term. The suggested idea was extended to models describing diffusion processes. At present, there are various generalizations of Fourier and Fick laws in the form of Lord-Schulman, Green-Linsdi, Green-Naghdi models, etc., which can be found in [21-28].

From the above list, it should be noted a number of publications that consider unsteady problems of thermomechanodiffusion for continuum [1, 4, 8, 10, 13]. Also of great interest are studies for unsteady thermomechanodiffusion processes considering electromagnetic fields [16, 17]. Both numerical finite difference methods [1] and finite element methods [14, 16] are used to solve the problems as well as analytical methods based on the Laplace transform [13, 17].

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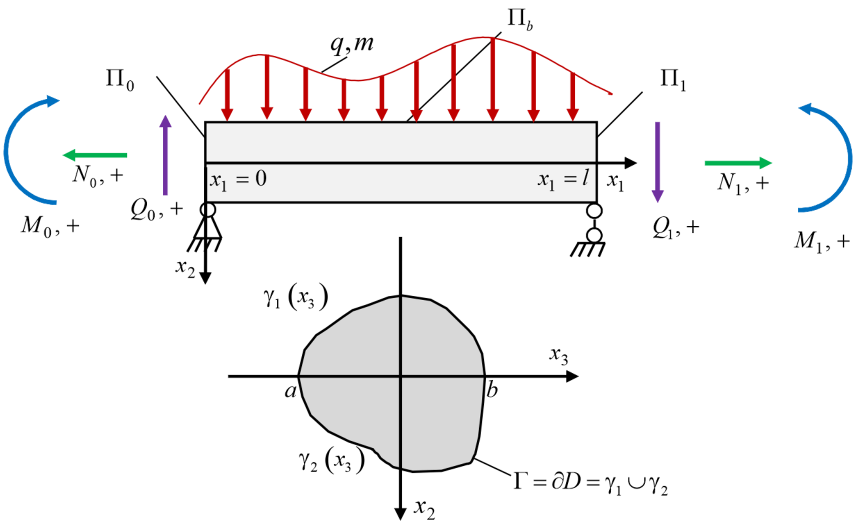

Fig. 1. Illustration to the problem formulation

To invert the Laplace transform scientists often use quadrature formulas based on orthogonal Legendre polynomials [13] or the Durbin method [17]. These methods have proven themselves in calculating the originals in a certain class of functions. However, these algorithms are not applicable to find the Green's functions because the Green's functions belong to the class of generalized functions - this complicates the use of numerical integration methods.

In addition, all above-mentioned studies consider models for binary medium, which is caused by known mathematical difficulties arising at solution of related multicomponent thermomechanodiffusion problems. There are no formulations of problems on unsteady thermoelastic diffusion vibrations of multicomponent beams and plates, as well as methods for their solution in the currently known publications.

2. Problem Formulation

The study is concerned to the problem of unsteady vibrations of a Bernoulli-Euler beam with consideration of relaxation of temperature and diffusion processes. The beam in the general case is under the action of longitudinal and transverse forces, bending moments, as well as und effects of temperature and diffusion fields. The scheme of the given forces is shown in the Fig. 1.

For the mathematical formulation of the problem we use a model of the thermoelastic diffusion processes in continuum in a rectangular Cartesian coordinate system. In the case of a homogeneous N+1-component medium the formulation has the form [28, 30]:

$$\ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} + F_i, \quad \dot{\theta} + B_{ij}\dot{\varepsilon}_{ij} + \sum_{q=1}^{N} \upsilon^{(q)}\eta^{(q)} + \frac{\partial q_i}{\partial x_i} = Q^{(J)}, \quad \dot{\eta}^{(q)} = -\frac{\partial J_i^{(q)}}{\partial x_i} + Y^{(q)} \quad \left(q = \overline{1,N}\right), \quad \eta^{(N+1)} = -\sum_{q=1}^{N}\eta^{(q)}. \tag{1}$$

where σ_{ij}, q_i and $J_i^{(q)}$ $\left(q = \overline{1,N}\right)$ - components of the stress tensor and thermal diffusion flux vectors, which are defined as follows

$$\sigma_{ij} = C_{ijkl}\frac{\partial u_k}{\partial x_l} - b_{ij}\theta - \sum_{q=1}^{N}\alpha_{ij}^{(q)}\eta^{(q)}, \quad \sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k q_i}{\partial \tau^k} = -\kappa_{ij}\frac{\partial \theta}{\partial x_j}, \quad \sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k J_i^{(q)}}{\partial \tau^k} = -\sum_{t=1}^{N} D_{ij}^{(q)} g^{(qt)}\frac{\partial \eta^{(t)}}{\partial x_j} + \Lambda_{ijkl}^{(q)}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + M_{ij}^{(q)}\frac{\partial \theta}{\partial x_j}. \tag{2}$$

The model used here takes into account a finite speed of propagation of thermal and diffusion perturbations. This is caused by the relaxation of thermal and diffusion fluxes. The upper summation limits K and M in the equations (2) are determined based on a given calculation accuracy. However, as the calculations in [28] show, that it is almost always possible to limit the value $K, M = 2$, and in most cases acceptable accuracy is provided even at $K, M = 1$ (Kataneo model). The case $K, M = 0$ corresponds to the classical model of heat and mass transfer with infinite speed of thermal and diffusion perturbations.

All quantities in (1) and (2) are dimensionless. The following notations are taken for them

$$x_i = \frac{x_i^*}{l}, \; u_i = \frac{u_i^*}{l}, \; \theta = \frac{T - T_0}{T_0}, \; \tau = \frac{Ct}{l}, \; C_{ijkl} = \frac{C_{ijkl}^*}{C_{1111}}, \; C^2 = \frac{C_{1111}^*}{\rho}, \; \alpha_{ij}^{(q)} = \frac{\alpha_{ij}^{*(q)}}{C_{1111}}, \; b_{ij} = \frac{b_{ij}^* T_0}{C_{1111}}, \; \kappa_{ij} = \frac{\kappa_{ij}^*}{\rho c_0 CL}, \; B_{ij} = \frac{b_{ij}^*}{\rho c_0}, \; \tau_0 = \frac{C\tau_t}{L},$$
$$\tau_q = \frac{C\tau^{(q)}}{L}, \; D_{ij}^{(q)} = \frac{D_{ij}^{*(q)}}{Cl}, \; \upsilon^{(q)} = \rho R T_0 \frac{\ln\left[n_0^{(q)}\gamma^{(q)}\right]}{m^{(q)}}, \; M_{ij}^{(q)} = \frac{n_0^{(q)} D_{ij}^{*(q)}}{T_0}\ln\left(n_0^{(q)}\gamma^{(q)}\right), \; \Lambda_{ijkl}^{(q)} = \frac{m^{(q)} D_{ij}^{*(q)}\alpha_{kl}^{*(q)} n_0^{(q)}}{\rho R T_0 Cl}, \; F_i = \frac{\rho l F_i^*}{C_{1111}}, \; Y^{(q)} = \frac{l Y^{*(q)}}{C}, \; Q^{(J)} = \frac{l Q^{*(J)}}{C T_0 c_0}. \tag{3}$$

where t is time; x_i^* are rectangular Cartesian coordinates; u_i^* are displacement vector components; l is beam length; $\eta_q = n^{(q)} - n_0^{(q)}$ is increment of concentration of q component substance in $N+1$ -component continuum; $n^{(q)}$ and $n_0^{(q)}$ - current and initial concentrations of qth substance; T is current temperature of medium, T_0 is initial temperature of medium; C_{ijkl}^* are components of the elastic constant tensor; ρ is density; κ_{ij}^* are components of the thermal conductivity tensor; c_0 is the specific heat capacity; b_{ij}^* are temperature coefficients characterizing deformation due to heating; $\alpha_{ij}^{*(q)}$ are coefficients characterizing volume expansion due to diffusion; $D_{ij}^{*(q)}$ are diffusion coefficients; $\upsilon^{(q)}$ are coefficients characterizing heating of medium due to diffusion; R is the universal gas constant; $m^{(q)}$ is the molar mass of qth substance; $\tau^{(q)}$ is the relaxation time of the diffusion fluxes; τ_t is the relaxation time of the diffusion fluxes; F_i^*, $Y^{*(q)}$ and $Q^{*(J)}$ are external mass perturbations.

The initial conditions are assumed to be zero. In this way, we assume that initial state of the medium is an unperturbed state:

$$u_i\big|_{\tau=0}=0,\quad \frac{\partial u_i}{\partial \tau}\bigg|_{\tau=0}=0,\quad \frac{\partial^m\theta}{\partial\tau^m}\bigg|_{\tau=0}=0,\quad \frac{\partial^k\eta^{(q)}}{\partial\tau^k}\bigg|_{\tau=0}=\eta_0^{(q)}\quad \left(m=\overline{0,M-1},\quad k=\overline{0,K-1},\quad q=\overline{1,N}\right). \tag{4}$$

Boundary conditions (G is the area filled by the beam material; n_i are the components of the unit vector of the external normal to the surface $\Pi=\partial G$):

- mechanical section ($\partial G=\Pi_u\cup\Pi_\sigma$, $\Pi_u\cap\Pi_\sigma=\varnothing$)

$$u_i\big|_{\Pi_u}=U_i,\quad \sigma_{ij}n_j\big|_{\Pi_\sigma}=P_i\quad (\tau>0), \tag{5}$$

- temperature section ($\partial G=\Pi_\theta\cup\Pi_q$, $\Pi_\theta\cap\Pi_q=\varnothing$)

$$\theta\big|_{\Pi_\theta}=\Theta,\quad \sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k q_i}{\partial\tau^k}\bigg|_{\Pi_q}=Q_i\quad (\tau>0), \tag{6}$$

- diffusion section ($\partial G=\Pi_\eta\cup\Pi_J$, $\Pi_\eta\cap\Pi_J=\varnothing$)

$$\eta^{(q)}\big|_{\Pi_\eta}=N^{(q)},\quad \sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k J_i^{(q)}}{\partial\tau^k}\bigg|_{\Pi_J}=I_i^{(q)}\quad (\tau>0), \tag{7}$$

The values in the right-hand sides of the boundary conditions (5)-(7) are surface kinematic U_i, Θ, $N^{(q)}$ and dynamic P_i, Q_i, $I_i^{(q)}$ perturbations.

3. Bernoulli-Euler Beam Unsteady Vibration Model

According to the D'Alembert's variational principle, the relations (1)-(7) are represented as the following variational equation:

$$\begin{gathered}\int_G\left(\frac{\partial^2 u_i}{\partial\tau^2}-\frac{\partial\sigma_{ij}}{\partial x_j}-F_i\right)\delta u_i dG+\sum_{q=1}^{N}\int_G\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k}{\partial\tau^k}\left(\frac{\partial\eta^{(q)}}{\partial\tau}+\frac{\partial J_i^{(q)}}{\partial x_i}-Y^{(q)}\right)\delta\eta^{(q)}dG+\\+\int_G\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k}{\partial\tau^k}\left(\frac{\partial\theta}{\partial\tau}+B_{ij}\frac{\partial\varepsilon_{ij}}{\partial\tau}+\frac{\partial q_i}{\partial x_i}+\sum_{q=1}^{N}\upsilon^{(q)}\frac{\partial\eta^{(q)}}{\partial\tau}-Q^{(J)}\right)\delta\theta dG+\iint_{\Pi_\sigma}\left(\sigma_{ij}n_j-P_i\right)\delta u_i dS+\\+\sum_{q=1}^{N}\iint_{\Pi_J}\left(\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k J_i^{(q)}}{\partial\tau^k}-I_i^{(q)}\right)n_i\delta\eta^{(q)}dS+\iint_{\Pi_q}\left(\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k q_i}{\partial\tau^k}-Q_i\right)n_i\delta\theta dS=0.\end{gathered} \tag{8}$$

Let's assume that:

1. The solution area of the problem is a cylinder $G=D\times[0,1]$, where D is the area filled by the beam's cross section. The section boundary $\Gamma=\partial D=\gamma_1(x_3)\cup\gamma_2(x_3)$ (see Fig. 1)
2. The axis Ox_3 is the central axis of the section. In this case

$$\iint_D x_2 dx_2 dx_3=0. \tag{9}$$

3. Beam surface $\Pi=\Pi_0\cup\Pi_1\cup\Pi_b$, where Π_0 is the end surface corresponding to $x_1=0$, Π_1 is the end surface corresponding to $x_1=1$, Π_b is the side surface that is free of mechanical loads:

$$\sigma_{ij}n_j\big|_{\Pi_b}=0. \tag{10}$$

Heat and mass transfer through the side surface is also absent

$$\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k J_i^{(q)}}{\partial\tau^k}\bigg|_{\Pi_b}=0,\quad \sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k q_i}{\partial\tau^k}\bigg|_{\Pi_b}=0. \tag{11}$$

4. The material of the beam is orthotropic. We will use Voigt notation to write the elastic physical constants. In this case

$$C_{\alpha\beta}=C_{\alpha\alpha\beta\beta},\quad C_{66}=C_{1212}. \tag{12}$$

5. We consider the bending of the beam in the plane x_1Ox_2. Then $u_k=u_k(x_1,x_2,\tau)$, $k=1,2$, $u_3=0$, $\varepsilon_{i3}=0$. Mass transfer and heat transfer are also performed in the plane x_1Ox_2, i.e. $\eta^{(q)}=\eta^{(q)}(x_1,x_2,\tau)$ and $\theta=\theta(x_1,x_2,\tau)$.
6. Transverse deflections are considered to be small. The sections perpendicular to the beam axis before the deformation remain flat even after the deformation (Euler–Bernoulli bending theory). Then the linearization of the quantities sought by the variable x_2 will have the form (the approximate equality is replaced by the exact one)

$$\begin{gathered}u_1(x_1,x_2,\tau)=u(x_1,\tau)-x_2\chi(x_1,\tau),\ u_2(x_1,x_2,\tau)=v(x_1,\tau)+x_2\psi(x_1,\tau),\\ \theta(x_1,x_2,\tau)=\varphi(x_1,\tau)+x_2\omega(x_1,\tau),\quad \eta^{(q)}(x_1,x_2,\tau)=N_q(x_1,\tau)+x_2H_q(x_1,\tau).\end{gathered} \tag{13}$$

7. We also assume that the cross sections remain normal to the beam bent axis after deformation. Taking into account (10) we will assume that there are no deformations along the x_2 axis. Then [18 – 20, 31]

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = C_{66}\left(-\chi + v'\right) = 0 \quad \Rightarrow \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \psi = 0 \Rightarrow \psi = 0.$$

Hence

$$\chi(x_1,\tau) = v'(x_1,\tau), \quad \psi = 0.$$

and the first two equalities in (13) will be written as follows:

$$u_1 = u - x_2 v', \quad u_2 = v, \quad u_3 = 0, \quad u = u(x_1,\tau), \quad v = v(x_1,\tau). \tag{14}$$

Here the prime means the derivative on the variable x_1.

The components of the stress tensor and the vectors of diffusion and heat fluxes based on (2) and considering (14) will be determined as follows

$$\begin{aligned}
&\sigma_{11} = C_{11}\frac{\partial u_1}{\partial x_1} + C_{12}\frac{\partial u_2}{\partial x_2} - b_1\theta - \sum_{q=1}^{N}\alpha_1^{(q)}\eta^{(q)} = (u' - x_2 v'') - b_1(\varphi + x_2\omega) - \sum_{q=1}^{N}\alpha_1^{(q)}(N_q + x_2 H_q),\\
&\sigma_{22} = C_{12}\frac{\partial u_1}{\partial x_1} + C_{22}\frac{\partial u_2}{\partial x_2} - b_2\theta - \sum_{q=1}^{N}\alpha_2^{(q)}\eta^{(q)} = C_{12}(u' - x_2 v'') - b_2(\varphi + x_2\omega) - \sum_{q=1}^{N}\alpha_2^{(q)}(N_q + x_2 H_q),\\
&\sigma_{12} = C_{66}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = C_{66}(-v' + v') = 0,\\
&\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k q_1}{\partial\tau^k} = -\kappa_1\frac{\partial\theta}{\partial x_1} = -\kappa_1(\varphi' + x_2\omega'), \qquad \sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k q_2}{\partial\tau^k} = -\kappa_2\frac{\partial\theta}{\partial x_2} = -\kappa_2\omega,\\
&\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k J_1^{(q)}}{\partial\tau^k} = -D_1^{(q)}\frac{\partial\eta^{(q)}}{\partial x_1} + \Lambda_{11}^{(q)}\frac{\partial^2 u_1}{\partial x_1^2} + \Lambda_{12}^{(q)}\frac{\partial^2 u_2}{\partial x_1\partial x_2} + M_1^{(q)}\frac{\partial\theta}{\partial x_1} = -D_1^{(q)}(N_q' + x_2 H_q') + \Lambda_{11}^{(q)}(u'' - x_2 v''') + M_1^{(q)}(\varphi' + x_2\omega'),\\
&\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k J_2^{(q)}}{\partial\tau^k} = -D_2^{(q)}\frac{\partial\eta^{(q)}}{\partial x_2} + \Lambda_{22}^{(q)}\frac{\partial^2 u_2}{\partial x_2^2} + \Lambda_{21}^{(q)}\frac{\partial^2 u_1}{\partial x_1\partial x_2} + M_2^{(q)}\frac{\partial\theta}{\partial x_2} = -D_2^{(q)}H_q - \Lambda_{21}^{(q)}v'' + M_2^{(q)}\omega,
\end{aligned} \tag{15}$$

Substituting the equalities (14), (15) into (8) given (9) we obtain ($dS = -dx_2dx_3$ when $x_1 = 0$ and $dS = dx_2dx_3$ when $x_1 = 1$, the dot above the symbol denotes the partial time derivative)

$$\begin{aligned}
&\int_0^1\left[F\left(\ddot{u} - u'' + b_1\varphi' + \sum_{q=1}^{N}\alpha_1^{(q)}N_q'\right) - n\right]\delta u dx_1 + \int_0^1\left[J_3\left(\ddot{v}' - v''' - b_1\omega' - \sum_{q=1}^{N}\alpha_1^{(q)}H_q'\right) + m\right]\delta v' dx_1 + \int_0^1(F\ddot{v} - q)\delta v dx_1 + \\
&+\int_0^1\left\{F\left[\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k}{\partial\tau^k}\left(\dot{\varphi} + B_1\dot{u}' + \sum_{q=1}^{N}\upsilon^{(q)}\dot{N}_q\right) - \kappa_1\varphi''\right] - q^{(J)}\right\}\delta\varphi dx_1 + \int_0^1\left\{J_3\left[\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k}{\partial\tau^k}\left(\dot{\omega} - B_1\dot{v}'' + \sum_{q=1}^{N}\upsilon^{(q)}\dot{H}_q\right) - \kappa_1\omega''\right] - q^{(Q)}\right\}\delta\omega dx_1 + \\
&+\sum_{q=1}^{N}\int_0^1\left[F\left(\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k\dot{N}_q}{\partial\tau^k} - D_1^{(q)}N_q'' + \Lambda_{11}^{(q)}u''' + M_1^{(q)}\varphi''\right) - y^{(q)}\right]\delta N_q dx_1 + \sum_{q=1}^{N}\int_0^1\left[J_3\left(\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k\dot{H}_q}{\partial\tau^k} - D_1^{(q)}H_q'' - \Lambda_{11}^{(q)}v^{IV} + M_1^{(q)}\omega''\right) - z^{(q)}\right]\delta H_q dx_1 + \\
&+\iint_D\left(u' - b_1\varphi - \sum_{q=1}^{N}\alpha_1^{(q)}N_q - P_1\right)\Bigg|_{x_1=0}^{x_1=1}\delta u dx_2dx_3 + \iint_D\left[x_2\left(v'' + b_1\omega + \sum_{q=1}^{N}\alpha_1^{(q)}H_q\right) + P_1\right]\Bigg|_{x_1=0}^{x_1=1} x_2\delta v' dx_2dx_3 - \iint_D P_2\delta v\big|_{x_1=0}^{x_1=1}dx_2dx_3 + \\
&+\sum_{q=1}^{N}\iint_D\left(-\sum_{r=1}^{N}D_1^{(qr)}N_r' + \Lambda_{11}^{(q)}u'' + M_1^{(q)}\varphi' - I_1^{(q)}\right)\Bigg|_{x_1=0}^{x_1=1}\delta N_q dx_2dx_3 - \sum_{q=1}^{N}\iint_D\left[\left(\sum_{r=1}^{N}D_1^{(qr)}H_r' + \Lambda_{11}^{(q)}v''' - M_1^{(q)}\omega'\right)\Bigg|_{x_1=0}^{x_1=1}x_2^2 + I_1^{(q)}x_2\right]\delta H_q dx_2dx_3 - \\
&-\iint_D\left(\kappa_1\omega' x_2^2 + Q_1 x_2\right)\big|_{x_1=0}^{x_1=1}n_1\delta\omega dx_2dx_3 - \iint_D\left(\kappa_1\varphi' + Q_1\right)\big|_{x_1=0}^{x_1=1}\delta\varphi dx_2dx_3.
\end{aligned} \tag{16}$$

where F is the cross-sectional area, J_3 is the moment of inertia of the beam relative to the axis Ox_3, n is the distributed longitudinal load, m is the distributed moment, q is the distributed shear load (Fig. 1), $q^{(J)}$, $y^{(q)}$, $q^{(Q)}$, $z^{(q)}$ are the linear densities of bulk sources of heat transfer and mass transfer. These values are defined as follows:

$$\begin{aligned}
&F = \iint_D dx_2dx_3, \; J_3 = \iint_D x_2^2 dx_2dx_3, \; n = \iint_D F_1 dx_2dx_3, \; m = \iint_D F_1 x_2 dx_2dx_3, \; q = \iint_D F_2 dx_2dx_3,\\
&q^{(J)} = \iint_D\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k Q^{(J)}}{\partial\tau^k}dx_2dx_3, \; q^{(Q)} = \iint_D\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k Q^{(J)}}{\partial\tau^k}x_2dx_2dx_3, \; y^{(q)} = \iint_D\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k Y^{(q)}}{\partial\tau^k}dx_2dx_3, \; z^{(q)} = \iint_D\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k Y^{(q)}}{\partial\tau^k}x_2dx_2dx_3.
\end{aligned}$$

Then, by partially integrating the third term in (16), we obtain

$$\begin{aligned}
&\int_0^1\left[F\left(\ddot{u} - u'' + b_1\varphi' + \sum_{q=1}^{N}\alpha_1^{(q)}N_q'\right) - n\right]\delta u dx_1 - \int_0^1\left[J_3\left(\ddot{v}'' - v^{IV} - b_1\omega'' - \sum_{q=1}^{N}\alpha_1^{(q)}H_q''\right) + m'\right]\delta v dx_1 + \int_0^1(F\ddot{v} - q)\delta v dx_1 + \\
&+\int_0^1\left\{F\left[\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k}{\partial\tau^k}\left(\dot{\varphi} + B_1\dot{u}' + \sum_{q=1}^{N}\upsilon^{(q)}\dot{N}_q\right) - \kappa_1\varphi''\right] - q^{(J)}\right\}\delta\varphi dx_1 + +\int_0^1\left\{J_3\left[\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k}{\partial\tau^k}\left(\dot{\omega} - B_1\dot{v}'' + \sum_{q=1}^{N}\upsilon^{(q)}\dot{H}_q\right) - \kappa_1\omega''\right] - q^{(Q)}\right\}\delta\omega dx_1\\
&+\sum_{q=1}^{N}\int_0^1\left[\left(\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k\dot{N}_q}{\partial\tau^k} - \sum_{r=1}^{N}D_1^{(qr)}N_r'' + \Lambda_{11}^{(q)}u''' + M_1^{(q)}\varphi''\right) - y^{(q)}\right]\delta N_q dx_1 + +\sum_{q=1}^{N}\int_0^1\left[J_3\left(\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k\dot{H}_q}{\partial\tau^k} - \sum_{r=1}^{N}D_1^{(qr)}H_r'' - \Lambda_{11}^{(q)}v^{IV} + M_1^{(q)}\omega''\right) - z^{(q)}\right]\delta H_q dx_1 + \\
&+\left[J_3\left(\ddot{v}' - v''' - b_1\omega' - \sum_{q=1}^{N}\alpha_1^{(q)}H_q'\right) + m\right]\delta v\Bigg|_{x_1=0}^{x_1=1} + \left\{\left[F\left(u' - b_1\varphi - \sum_{q=1}^{N}\alpha_1^{(q)}N_q\right) - N\right]\delta u + \left[J_3\left(v'' + b_1\omega + \sum_{q=1}^{N}\alpha_1^{(q)}H_q\right) + M\right]\delta v' - Q\delta v\right\}\Bigg|_{x_1=0}^{x_1=1}\\
&-\left(F\kappa_1\varphi' + \Gamma\right)\big|_{x_1=0}^{x_1=1}\delta\varphi - \left(J_3\kappa_1\omega' + \Omega\right)\big|_{x_1=0}^{x_1=1}n_1\delta\omega + +\sum_{q=1}^{N}\left[F\left(-\sum_{r=1}^{N}D_1^{(qr)}N_r' + \Lambda_{11}^{(q)}u'' + M_1^{(q)}\varphi'\right) - \Gamma^{(q)}\right]\delta N_q\Bigg|_{x_1=0}^{x_1=1} - \sum_{q=1}^{N}\left[J_3\left(\sum_{r=1}^{N}D_1^{(qr)}H_r' + \Lambda_{11}^{(q)}v''' - M_1^{(q)}\omega'\right) + \Omega^{(q)}\right]\delta H_q\Bigg|_{x_1=0}^{x_1=1}.
\end{aligned} \tag{17}$$

where

$$N(x_1,\tau)=\iint_D P_1(x_1,x_2,x_3,\tau)dx_2dx_3,\ M(x_1,\tau)=\iint_D P_1(x_1,x_2,x_3,\tau)x_2dx_2dx_3,\ Q(x_1,\tau)=\iint_D P_2(x_1,x_2,x_3,\tau)dx_2dx_3,$$
$$\Gamma^{(q)}(x_1,\tau)=\iint_D I_1^{(q)}(x_1,x_2,x_3,\tau)dx_2dx_3,\ \Omega^{(q)}(x_1,\tau)=\iint_D x_2I_1^{(q)}(x_1,x_2,x_3,\tau)dx_2dx_3, \quad (18)$$
$$\Gamma(x_1,\tau)=\iint_D Q_1(x_1,x_2,x_3,\tau)dx_2dx_3,\ \Omega(x_1,\tau)=\iint_D x_2Q_1(x_1,x_2,x_3,\tau)dx_2dx_3.$$

From the variational equality (8) under the conditions (9) – (13) we obtain:

- Boundary problems for longitudinal thermomechanodiffusive vibrations of the beam

$$\ddot{u}=u''-b_1\varphi'-\sum_{q=1}^{N}\alpha_1^{(q)}N_q'+\frac{n}{F},\quad N_{N+1}=-\sum_{q=1}^{N}N_q,$$
$$\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k}{\partial\tau^k}\left(\dot{\varphi}+B_1\dot{u}'+\sum_{q=1}^{N}\upsilon^{(q)}\dot{N}_q\right)=\kappa_1\varphi''+\frac{q^{(J)}}{F}, \quad (19)$$
$$\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k\dot{N}_q}{\partial\tau^k}=\sum_{r=1}^{N}D_1^{(qr)}N_r''-\Lambda_{11}^{(q)}u'''-M_1^{(q)}\varphi''+\frac{y^{(q)}}{F};$$

$$\left.\left(u'-b_1\varphi-\sum_{q=1}^{N}\alpha_1^{(q)}N_q\right)\right|_{x_1=0}=\frac{N_0}{F},\quad \left.\left(u'-b_1\varphi-\sum_{q=1}^{N}\alpha_1^{(q)}N_q\right)\right|_{x_1=1}=\frac{N_1}{F}; \quad (20)$$

$$\left.T'\right|_{x_1=0}=-\frac{\Gamma_0}{F\kappa_1},\quad \left.T'\right|_{x_1=1}=-\frac{\Gamma_1}{F\kappa_1}; \quad (21)$$

$$\left.\left(-\sum_{r=1}^{N}D_1^{(qr)}N_r'+\Lambda_{11}^{(q)}u''+M_1^{(q)}\varphi'\right)\right|_{x_1=0}=\frac{\Gamma_0^{(q)}}{F},\quad \left.\left(-\sum_{r=1}^{N}D_1^{(qr)}N_r'+\Lambda_{11}^{(q)}u''+M_1^{(q)}\varphi'\right)\right|_{x_1=1}=\frac{\Gamma_1^{(q)}}{F}; \quad (22)$$

- Boundary problems for transverse thermomechanodiffusive vibrations of a beam

$$\ddot{v}''-\frac{F}{J_3}\ddot{v}=v^{IV}+b_1\omega''+\sum_{j=1}^{N}\alpha_1^{(j)}H_j''-\frac{q+m'}{J_3},\quad H_{N+1}=-\sum_{q=1}^{N}H_q,$$
$$\sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k}{\partial\tau^k}\left(\dot{\omega}-B_1\dot{v}''+\sum_{q=1}^{N}\upsilon^{(q)}\dot{H}_q\right)=\kappa_1\omega''+\frac{q^{(Q)}}{J_3}, \quad (23)$$
$$\sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k\dot{H}_q}{\partial\tau^k}=\sum_{r=1}^{N}D_1^{(qr)}H_r''+\Lambda_{11}^{(q)}v^{IV}-M_1^{(q)}\omega''+\frac{z^{(q)}}{J_3};$$

$$\left.\left(v''+b_1\omega+\sum_{j=1}^{N}\alpha_1^{(q)}H_j\right)\right|_{x_1=0}=-\frac{M_0}{J_3},\quad \left.\left(v''+b_1\omega+\sum_{j=1}^{N}\alpha_1^{(q)}H_j\right)\right|_{x_1=1}=-\frac{M_1}{J_3}; \quad (24)$$

$$\left.\left(v'''+b_1\omega'+\sum_{j=1}^{N}\alpha_1^{(q)}H_j'-\ddot{v}'\right)\right|_{x_1=0}=\frac{-Q_0+m|_{x_1=0}}{J_3},\quad \left.\left(v'''+b_1\omega'+\sum_{j=1}^{N}\alpha_1^{(q)}H_j'-\ddot{v}'\right)\right|_{x_1=1}=\frac{-Q_1+m|_{x_1=1}}{J_3}; \quad (25)$$

$$\left.\omega'\right|_{x_1=0}=-\frac{\Omega_0}{J_3\kappa_1},\quad \left.\omega'\right|_{x_1=1}=-\frac{\Omega_1}{J_3\kappa_1}; \quad (26)$$

$$\left.\left(\sum_{r=1}^{N}D_1^{(qr)}H_r'+\Lambda_{11}^{(q)}v'''-M_1^{(q)}\omega'\right)\right|_{x_1=0}=-\frac{\Omega_0^{(q)}}{J_3},\quad \left.\left(\sum_{r=1}^{N}D_1^{(qr)}H_r'+\Lambda_{11}^{(q)}v'''-M_1^{(q)}\omega'\right)\right|_{x_1=1}=-\frac{\Omega_1^{(q)}}{J_3}, \quad (27)$$

where, in accordance with Fig. 1, the following designations are accepted

$$N_0(\tau)=N(0,\tau),\quad N_1(\tau)=N(1,\tau),\quad M_0(\tau)=M(0,\tau),\quad M_1(\tau)=M(1,\tau),\quad Q_0(\tau)=Q(0,\tau),\quad Q_1(\tau)=Q(1,\tau),\quad \Gamma_0(\tau)=\Gamma(0,\tau),$$
$$\Gamma_1(\tau)=\Gamma(1,\tau),\quad \Omega_0(\tau)=\Omega(0,\tau),\quad \Omega_1(\tau)=\Omega(1,\tau),\quad \Gamma_0^{(q)}(\tau)=\Gamma^{(q)}(0,\tau),\quad \Gamma_1^{(q)}(\tau)=\Gamma^{(q)}(1,\tau),\quad \Omega_0^{(q)}(\tau)=\Omega^{(q)}(0,\tau),\quad \Omega_1^{(q)}(\tau)=\Omega^{(q)}(1,\tau).$$

According to the variational Lagrange principle, the boundary conditions (20) – (22) and (24) – (27) are considered together with the kinematic boundary conditions

$$\left.u\right|_{x_1=0}=U_0,\quad \left.u\right|_{x_1=1}=U_1; \quad (28)$$

$$\left.\varphi\right|_{x_1=0}=\varphi_0,\quad \left.\varphi\right|_{x_1=1}=\varphi_1,\quad \left.N_q\right|_{x_1=0}=N_{q0},\quad \left.N_q\right|_{x_1=1}=N_{q1}; \quad (29)$$

$$\left.v'\right|_{x_1=0}=V_0',\quad \left.v'\right|_{x_1=1}=V_1'; \quad (30)$$

$$\left.v\right|_{x_1=0}=V_0,\quad \left.v\right|_{x_1=1}=V_1,\quad \left.\omega\right|_{x_1=0}=\omega_0,\quad \left.\omega\right|_{x_1=1}=\omega_1,\quad \left.H_q\right|_{x_1=0}=H_{q0},\quad \left.H_q\right|_{x_1=1}=H_{q1}. \quad (31)$$

The equations (23) combined with the boundary conditions (24) and (31) constitute a mathematical model of the transverse thermoelastic diffusion vibrations of a freely supported Bernoulli-Euler beam. The initial conditions according to (4) are assumed to be zero.

4. Integral Representation of the Solution

The solutions of (23), (24) and (31) are represented as ($q=\overline{1,N}$):

$$\begin{Bmatrix} v(x,\tau) \\ \omega(x,\tau) \\ H_q(x,\tau) \end{Bmatrix} = \sum_{l=1}^{2}\sum_{k=1}^{N+3}\int_0^{\tau}\begin{Bmatrix} G_{1kl}(x,\tau-t) \\ G_{2kl}(x,\tau-t) \\ G_{q+2,kl}(x,\tau-t) \end{Bmatrix} f_{kl}(t)\,dt. \tag{32}$$

Here $x=x_1$, $f_{kl}(t)$ are the external influences at the edges of the beam, which have the form:

$$f_{11}(\tau)=-\frac{M_0(\tau)}{J_3},\ f_{12}(\tau)=-\frac{M_1(\tau)}{J_3},\ f_{21}(\tau)=V_0(\tau),\ f_{22}(\tau)=V_1(\tau),\ f_{31}(\tau)=\omega_0(\tau),\ f_{32}(\tau)=\omega_1(\tau),\ f_{q+3,1}(\tau)=H_{q0}(\tau),\ f_{q+3,2}(\tau)=H_{q1}(\tau). \tag{33}$$

where G_{1kl} are surface Green's functions satisfying the equations

$$\begin{gathered} \ddot{G}''_{1kl}-a\ddot{G}_{1kl}=G^{IV}_{1kl}+b_1G''_{2kl}+\sum_{j=1}^{N}\alpha_1^{(j)}G''_{j+2,kl},\quad a=\frac{F}{J_3}, \\ \sum_{k=0}^{M}\frac{\tau_0^k}{k!}\frac{\partial^k}{\partial\tau^k}\left(\dot{G}_{2kl}-B_1\dot{G}''_{1kl}+\sum_{q=1}^{N}\upsilon^{(q)}\dot{G}_{q+2,kl}\right)=\kappa_1G''_{2kl}, \\ \sum_{k=0}^{K}\frac{\tau_q^k}{k!}\frac{\partial^k\dot{G}_{q+2,kl}}{\partial\tau^k}=\sum_{r=1}^{N}D_1^{(qr)}G''_{r+2,kl}+\Lambda_{11}^{(q)}G^{IV}_{1kl}-M_1^{(q)}G''_{2kl}. \end{gathered} \tag{34}$$

and boundary conditions:

$$\begin{gathered} \left(G''_{1kl}+b_1G_{2kl}+\sum_{j=1}^{N}\alpha_1^{(j)}G_{j+2,kl}\right)\Bigg|_{x=0}=\delta_{1k}\delta_{1l}\delta(\tau),\quad \left(G''_{1kl}+b_1G_{2kl}+\sum_{j=1}^{N}\alpha_1^{(j)}G_{j+2,kl}\right)\Bigg|_{x=1}=\delta_{1k}\delta_{2l}\delta(\tau),\quad G_{q+2,kl}\big|_{x=0}=\delta_{q+3,k}\delta_{1l}\delta(\tau), \\ G_{1kl}\big|_{x=0}=\delta_{2k}\delta_{1l}\delta(\tau),\quad G_{1kl}\big|_{x=1}=\delta_{2k}\delta_{2l}\delta(\tau),\quad G_{2kl}\big|_{x=0}=\delta_{3k}\delta_{1l}\delta(\tau),\quad G_{2kl}\big|_{x=1}=\delta_{3k}\delta_{2l}\delta(\tau),\quad G_{q+2,kl}\big|_{x=1}=\delta_{q+3,k}\delta_{2l}\delta(\tau). \end{gathered} \tag{35}$$

By replacing the variables $y=1-x$ it can be shown that

$$G_{mk2}(x,\tau)=G_{mk1}(1-x,\tau). \tag{36}$$

In this case, the solution of (23), (24) and (31) will be written as follows

$$\begin{Bmatrix} v(x,\tau) \\ \omega(x,\tau) \\ H_q(x,\tau) \end{Bmatrix} = \sum_{k=1}^{N+3}\int_0^{\tau}\begin{Bmatrix} G_{1k}(x,\tau-t)f_{k1}(t)+G_{1k}(1-x,\tau-t)f_{k2}(t) \\ G_{2k}(x,\tau-t)f_{k1}(t)+G_{2k}(1-x,\tau-t)f_{k2}(t) \\ G_{q+2,k}(x,\tau-t)f_{k1}(t)+G_{q+2,k}(1-x,\tau-t)f_{k2}(t) \end{Bmatrix}dt. \tag{37}$$

where $G_{mk2}(x,\tau)=-G_{mk1}(1-x,\tau)$.

$$\begin{Bmatrix} v(x,\tau) \\ \omega(x,\tau) \\ H_q(x,\tau) \end{Bmatrix} = \sum_{k=1}^{N+3}\int_0^{\tau}\begin{Bmatrix} G_{1k}(x,\tau-t)f_{k1}(t)+G_{1k}(1-x,\tau-t)f_{k2}(t) \\ G_{2k}(x,\tau-t)f_{k1}(t)+G_{2k}(1-x,\tau-t)f_{k2}(t) \\ G_{q+2,k}(x,\tau-t)f_{k1}(t)+G_{q+2,k}(1-x,\tau-t)f_{k2}(t) \end{Bmatrix}dt. \tag{37}$$

5. Finding Green's Functions

We assume that the beam material is an ideal solid solute from the point of view of the mass-transfer phenomenon. In this case [18 – 20, 28 - 30]

$$g^{(qr)}=\delta_{qr},\quad D_i^{(qr)}=D_i^{(q)}g^{(qr)}=D_i^{(q)}\delta_{qr}=D_i^{(q)}.$$

where δ_{qr} is the Kronecker symbol.

Applying to (34), (35) the Laplace transform over time, we obtain (s is the Laplace transform parameter, the upper index L denotes the Laplace transform):

$$\begin{gathered} s^2G''^L_{1k}(x,s)-as^2G^L_{1k}(x,s)=\left[G^{IV}_{1k}(x,s)\right]^L+b_1G''^L_{2k}+\sum_{j=1}^{N}\alpha_1^{(j)}G''^L_{j+2,k}(x,s), \\ \sum_{k=0}^{M}\frac{s^{k+1}\tau_0^k}{k!}\left[G^L_{2k}(x,s)-B_1G''^L_{1k}(x,s)+\sum_{j=1}^{N}\upsilon^{(j)}G^L_{j+2,k}(x,s)\right]=\kappa_1G''_{2kl}, \\ \sum_{k=0}^{K}\frac{s^{k+1}\tau_q^k}{k!}G^L_{q+2,k}(x,s)=\Lambda_{11}^{(q)}\left[G^{IV}_{1k}(x,s)\right]^L-M_1^{(q)}G''^L_{2k}+D_1^{(q)}G''^L_{q+2,kl}(x,s). \end{gathered} \tag{38}$$

$$\left(G''^{L}_{1k}+b_1G^{L}_{2k}+\sum_{j=1}^{N}\alpha_1^{(j)}G^{L}_{j+2,k}\right)\Bigg|_{x_1=0}=\delta_{1k},\quad \left(G''^{L}_{1k}+b_1G^{L}_{2k}+\sum_{j=1}^{N}\alpha_1^{(j)}G^{L}_{j+2,k}\right)\Bigg|_{x_1=1}=0,$$
$$G^{L}_{1k}\big|_{x_1=0}=\delta_{2k},\quad G^{L}_{1k}\big|_{x_1=1}=0,\quad G^{L}_{2k}\big|_{x_1=0}=\delta_{3k},\quad G^{L}_{2k}\big|_{x_1=1}=0,\quad G^{L}_{q+2,k}\big|_{x_1=0}=\delta_{q+3,k},\quad G^{L}_{q+2,k}\big|_{x_1=1}=0. \tag{39}$$

By multiplying each equation by $\sin\lambda_n x$ ($\lambda_n=\pi n$) and integrating over the interval $[0,1]$ we obtain the following system of linear algebraic equations

$$\begin{gathered}k_{1n}(s)G^{L}_{1kn}(s)-b_1\lambda_n^2G^{L}_{2kn}(s)-\lambda_n^2\sum_{j=1}^{N}\alpha_1^{(j)}G^{L}_{j+2,kn}(s)=F_{1n}(s),\\ B_1k_0(s)\lambda_n^2G^{L}_{1kn}(s)+k_{2n}(s)G^{L}_{2kn}(s)+k_0(s)\sum_{j=1}^{N}\upsilon^{(j)}G^{L}_{j+2,kn}(s)=F_{2n},\\ -\Lambda_{11}^{(q)}\lambda_n^4G^{L}_{1kn}(s)-M_1^{(q)}\lambda_n^2G^{L}_{2kn}(s)+k_{q+2,n}(s)G^{L}_{q+2,kn}=F_{q+2,n}.\end{gathered} \tag{40}$$

here,

$$G^{L}_{ikn}(s)=2\int_0^1G^{L}_{ik}(x,s)\sin\lambda_n x\,dx,\quad G^{L}_{ik}(x,s)=\sum_{n=1}^{\infty}G^{L}_{ikn}(s)\sin\lambda_n x\quad(\lambda_n=\pi n). \tag{41}$$

and the other values are determined as follows:

$$\begin{gathered}k_{1n}(s)=s^2\lambda_n^2+as^2+\lambda_n^4,\quad k_{2n}(s)=k_0(s)+\kappa_1\lambda_n^2,\quad k_{q+2,n}(s)=\sum_{k=0}^{K}\frac{s^{k+1}\tau_q^k}{k!}+D_1^{(q)}\lambda_n^2,\quad k_0(s)=\sum_{k=0}^{M}\frac{s^{k+1}\tau_0^k}{k!},\\ F_{1n}(s)=2\lambda_n\left(s^2+\lambda_n^2\right)\delta_{2k}-2\lambda_n\delta_{1k},\quad F_{2n}=2\kappa_1\lambda_n\delta_{3k}+2B_1sk_0(s)\lambda_n\delta_{2k},\\ F_{q+2,n}=2\Lambda_{11}^{(q)}\lambda_n\delta_{1k}-2\Lambda_{11}^{(q)}\lambda_n^3\delta_{2k}-2\left(M_1^{(q)}+b_1\Lambda_{11}^{(q)}\right)\lambda_n\delta_{3k}+2\lambda_n\left(D_1^{(q)}\delta_{q+3,k}-\Lambda_{11}^{(q)}\sum_{j=1}^{N}\alpha_1^{(j)}\delta_{j+3,k}\right).\end{gathered}$$

The solution of the system (40) is

$$\begin{gathered}G^{L}_{ikn}(s)=\frac{P_{ikn}(s)}{P_n(s)}\quad(i=1,2),\\ G^{L}_{q+2,kn}(s)=\frac{P_{q+2,kn}(s)}{Q_{qn}(s)}+2\lambda_n\frac{\Lambda_{11}^{(q)}\delta_{1k}-\Lambda_{11}^{(q)}\lambda_n^2\delta_{2k}-\left(M_1^{(q)}+b_1\Lambda_{11}^{(q)}\right)\delta_{3k}+D_1^{(q)}\delta_{q+3,k}-\Lambda_{11}^{(q)}\sum_{j=1}^{N}\alpha_1^{(j)}\delta_{j+3,k}}{k_{q+2,n}(s)}.\end{gathered} \tag{42}$$

where

$$\begin{gathered}P_n(s)=\left[k_{1n}(s)k_{2n}(s)+k_0(s)\lambda_n^4B_1b_1\right]\Pi_n(s)+k_{1n}(s)k_0(s)\lambda_n^2\sum_{j=1}^{N}\upsilon^{(j)}M_1^{(j)}\Pi_{jn}(s)-k_{2n}(s)\lambda_n^6\sum_{j=1}^{N}\alpha_1^{(j)}\Lambda_{11}^{(j)}\Pi_{jn}(s)+\\ +k_0(s)\lambda_n^6B_1\sum_{j=1}^{N}\alpha_1^{(j)}M_1^{(j)}\Pi_{jn}(s)+k_0(s)\lambda_n^6b_1\sum_{j=1}^{N}\upsilon^{(j)}\Lambda_{11}^{(j)}\Pi_{jn}(s)+k_0(s)\lambda_n^8\sum_{i=1}^{N}\sum_{j=1}^{N}\upsilon^{(i)}\alpha_1^{(j)}M^{(ji)}\Pi_{ijn}(s),\\ Q_{qn}(s)=k_{q+2,n}(s)P_n(s);\end{gathered} \tag{43}$$

$$\begin{gathered}\Pi_n(s)=\prod_{j=1}^{N}k_{j+2,n}(s),\quad \Pi_{qn}(s)=\prod_{j=1,j\neq q}^{N}k_{j+2,n}(s),\quad \Pi_{qpn}(s)=\prod_{j=1,j\neq q,p}^{N}k_{j+2,n}(s),\\ C_{1n}^{(j)}(s)=k_{1n}(s)\upsilon^{(j)}+\lambda_n^4B_1\alpha_1^{(j)},\quad C_{2n}^{(j)}(s)=k_{2n}(s)\alpha_1^{(j)}-b_1k_0(s)\upsilon^{(j)},\quad S^{(ij)}=\alpha_1^{(i)}\upsilon^{(j)}-\alpha_1^{(j)}\upsilon^{(i)},\quad M^{(qi)}=M_1^{(q)}\Lambda_{11}^{(i)}-\Lambda_{11}^{(q)}M_1^{(i)}.\end{gathered} \tag{42}$$

The Laplace originals (time domain) of Green's functions in (42) are from the tables of the operational calculus [32]

$$\begin{gathered}G_{ikn}(\tau)=\sum_{j=1}^{\Sigma}A_{ikn}^{(j)}e^{s_{jn}\tau},\quad\left(i=1,2,\quad \Sigma=(K+1)N+M+3\right),\\ G_{q+2,1n}(\tau)=\sum_{j=1}^{\Sigma}A_{q+2,1n}^{(j)}e^{s_{jn}\tau}+\sum_{l=1}^{K+1}\left[\frac{2\Lambda_{11}^{(q)}\lambda_n}{k'_{q+2,n}\left(\xi_{qln}\right)}+A_{q+2,1n}^{(\Sigma+l)}\right]e^{\xi_{qln}\tau},\quad G_{q+2,2n}(\tau)=\sum_{j=1}^{\Sigma}A_{q+2,2n}^{(j)}e^{s_{jn}\tau}+\sum_{l=1}^{K+1}\left[\frac{-2\Lambda_{11}^{(q)}\lambda_n^3}{k'_{q+2,n}\left(\xi_{qln}\right)}+A_{q+2,2n}^{(\Sigma+l)}\right]e^{\xi_{qln}\tau},\\ G_{q+2,3n}(\tau)=\sum_{j=1}^{\Sigma}A_{q+2,2n}^{(j)}e^{s_{jn}\tau}+\sum_{l=1}^{K+1}\left[-\frac{2\left(M_1^{(q)}+b_1\Lambda_{11}^{(q)}\right)\lambda_n}{k'_{q+2,n}\left(\xi_{qln}\right)}+A_{q+2,2n}^{(\Sigma+l)}\right]e^{\xi_{qln}\tau},\\ G_{q+2,p+3}(\tau)=\sum_{j=1}^{\Sigma}A_{q+2,p+3,n}^{(j)}e^{s_{jn}\tau}+\sum_{l=1}^{K+1}\left[\frac{2\left(D_1^{(q)}\delta_{qp}-\Lambda_{11}^{(q)}\alpha_1^{(p)}\right)\lambda_n}{k'_{q+2,n}\left(\xi_{qln}\right)}+A_{q+1,p+2,n}^{(\Sigma+l)}\right]e^{\xi_{qln}\tau},\\ A_{ikn}^{(j)}=\frac{P_{ikn}\left(s_{jn}\right)}{P'_n\left(s_{jn}\right)}\quad\left(j=\overline{1,\Sigma},\quad k=\overline{1,N+3},\quad i=1,2\right),\quad A_{q+2,k}^{(r)}(\lambda_n)=\frac{P_{q+2,kn}\left(s_{rn}\right)}{Q'_{qn}\left(s_{rn}\right)}\quad\left(r=\overline{1,\Sigma+K+1},\quad q=\overline{1,N}\right),\\ A_{q+2,k}^{(2N+4+l)}=\frac{P_{q+2,kn}\left(\xi_{qln}\right)}{Q'_{qn}\left(\xi_{qln}\right)}\quad\left(l=\overline{1,K}\right).\end{gathered} \tag{42}$$

where s_{jn}, $j=\overline{1,\Sigma}$ are zeros of the polynomial $P_n(s)$, χ_{rqn} are zeros of the polynomial $k_{q+2,n}(s)$. At $K=1$ they have the form:

$$\xi_{1qn}=\frac{-1-\sqrt{1-4\tau_q D_1^{(q)}\lambda_n^2}}{2\tau_q},\quad \xi_{2qn}=\frac{-1+\sqrt{1-4\tau_q D_1^{(q)}\lambda_n^2}}{2\tau_q}.$$

6. Calculation Example

For the calculation example, consider a beam of length $l=0{,}01\,m$, rectangular cross-section $h\times b=0{,}05l\times 0{,}05l$ duralumin, where the independent components are zinc (component 1) and copper (component 2) [33]:

$$C_{1122}^{*}=6.93\times 10^{10}\frac{N}{m^2},\quad C_{1212}^{*}=2.56\times 10^{10}\frac{N}{m^2},\quad T_0=800\,K,\quad D_{11}^{*(1)}=7.74\times 10^{-14}\frac{m^2}{sec},\quad D_{11}^{*(2)}=6.67\times 10^{-14}\frac{m^2}{sec},\quad l=0.01\,m,$$
$$\alpha_{11}^{*(1)}=1.55\times 10^{7}\frac{J}{m^3},\quad \alpha_{11}^{*(2)}=6.14\times 10^{7}\frac{J}{m^3},\ \rho=2780\frac{kg}{m^3},\quad m^{(1)}=0.027\frac{kg}{mol},\quad m^{(2)}=0.064\frac{kg}{mol},\ n_0^{(1)}=0.935,\ n_0^{(2)}=0.045.$$

For the first example, we assume that the beam is under the action of a pair of bending moments applied to its ends. This corresponds to the following values of the external action parameters in (33)

$$f_{11}(\tau)=-\frac{M_0(\tau)}{J_3}=0.01\times H(\tau),\ f_{12}(\tau)=-\frac{M_1(\tau)}{J_3}=0.01\times H(\tau),\quad f_{21}(\tau)=V_0(\tau)=0,\quad f_{22}(\tau)=V_1(\tau)=0,$$
$$f_{31}(\tau)=\omega_0(\tau)=0,\quad f_{32}(\tau)=\omega_1(\tau)=0,\quad f_{q+2,1}(\tau)=H_{q0}(\tau)=0,\quad f_{q+2,2}(\tau)=H_{q1}(\tau)=0. \tag{43}$$

Then, according to (37) with (44) and (45) we have

$$v(x,\tau)=\int_0^{\tau}\left[G_{11}(x,\tau-t)+G_{11}(1-x,\tau-t)\right]H(t)dt=2\sum_{n=1}^{\infty}\sin\frac{\lambda_n}{2}\cos\left[\lambda_n\left(\frac{1}{2}-x\right)\right]\sum_{j=1}^{2N+4}A_{11n}^{(j)}\frac{\exp(s_{jn}\tau)-1}{s_{jn}},$$
$$\omega(x,\tau)=\int_0^{\tau}\left[G_{21}(x,\tau-t)+G_{21}(1-x,\tau-t)\right]H(t)dt=2\sum_{n=1}^{\infty}\sin\frac{\lambda_n}{2}\cos\left[\lambda_n\left(\frac{1}{2}-x\right)\right]\sum_{j=1}^{2N+4}A_{21n}^{(j)}\frac{\exp(s_{jn}\tau)-1}{s_{jn}},$$
$$H_q(x,\tau)=\int_0^{\tau}\left[G_{q+2,1}(x,\tau-t)+G_{q+2,1}(1-x,\tau-t)\right]H(t)dt=$$
$$=2\sum_{n=1}^{\infty}\sin\frac{\lambda_n}{2}\cos\left[\lambda_n\left(\frac{1}{2}-x\right)\right]\left[\sum_{l=1}^{2}\left(\frac{2\Lambda_{11}^{(q)}\lambda_n}{k'_{q+2,n}(\xi_{qln})}+A_{q+2,1n}^{(2N+4+l)}\right)\frac{\exp(\xi_{qln}\tau)-1}{\xi_{qln}}+\sum_{j=1}^{2N+4}A_{q+2,1n}^{(j)}\frac{\exp(s_{jn}\tau)-1}{s_{jn}}\right].$$

For the calculations, we are limited to the case $K,M=1$ (Cattaneo model). The results of the calculations are shown in Fig. 2 - Fig. 8.

The deflections presented in Fig. 2 and the bending-initiated deformation increments of zinc (Fig. 4) and copper (Fig. 5) concentrations agree with the results obtained in [18] with a sufficient accuracy. The heating of the beam due to bending is demonstrated in figure Fig. 3.

At the considered time interval, the maximum temperature increment $\omega=0{,}012$, which corresponds to $8{,}4\,K$ at an initial temperature of $700\,K$ (Fig. 3). Thus, the heat transfer caused by the unsteady bending of the beam is small. The mass transfer under a given loading is negligible Fig. 4, 5. Figures 6, 7 demonstrate the effect of relaxation processes on the kinetics of mass transfer. Comparing the graphs in Fig. 6 and Fig. 7, we can see that over time, the influence of relaxation processes on the diffusion field disappears. Calculations show that under given loads, there is no effect of relaxation of heat and diffusion flows on beam deflections. The influence of relaxation effects on heat transfer is very insignificant and lasts only for a very short period of time (Fig. 8).

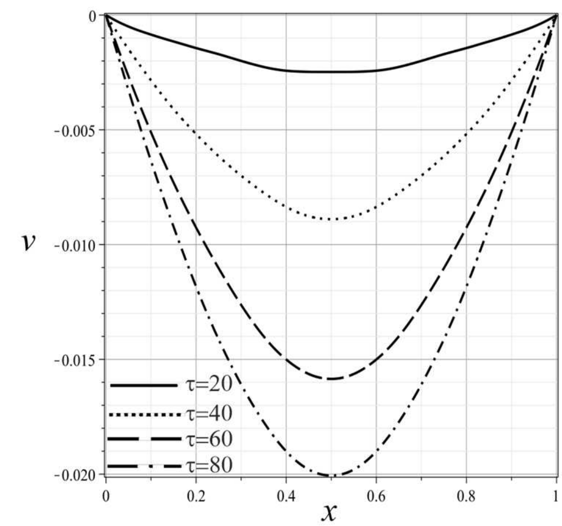

Fig. 2. Beam deflection $v(x,\tau)$

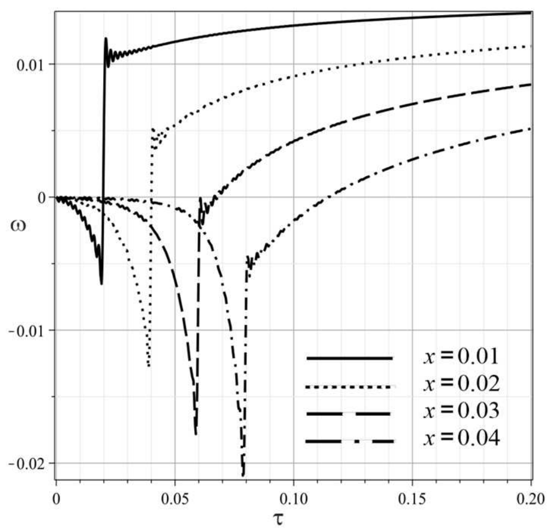

Fig. 3. Linear density of temperature increment $\omega(x,\tau)$

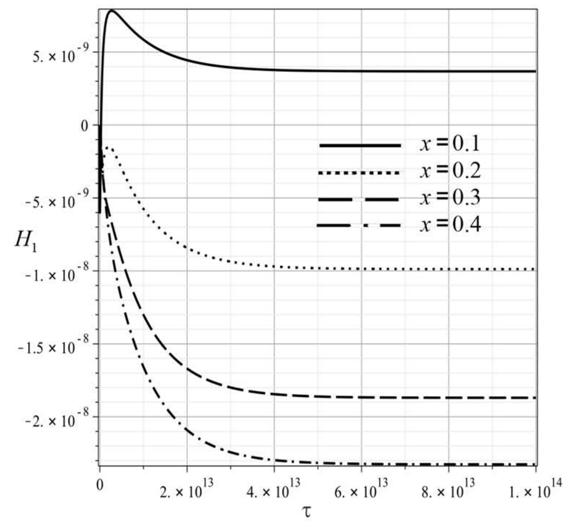

Fig. 4. Linear density of zinc concentration increment $H_1(x,\tau)$

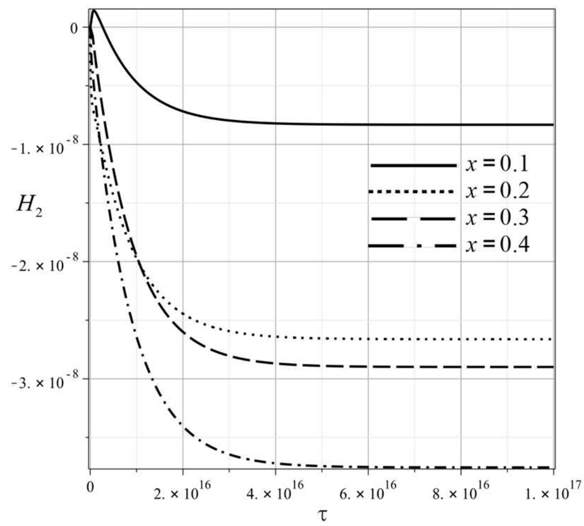

Fig. 5. Linear density of copper concentration increment $H_2(x,\tau)$

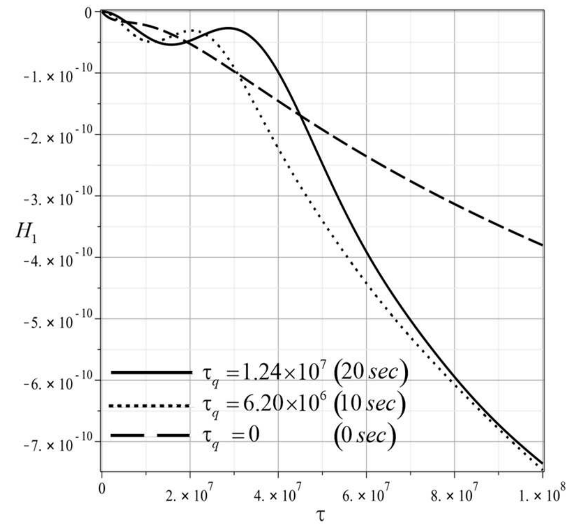

Fig. 6. Linear density of zinc concentration increment $H_1(0.1,\tau)$

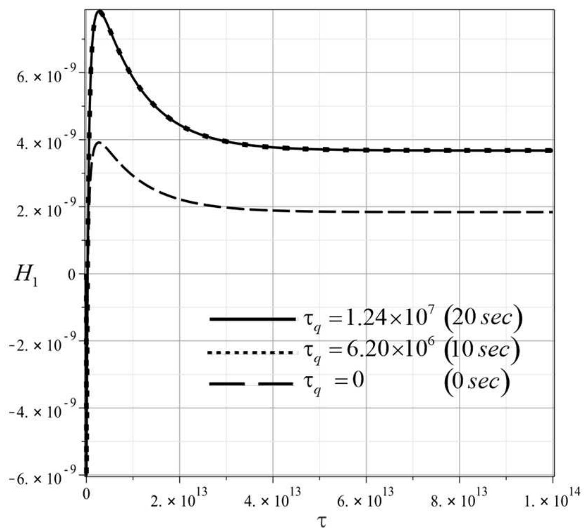

Fig. 7. Linear density of zinc concentration increment $H_1(0.1,\tau)$

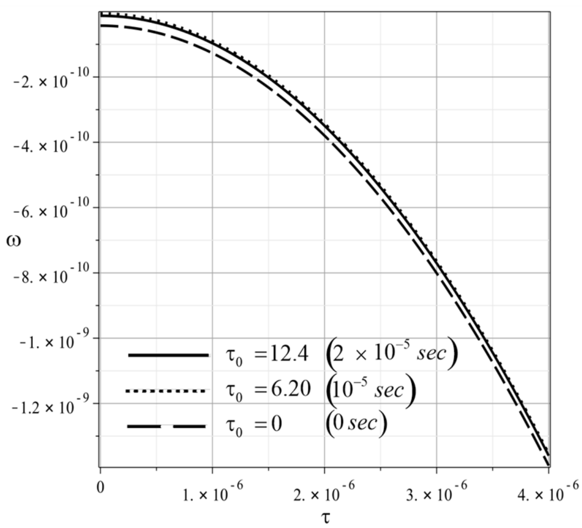

Fig. 8. Linear density of temperature increment $\omega(0.1,\tau)$

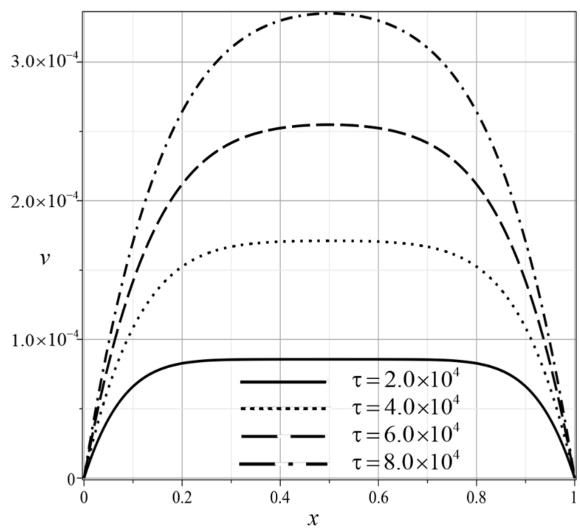

Fig. 9. Beam deflection $v(x,\tau)$

As a second example, consider a simple supported beam, at the ends of which the temperature regime is set. This corresponds to the following values of the external action parameters in (33).

$$\begin{aligned} &f_{11}(\tau) = -\frac{M_0(\tau)}{J_3} = 0,\ f_{12}(\tau) = -\frac{M_1(\tau)}{J_3} = 0, \quad f_{21}(\tau) = V_0(\tau) = 0, \quad f_{22}(\tau) = V_1(\tau) = 0, \\ &f_{31}(\tau) = \omega_0(\tau) = 0.1\times H(\tau), \quad f_{32}(\tau) = \omega_1(\tau) = 0.1\times H(\tau), \quad f_{q+2,1}(\tau) = H_{q0}(\tau) = 0, \quad f_{q+2,2}(\tau) = H_{q1}(\tau) = 0. \end{aligned} \tag{44}$$

Then, according to (37) with (44) and (45) we have

$$v(x,\tau) = \int_0^\tau \left[G_{13}(x,\tau-t) + G_{13}(1-x,\tau-t)\right]H(t)\,dt = 2\sum_{n=1}^{\infty}\sin\frac{\lambda_n}{2}\cos\left[\lambda_n\left(\frac{1}{2}-x\right)\right]\sum_{j=1}^{2N+4}A_{13n}^{(j)}\frac{\exp(s_{jn}\tau)-1}{s_{jn}},$$

$$\omega(x,\tau) = \int_0^\tau \left[G_{23}(x,\tau-t) + G_{23}(1-x,\tau-t)\right]H(t)\,dt = 2\sum_{n=1}^{\infty}\sin\frac{\lambda_n}{2}\cos\left[\lambda_n\left(\frac{1}{2}-x\right)\right]\sum_{j=1}^{2N+4}A_{23n}^{(j)}\frac{\exp(s_{jn}\tau)-1}{s_{jn}},$$

$$H_q(x,\tau) = \int_0^\tau \left[G_{q+2,3}(x,\tau-t) + G_{q+2,3}(1-x,\tau-t)\right]H(t)\,dt =$$

$$= 2\sum_{n=1}^{\infty}\sin\frac{\lambda_n}{2}\cos\left[\lambda_n\left(\frac{1}{2}-x\right)\right]\left[\sum_{l=1}^{2}\left(-\frac{2\left(M_1^{(q)}+b_1\Lambda_{11}^{(q)}\right)\lambda_n}{k'_{q+2,n}\left(\xi_{qln}\right)} + A_{q+2,3n}^{(2N+4+l)}\right)\frac{\exp(\xi_{qln}\tau)-1}{\xi_{qln}} + \sum_{j=1}^{2N+4}A_{q+2,3n}^{(j)}\frac{\exp(s_{jn}\tau)-1}{s_{jn}}\right].$$

For the calculations, we are limited to the case $K, M = 1$. The results of the calculations are shown in Figs. 9 - 14.

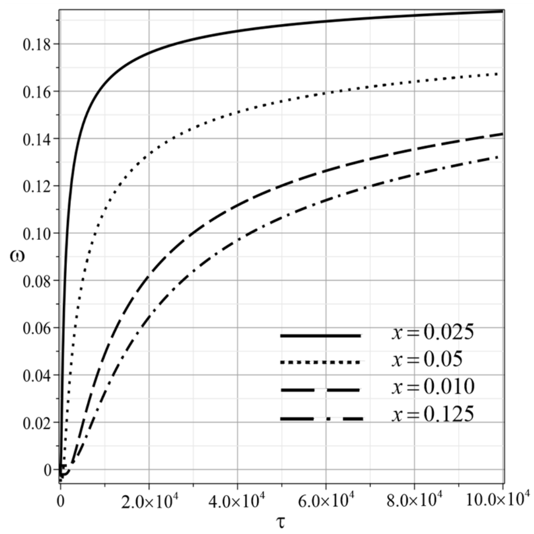

Fig. 10. Linear density of temperature increment $\omega(x,\tau)$

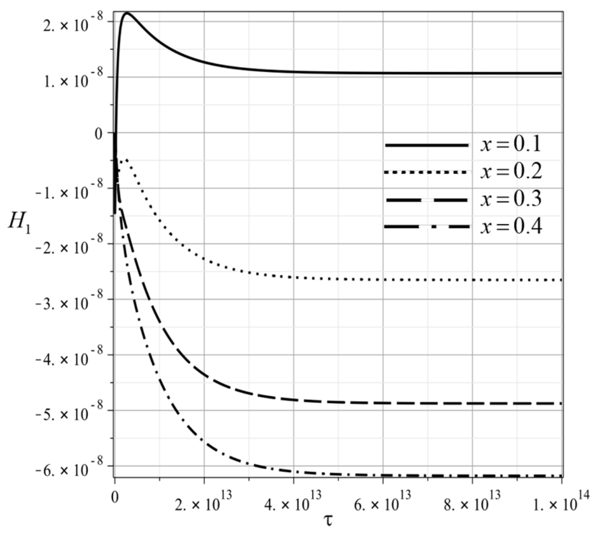

Fig. 11. Linear density of zinc concentration increment $H_1(x,\tau)$

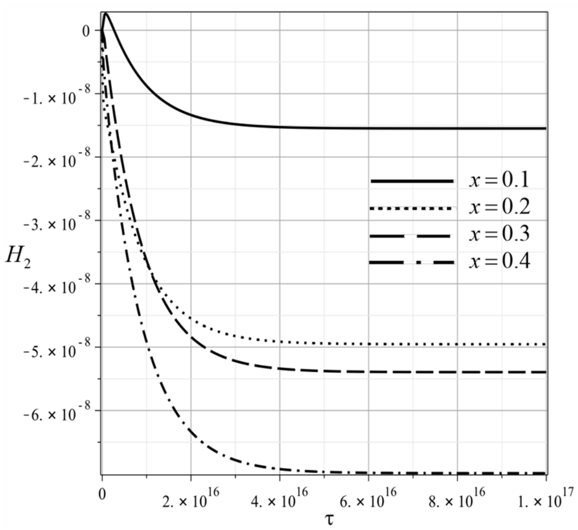

Fig. 12. Linear density of copper concentration increment $H_2(x,\tau)$

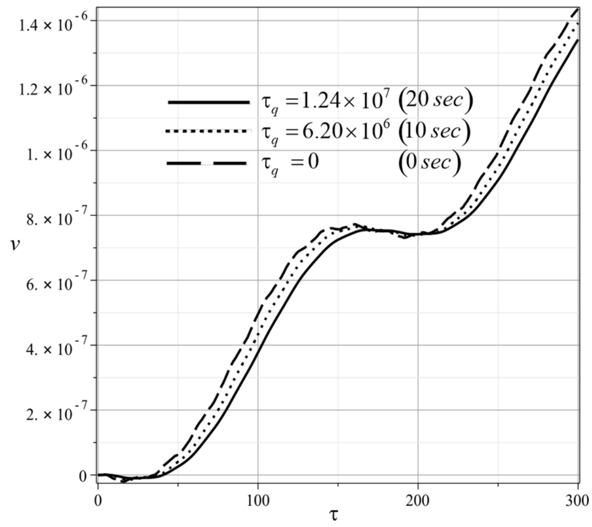

Fig. 13. Beam deflection $v(0.1,\tau)$

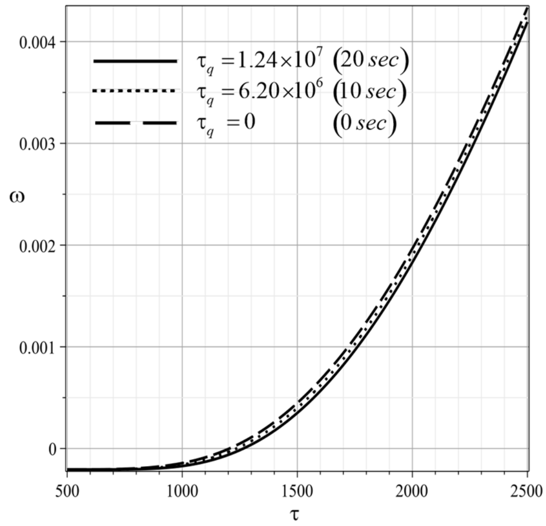

Fig. 14. Linear density of temperature increment $\omega(0.1,\tau)$

It shows the temperature field (Fig. 10), as well as the fields of displacements and increments of zinc and copper concentration (Fig. 9, 11, 12), which are initiated by the temperature regime set at the ends of the beam (44). As can be seen, even at these boundary perturbations, the effect of field interaction manifests itself rather weakly. This can be judged by the orders of magnitude in Figs. 9, 11, 12. It should also be noted that the plots for incremental concentrations of zinc and copper in Figs. 11 and 12 are very similar to those shown in Fig. 4 and 5. Based on this, we can conclude that boundary perturbations (43) and (44) have approximately the same effect on the kinetics of mass transfer in the beam.

The influence of relaxation effects on the mechanical and temperature fields is shown in Figs. 13 and 14. Similar results for concentration increments look very close to what is shown in Figs. 6 and 8.

This agrees with the results of experimental studies confirming that the interaction of the mechanical field with the diffusion field occurs to a large extent only at plastic deformations or at high-frequency pulse impacts [34 - 36].

7. Conclusion

The mathematical model of thermoelastic diffusion unsteady longitudinal and transverse vibrations of a Bernoulli-Euler beam is developed with the help of the D'Alembert's variational principle. It describes the interaction between mechanical, temperature, and diffusion fields in continuum. The model includes relaxation effects, which cause finite speed propagation of thermal and diffusion perturbations. We formulated the initial boundary value problem of bending of a freely supported beam and proposed a solution algorithm based on the Green's functions method. The Laplace transform and Fourier series expansion were used to find these functions. This approach has the advantage that it allows us to find the fundamental solution of the problem for the freely supported beam in the explicit form. This solution can be used further in modeling of thermomechanodiffusion vibrations of the beam under various loading modes. Effects of interaction of mechanical, thermal and diffusion fields have been demonstrated by the example of the freely supported beam under the action of a pair of suddenly applied nonstationary bending moments. The performed calculations confirm that the unsteady bending generates unsteady heat and mass transfer, and the relaxation thermodiffusion effects decrease over time. On the other hand, the results indicate that the influence of elastic strain on the temperature and diffusion fields is negligible. This is consistent with the results of experimental studies by other authors.

Author Contributions

A.V. Zemskov has developed a model of thermoelastic-diffusion vibrations of a Bernoulli-Euler beam. L.V. Hao developed an algorithm for constructing Green's functions for the problem of unsteady bending of a simple supported beam. D.V. Tarlakovskii wrote a program calculating displacements, temperature increments, and concentration increments.

Acknowledgments

Not applicable.

Conflict of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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