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Research Paper

# An Alternative Procedure for Longitudinal Vibration Analysis of Bars with Arbitrary Boundary Conditions

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**Abstract.** The present work aims at generating a systematic way for longitudinal vibration (LV) analysis of bars (or rods) with arbitrary boundary conditions (BCs) by mixed-type finite element (MFE) method using the Gâteaux differential. Both materials and geometrical properties of the bar are uniform along the longitudinal direction. The problem is reduced to solution of the classical eigenvalue problem in dynamic analysis. The axial (normal) load and the displacement along the bar are the basic unknowns of the mixed element. The element formulation for the shape function must satisfy only $C^0$ class continuity since the first derivatives of the variables exist in the functional. The functional governed with proper dynamic and geometric BCs of the problem. Results of the recommended method are benchmarked and verified via numerous problems present in the literature. The unique aspects of this study and the possible contributions of the proposed method to the literature can be summarized as follows: by using this new functional, displacements and internal force values can be obtained directly without any mathematical operation. In addition, geometric and dynamic BCs can be obtained easily and a field variable can be included to the functional systematically. To examine the effects of BCs on the longitudinal vibratory motion of a uniform elastic bar and to give a better insight into LV analysis of bars with arbitrary BCs, a set of numerical examples are presented.

**Keywords:** Longitudinal vibration, Gâteaux differential, arbitrary boundary condition, mixed finite element formulation.

## 1. Introduction

Oscillatory motion of dynamic systems is the subject of vibration. All bodies which possess mass and elasticity are capable of relative motion. The dynamic system considered may be in the form of a structure, a machine or its components. To control the vibration when it is undesirable and to utilize the vibration when it is desirable are the objectives of the designer.

A system generally consists of many mass particles. In order to define the configuration of the system, only one spatial coordinate is required it is called one-degree of freedom (DOF) system. In engineering, many dynamic (vibrating) systems can be represented or approximated by one-DOF systems. The spring-mass system can be given as a simple one-DOF system. In order to assess the dynamic response of a linear one-DOF system, the differential equation of motion must be solved. The dynamic response of a one-DOF system is calculated under several types of conditions. A one-DOF system undergoes free vibration in the absence of external loading and a system is said to undergo forced vibration when the mass is subjected to some external dynamic force. The loading may be periodic or nonperiodic. Solution of the equation of motion developed for cases in which damping is and is not present. The system is said to be overdamped if the damping factor is greater than one, critically damped if the damping factor is equal to one and underdamped if the damping factor is less than one. Vibratory motion exists only if the system is underdamped. The vibratory motion can be categorized into three types depending on the nature of vibrations: i) Longitudinal vibration, ii) Lateral vibration and iii) Torsional vibration.

Basic ideas relating to LV characteristics are covered in vibration books of Bishop and Johnson [1], Timoshenko [2] and Strutt and Rayleigh [3]. There are also many studies in literature of which the LV characteristics of a uniform and non-uniform rods (or bars) are reported [4-13]. In recent years, a number of exquisite studies have been reported on the bending, vibration, and stability analyses of bars [14-19]. In the last decade, the study of bar as an elementary structural element with variable cross-section and different BCs undergoing LV has drawn considerable attentions in the case of engineering applications. Udwadia [20] studied the LV of a bar with dampers (viscous BCs) at its two ends. A closed form solution was obtained for the system subjected to initial conditions and external excitation. Jovanovic [21] investigated the longitudinal vibrations (LVs) of a bar with viscous BCs at each end. In order to determine complex-valued eigenvalues and eigenfunctions, a boundary value problem was derived. Presented generalized Fourier series solution was verified for the LV of a free-free, fixed-damper, fixed-fixed and fixed-free bar cases. Gan et al. [22] investigated the propagation of longitudinal wave in rods with the varying cross-sections in the exponential and the

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polynomial forms. In order to derive the equations of motion; three different kinds of wave theories were employed. Transfer matrices of the longitudinal wave and their corresponding eigenvalues were studied analytically. A lie symmetry method-based approach proposed by Nunes et al. [23] for the mode shapes of longitudinally vibrating non-uniform rods. The mode shape equation was modeled by the elementary rod theory with polynomial, exponential, trigonometric and hyperbolic cross-section variations. Krawczuk and Palacz [24] determined optimal mass matrix for a rod finite element in order to use in the analysis of LVs of rods. Peck et al. [25] considered free LVs of a slender viscoelastic rod growing in both lateral and axial directions. Left end of the rod was permanently fixed and its right end was free. Kelvin's model of linear viscoelasticity was assumed. Liu et al. [26] proposed closed-form dynamic stiffness method for exact longitudinal free vibration analyses of rods and trusses based on the four rod theories. General solutions of the governing differential equations of motion used as exact shape functions. Demir [27] investigated LV behavior of temperature dependent bar with variable cross-section. Temperature variation, geometric and slenderness ratios and mode numbers variation effects on the natural frequency of the bar discussed. Mei [28] analyzed dynamics of longitudinally vibrating uniform and stepped rods. Four rod theories were considered and the motion of the rod was described in terms of waves. The wave propagation, reflection and transmission relations derived. Xu et al. [29] presented Fourier series solution for the free LV of general non-uniform rods with arbitrary end restraints. In order to formulate the problem, energy principle in combination of Rayleigh-Ritz procedure was applied. Numerical examples presented for the natural frequency and mode shapes of non-uniform rod of free and clamped BCs. Influence of cross-section area variation on vibration characteristics of non-uniform rods discussed. Turdibekov and Aliyev [30] considered the solution of the problem of longitudinal oscillations of a round elastic rigidly fixed rod under kinematic excitation of the free end. Finite difference method applied for the solution and rotatory inertia of the body was considered. Utyashev [31] determined the natural frequencies of LVs. It was assumed that the density changes along the axis in the form of a polynomial function. Pala and Kahya [32] studied the LV of rods with variable cross-sections. A new analytical method based on Ricatti differential equation presented.

Although many techniques have been developed for the LV analysis of rods, being able to characterize the LV of rods via a numerical methodology specifically simple, reliable and efficient in computations for different geometrical and material properties becomes increasingly important. Due to the difficulty of obtaining closed-form solutions for the problems which have complex geometries and loading conditions, numerical solution techniques are employed. The finite element method is widely used because it is systematic and suitable for programming and it can be easily applied to solve complex problems in engineering disciplines. In this sense, the main objective of this research is to investigate the LV characteristics of bars numerically. The model will serve to analyze the natural longitudinal frequency of an elastic bar system and the free vibration response of the model. For the analysis, a MFE formulation is developed. Researchers suggested many principles to formulate MFE formulations. The Hellinger-Reissner and the Hu-Washizu principles are more popular ones and these variational principles are based on the minimization of the energy functional. Hellinger-Reissner principle involves stresses and displacements as fundamental unknown variables. Hu-Washizu principle is a generalization of Hellinger-Reissner principle and involves stresses, strains and displacements as functional arguments. In derivation of a functional, the Gâteaux differential method is more powerful and efficient variational tool when compared to conventional variational principles, Hellinger-Reissner and Hu-Washizu. By applying the systematic procedure called the Gâteaux differential, a new functional that includes the geometric (essential) and dynamic (natural) BCs of the problem is proposed. In this functional, there exists two independent variables such as axial (normal) load and longitudinal displacement. The governing equations of the problem arranged in order to satisfy the potential operator condition. For the analysis, a special MFE program is developed. Due to the existence of first derivatives of the variables in the functional, the element shape (interpolation) function must satisfy only $C^0$ class continuity. Two variables in the functional are approximated by using the interpolation functions. After minimization of the functional, element matrix for the MFE formulation of the longitudinal vibratory motion of elastic bars is derived. Natural longitudinal frequencies of a uniform elastic bar with different BCs are calculated. All analysis are performed by writing the Fortran code. By using the Gâteaux differential, free vibration (FV) analyses of laminated composite curved beams [33], of cross-ply and angle-ply laminated composite beams [34, 35] and buckling of laminated composite beams [36] studied by Kadıoğlu and his co-workers.

## 2. Methodology

Consider axially vibrating homogenous elastic bar has a length of $L$ and constant cross-sectional area of $A$ (see Fig. 1). The longitudinal stiffness of the bar is $EA$ where $E$ is the Young's (or elastic) modulus of the bar material. Let $u(z, t)$ be the longitudinal displacement at a point $z \in [0, L]$ along the bar. Differential form of governing equation for the longitudinal oscillating motion of a bar can be governed as follows:

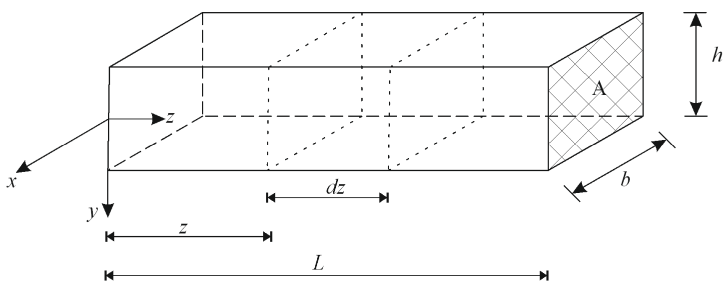


**Fig. 1.** Uniform bar and associated (x, y, z) coordinate system.

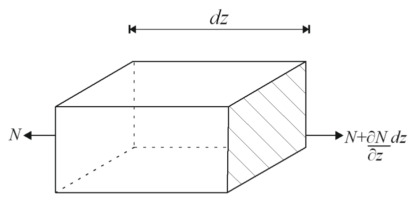


**Fig. 2.** Infinitesimal bar segment of length $dz$.

$$\frac{\partial^2 u}{\partial z^2} = \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $\rho$ is density of the rod. Other form of Eq. (1) in terms of axial (normal) force *N* (see Fig. 2) can be written as:

$$\frac{\partial N}{\partial z} = \rho A \frac{\partial^2 u}{\partial t^2} \tag{2}$$

by performing integration over the area. If harmonic solutions are assumed for both displacement and axial load:

$$u(z,t) = u\, e^{i\omega t} \tag{3}$$

$$N(z,t) = N\, e^{i\omega t} \tag{4}$$

where *u* and *N* are displacement and normal force as function of z, respectively and $\omega$ is the circular natural frequency of the harmonic vibrations in rad/sec. By using Eqs. (3) and (4), the following ordinary differential equations are obtained:

$$-\frac{\partial u}{\partial z} + \frac{N}{EA} = 0 \tag{5}$$

$$\rho A\, \omega^2 u + \frac{\partial N}{\partial z} = 0 \tag{6}$$

Field equations (5) and (6) can be written in operator form as:

$$\mathbf{Q} = \boldsymbol{L}\boldsymbol{v} - \boldsymbol{f} \tag{7}$$

where $\boldsymbol{L}$ represents the coefficient matrix, $\boldsymbol{v}$ represents unknown vectors $\boldsymbol{v} = \{N, u\}$ and $\boldsymbol{f}$ represents the load vector. Here, **Q** is a continuous operator which has a linear Gâteaux differential. The necessary and sufficient condition that **Q** be potential operator is that

$$\langle d\mathbf{Q}(\boldsymbol{v},\bar{\boldsymbol{v}}),\boldsymbol{v}^*\rangle = \langle d\mathbf{Q}(\boldsymbol{v},\boldsymbol{v}^*),\bar{\boldsymbol{v}}\rangle \tag{8}$$

This operator can be written explicitly in matrix form as:

$$\begin{bmatrix} \rho A\omega^2 & \frac{\partial}{\partial z} & 0 & 0 \\ -\frac{\partial}{\partial z} & \frac{1}{EA} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ N \\ u \\ N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\hat{N} \\ \hat{u} \end{bmatrix} \tag{9}$$

As seen from Eq. (9), BC terms are also included. Since potential condition is satisfied with the application of a related mathematical procedure, the functional corresponds to the field equations must be governed. According to the Oden and Reddy [37], the functional $I(\boldsymbol{v})$ whose gradient is the operator **Q** can be given by:

$$I(\boldsymbol{v}) = \int_0^1 \langle \mathbf{Q}(s\boldsymbol{v},\boldsymbol{v}),\boldsymbol{v}\rangle ds \tag{10}$$

If so the operator **Q** is potential, then the functional corresponds to the field equations can be written in explicit form as:

$$I(\boldsymbol{v}) = \frac{1}{2}\rho A\omega^2[u,u] - [N,u'] + \frac{1}{2EA}[N,N] + [u,N]_\varepsilon + [\hat{N},u]_\sigma - [\hat{u},N]_\varepsilon \tag{11}$$

Here, square parenthesis is defined as:

$$[f,g] = \int_z fg\,dz \tag{12}$$

In Eq. (11), subscripts $\sigma$ and $\varepsilon$ outside the square parentheses indicate dynamic and geometric BCs, respectively. For one-dimensional domain, the finite element approximation $\Phi^e(z)$ of *u*(z) can be given in the following complete linear polynomial form:

$$\Phi^e(z) = a_1^e + a_2^e z \tag{13}$$

where $a_1^e$ and $a_2^e$ are constants. After the necessary conditions are satisfied, $a_1^e$ and $a_2^e$ yield the following linear finite element approximation functions:

$$\Psi_i = \frac{z_j - z}{z_j - z_i} = \frac{z_j - z}{L_e}, \qquad \Psi_j = \frac{z - z_i}{z_j - z_i} = \frac{z - z_i}{L_e} \tag{14}$$

Here, $\Psi_{i,j}$ are also called finite element interpolation functions. $L_e$ is the length of corresponding element, i and j subscripts represent the left and right nodal coordinates of an finite element, respectively. The finite element approximation must satisfy the differential equations and appropriate BCs. Since there are two unknown parameters, we need two relations to determine their values. With substitution of the following approximate solutions:

$$u = u_i \Psi_i + u_j \Psi_j$$
$$\mathrm{N} = \mathrm{N}_i \Psi_i + \mathrm{N}_j \Psi_j \tag{15}$$

and their derivatives with respect to z,

$$\frac{du}{dz} = u' = u_i \Psi_i' + u_j \Psi_j'$$
$$\frac{du}{dz} = \mathrm{N}' = \mathrm{N}_i \Psi_i' + \mathrm{N}_j \Psi_j' \tag{16}$$

into the functional $I(\boldsymbol{v})$, the necessary algebraic equations will be obtained. After minimization of the functional, the element matrix of the considered problem can be derived.

$$\begin{bmatrix} (\rho A\omega^2)\frac{L}{3} & \frac{1}{2} & (\rho A\omega^2)\frac{L}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{L}{3EA} & -\frac{1}{2} & \frac{L}{6EA} \\ (\rho A\omega^2)\frac{L}{6} & -\frac{1}{2} & (\rho A\omega^2)\frac{L}{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{L}{6EA} & -\frac{1}{2} & \frac{L}{3EA} \end{bmatrix} \begin{bmatrix} u_i \\ N_i \\ u_j \\ N_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

As noticed that, presented element matrix can be used for the LV analysis of bars with different BCs.

## 3. Eigenvalue Problem for Free Vibration Analysis

The equation of motion of an undamped multi-degree of freedom (MDOF) system can be written as:

$$[\boldsymbol{M}]\ddot{\boldsymbol{u}} + [\boldsymbol{K}]\boldsymbol{u} = 0 \tag{18}$$

where $[K]$ denotes the system stiffness matrix and $[M]$ is the mass matrix of the system and $\boldsymbol{u}$ is a vector of generalized displacement coordinates.

Harmonic motion given by:

$$\boldsymbol{u} = \varphi e^{i\omega t} \tag{19}$$

may be substituted into Eq. (18) to give the algebraic eigenvalue problem as:

$$([\mathbf{K}] - \omega^2[\mathbf{M}])\{\boldsymbol{\varphi}\} = 0 \tag{20}$$

For there to be a nontrivial solution of Eq. (20), it is necessary that:

$$\det\|[\mathbf{K}] - \omega^2[\mathbf{M}]\| = 0 \tag{21}$$

This is called the characteristic equation. It results a polynomial equation in $\omega^2$ whose roots are the eigenvalues (or squared natural frequencies) after expanded form of the determinant of Eq. (21) is obtained. Corresponding to each eigenvalue, there will be an eigenvector, or natural mode.

## 4. Numerical Examples and Verification

In this section, three study cases examining the LV characteristics of a) clamped-free (C-F) bar b) clamped-clamped (C-C) bar and c) bar with a spring at the tip are considered to validate the efficiency and accuracy of the developed MFE formulation. The results of the first study case are compared with the results available in literature reported by Eisenberger [6] and the exact result. For the second study case, numerical results are compared with the exact result. In the last study case, natural longitudinal frequencies of vibration are calculated for different $k/_{EA}$ ratios and different orders of the mesh scheme, in other words, different number of elements of which the bar is artificially divided.

### 4.1 1st study case

In this example, the geometric properties of the bar are $L$ = 1 m and A = 1 m$^2$ and the material properties of the bar are E = 1 N/m$^2$ and $\rho$ = 1 kg/m$^3$. In the analysis, the bar is artificially divided into 1, 2, 5, 10, 25 and 50 equal elements and the convergence of the results for the natural frequency of LV for a C-F uniform bar is presented in Table 1 with the results of reference study in literature and the exact result. Exact result is calculated by considering the BCs of a C-F bar as: $u(0,t) = 0$ and $EA\partial u(L,t)/\partial z = 0$. The solution of the differential equation gives the fundamental equation of a C-F uniform bar, which corresponds to the characteristic equation for axial motion of a C-F bar as $\cos(\omega L / c_o) = 0$ where $c_o = \sqrt{E/\rho}$ in m/sec.

Based on comparison results, a good agreement can be observed between the results of the present and reference studies. In addition, the numerical results obtained using dense mesh also show good agreement with the reference data. For instance, the result of a beam artificially divided into 5 equal elements and exact result accurate to three decimal places. Increasing the number of elements (i.e., decrease the size of the elements) will lead to a more accurate solution. As can be seen from Table 1, the difference with the exact solution decreases as the number of elements increases. Also, there is more convergence with the values in Eisenberger [6].

**Table 1.** Convergence test of the natural frequency of LV for a C-F uniform bar.

| Number of Elements | Present | Eisenberger [6] | Exact Result |
|---|---|---|---|
| 1 | 1.5 | 1.73205 | 1.57080 |
| 2 | 1.567223 | 1.61142 | |
| 5 | 1.570708 | 1.57726 | |
| 10 | 1.570798 | 1.57241 | |
| 25 | 1.570788 | 1.57105 | |
| 50 | 1.570788 | 1.57086 | |

**Table 2.** Convergence test of the natural frequency of LV for a C-C uniform bar.

| Number of Elements | Present | Exact Result |
|---|---|---|
| 1 | 3 | 3.141592 |
| 2 | 3 | |
| 5 | 3.138740 | |
| 10 | 3.141417 | |
| 25 | 3.141596 | |
| 50 | 3.141588 | |

### 4.2 2nd study case

Natural frequency of LV for a C-C uniform bar is considered. Results are calculated using a uniform mesh with 1, 2, 5, 10, 25 and 50 elements for a bar. Notice that, the same geometrical and material properties used in the previous example are considered for this study case. Results are tabulated with the exact result in Table 2. Exact result is calculated by considering the BCs of a C-C bar as: $u(0,t)=0$ and $u(L,t)=0$. The solution of the differential equation gives the fundamental equation of a C-C uniform bar, which corresponds to the characteristic equation for axial motion of a C-C bar as $\sin(\omega L/c_o)=0$ where $c_o=\sqrt{E/\rho}$ in m/sec. The numerical results obtained using dense mesh (i.e., 10) also show good agreement with the exact result. They are accurate to nearly three decimal places. Comparison of Tables 1-2 verify the expected result that the natural frequency value of LV for a bar with two ends clamped two times bigger than a bar with one end clamped and the other end is free.

### 4.3 3rd study case

Figure 3 shows a uniform elastic bar of length $L$, cross-section $A$. Its mass per unit length is $\mu$ (in kg/m). The material of the bar with a spring (a spring constant is $k$ in N/m) at the tip has an elastic modulus $E$.

Without external forces, partial differential equation (PDE) of motion reduces to:

$$-\mu\frac{\partial^2 u}{\partial t^2}+EA\frac{\partial^2 u}{\partial z^2}=0 \tag{22}$$

for small values of elastic displacements along the bar. The solution of PDE is of the form:

$$u(z,t)=u\sin\omega t \tag{23}$$

Substituting Eq. (23) into Eq. (22) gives:

$$\omega^2\mu u+EA\frac{\partial^2 u}{\partial z^2}=0 \tag{24}$$

The general solution of Eq. (24) is:

$$u=C_1\cos\frac{\omega z}{L}+C_2\sin\frac{\omega z}{L} \tag{25}$$

The coefficient $C_1$ and $C_2$ are determined from satisfying the following BCs for the specific bar configuration. Eq. (25) is the fundamental equation for a uniform elastic bar and it contains the information on natural frequencies. After substituting the following BCs:

$$\begin{aligned} &u(0,t)=0 \\ &-ku(L,t)=EA\frac{\partial u(L,t)}{\partial z} \end{aligned} \tag{26}$$


**Fig. 3.** Schematic representation of a bar with a spring at the tip.

**Table 3.** Natural frequencies of LV for different *k*/*EA* ratios and number of elements.

| *k*/*EA* | | 0.0 | 0.1 | 0.3 | 0.5 | 1 | 10 | ∞ |
|---|---|---|---|---|---|---|---|---|
| Number of Elements | 10 | π/2 | 1.631988 | 1.741391 | 1.836584 | 2.028739 | 2.8626 | π |
| | 20 | π/2 | 1.632016 | 1.741387 | 1.836586 | 2.028745 | 2.8627 | π |
| | 40 | π/2 | 1.632016 | 1.741387 | 1.836586 | 2.028735 | 2.8627 | π |

the characteristic equation for axial motion of the bar with a spring at the tip is obtained:

$$\frac{k}{EA}\sin\omega + \frac{\omega}{L}\cos\omega = 0 \tag{27}$$

With the help of a computer program written in Fortran language, the analyses are performed for different *k*/*EA* ratios and number of elements. Results are tabulated in Table 3. All results given in Table 3 satisfy the exact solution of characteristic equation, see Eq. (27). The exact solution of Eq. (27) has already been presented by Kadıoğlu and Tekin [38] in literature. It should be noticed that the natural frequency values of LV for a bar with a spring at the tip are directly proportional to the *k*/*EA* ratio.

## 5. Conclusion

This current research intended to propose a new functional to analyze the LV of bars with arbitrary BCs. By using the Gâteaux differential method, it was easy to obtain a functional which includes all field equations and BCs of the problem. Mixed formulation produced by Gâteaux differential approach was more efficient and robust compared with the conventional variational principles such as Hellinger-Reissner and Hu-Washizu. The main characteristic of the MFE is that the displacement, internal force and frequency values can be obtained directly. A mathematical operation like back substitution process was not required. The efficiency of the suggested solution procedure demonstrated with a high accuracy achieved in the numerical examples which were solved in literature. Also it was noticed that, the proposed method can produce satisfactory results even if dense mesh employed in the solution. Uniqueness and the theoretical contributions of this research to existing literature can be summarized as:

- Fine meshing is often a time-consuming. This method still gives satisfactory results with using dense mesh and saving time.
- A new functional is proposed for LV analysis of bars with different end restraints through a systematic procedure called the Gâteaux Differential. Two independent variables in addition to geometric and dynamic BCs of the problem included in this functional.
- A special MFE program is developed. Due to the existence of first derivatives of the variables in the functional, the element shape (interpolation) function must satisfy only $C^0$ class continuity.
- The developed MFE is capable of predicting displacements and internal forces directly. A mathematical operation like back substitution process is not required.
- The LV analysis of bars are investigated for arbitrary BCs, such as C-F, C-C and a bar with a spring at the tip. The natural frequencies of LV are obtained by the proposed MFE formulation. The natural frequency value of LV for a C-C bar is nearly two times of a C-F bar, as expected. Moreover, the natural frequency values of LV for a bar with a spring at the tip are directly proportional to the *k*/*EA* ratio. Results are in good agreement with the literature for different *k*/*EA* ratios.
- The approach introduced can be applied for different types of rods like curved and helical rods. Following the described methodology, some of these problems are under study.
- It is expected that the results reported in this study may serve as a benchmark for future studies.

## Author Contributions

G. Tekin and F. Kadıoğlu planned the scheme, developed the mathematical modeling, coded computer program in Fortran language, examined the theory validation, wrote original draft, reviewed and editing the manuscript; S. Ecer analyzed the results using coded computer program. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.


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## Conflict of Interest

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## Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

## Nomenclature

| | | | |
|---|---|---|---|
| A | Area of cross-section | **M** | Mass matrix |
| BCs | Boundary conditions | MFE | Mixed finite element |
| DOF | Degree of freedom | *N* | Normal force |
| *E* | Modulus of elasticity | **Q** | Continuous operator |
| ***f*** | Load vector | *u* | Longitudinal displacement |
| *I* | Functional | ***v*** | Unknown vectors |

| | | | |
|---|---|---|---|
| $k$ | Spring constant | $\rho$ | Density of the rod |
| $\mathbf{K}$ | Stiffness matrix | $\Psi$ | Interpolation (Shape) function |
| $L$ | Length of a bar | t | Time |
| $L_e$ | Length of an element | $\omega$ | Circular natural frequency |
| LV | Longitudinal vibration | $\mu$ | The mass per unit length |
| $\mathbf{L}$ | Coefficient matrix | | |

## References


[1] Bishop, R.E.D., Johnson, D.C., *The Mechanics of Vibration (1st ed.)*, Cambridge University Press, 1960.
[2] Timoshenko, S., *Vibration Problems in Engineering (2nd ed.)*, D. Van Nostrand Company, Inc., 1937.
[3] Strutt, J.W., Rayleigh, B., *The Theory of Sound*, Macmillan and Co., London, 1877.
[4] Kohoutek, R., Natural Longitudinal Frequencies of a Uniform Rod with a Tip Mass or a Spring, *Journal of Sound and Vibration*, 77(1), 1981, 147-148.
[5] Singh, R., Lyons, W.M., Prater, Jr., Complex Eigensolution for Longitudinally Vibrating Bars with a Viscously Damped Boundary, *Journal of Sound and Vibration*, 133(2), 1989, 364-367.
[6] Eisenberger, M., Exact Longitudinal Vibration Frequencies of a Variable Cross-section Rod, *Applied Acoustics*, 34(2), 1991, 123-130.
[7] Hull, A.J., A Closed Form Solution of a Longitudinal Bar with a Viscous Boundary Condition, *Journal of Sound and Vibration*, 169(1), 1994, 19-28.
[8] Rossit, C.A., Bambill, D.V., Laura, P.A.A., Longitudinal Vibrations of a Prismatic Bar Suddenly Subjected to a Tensile Load at One End when the Other is Elastically Restrained, *Journal of Sound and Vibration*, 188(1), 1996, 145-148.
[9] Kumar, B.M., Sujith, R.I., Exact Solutions for the Longitudınal Vibration of Non-Uniform Rods, *Journal of Sound and Vibration*, 207(5), 1997, 721-729.
[10] Li, Q.S., Exact Solutions for Free Longitudinal Vibrations of Non-Uniform Rods, *Journal of Sound and Vibration*, 234(1), 2000, 1-19.
[11] Li, Q.S., Free Longitudinal Vibration Analysis of Multi-Step Non-Uniform Bars Based on Piecewise Analytical Solutions, *Engineering Structures*, 22(9), 2000, 1205-1215.
[12] Yardımoğlu, B., Exact Solutions for the Longitudinal Vibration of Non-Uniform Rods, *Journal of Sound and Vibration*, 329, 2010, 4107-4107.
[13] Yardımoğlu, B. Aydın, L., Exact Longitudinal Vibration Characteristics of Rods with Variable Cross-Sections, *Shock and Vibration*, 18, 2011, 555-562.
[14] Numanoglu, H.M., Akgoz, B., Civalek, O., On Dynamic Analysis of Nanorods, *International Journal of Engineering Science*, 130, 2018, 33–50.
[15] Polit, O., Merzouki, T., Ganapathi, M., Elastic Stability of Curved Nanobeam Based on Higher-order Shear Deformation Theory and Nonlocal Analysis by Finite Element Approach, *Finite Elements in Analysis and Design*, 146, 2018, 1–15.
[16] Singh, K.V., Li, G., Buckling of Functionally Graded and Elastically Restrained Non-uniform Columns, *Composites Part B: Engineering*, 40(5), 2009, 393-403.
[17] Zenkour, A.M., Allam, M.N.M., Sobhy, M., Bending Analysis of FG Viscoelastic Sandwich Beams with Elastic Cores Resting on Pasternak's Elastic Foundations, *Acta Mechanica*, 212(3), 2010, 233-252.
[18] Shahba, A., Atternejad, R., Tavanaie Marvi, M., Hajilar, S., Free Vibration of Stability Analysis of Axially Functionally Graded Tapered Timoshenko Beams with Classical and Non-Classical Boundary Conditions, *Composites Part B: Engineering*, 42(4), 2011, 801-808.
[19] Sobhy, M., Zenkour, A.M., Magnetic Field Effect on Thermomechanical Buckling and Vibration of Viscoelastic Sandwich Nanobeams with CNT Reinforced Face Sheets on A Viscoelastic Substrate, *Composites Part B: Engineering*, 154, 2018, 492-506.
[20] Udwadia, F. E., On the Longitudinal Vibrations of a Bar with Viscous Boundaries: Super-stability, super-instability, and loss of damping, *International Journal of Engineering Science*, 50(1), 2012, 79-100.
[21] Jovanovic, V., A Fourier Series Solution for the Longitudinal Vibrations of a Bar with Viscous Boundary Conditions at Each End, *Journal of Engineering Mathematics*,79, 2013, 125-142.
[22] Gan, C., Wei, Y., Yang, S., Longitudinal Wave Propagation in a Rod with Variable Cross-section, *Journal of Sound and Vibration*, 33(2), 2014, 434–445.
[23] Nunes, A. W., Silva, S. da, Ruiz , A., Exact General Solutions for the Mode Shapes of Longitudinally Vibrating Non-Uniform Rods via Lie Symmetries, *Journal of Sound and Vibration*, 538, 2022,117216.
[24] Krawczuk, M., Palacz, M., An Optimal Form of the Finite Element Mass Matrix in the Analysis of Longitudinal Vibrations of Rods, *Finite Elements in Analysis and Design*, 207, 2022, 103763.
[25] Peck, A.J., Shatalov, M.Y. Joubert, S.V., Longitudinal Vibrations of a Lightly Damped Viscoelastic Rod Growing in both Lateral and Axial Directions, *Applied Mathematical Modelling*, 101, 2022, 332–352.
[26] Liu, X., Zhao, Y., Zhou, W., Banerjee, J.R., Dynamic Stiffness Method for Exact Longitudinal Free Vibration of Rods and Trusses Using Simple and Advanced Theories, *Applied Mathematical Modelling*, 104, 2022, 401–420.
[27] Demir, E., Longitudinal Vibration of Temperature Dependent Bar with Variable Cross-section, *Journal of Polytechnic*, 21(4), 2018, 813-819.
[28] Mei, C., Comparison of the Four Rod Theories of Longitudinally Vibrating Rods, *Journal of Vibration and Control*, 21(8), 2015, 1639–1656.
[29] Xu, D., Du, J., Liu, Z., An Accurate and Efficient Series Solution for the Longitudinal Vibration of Elastically Restrained Rods with Arbitrarily Variable Cross sections, *Journal of Low Frequency Noise, Vibration and Active Control*, 38(2), 2019, 403–414.
[30] Turdibekov, J., Aliyev, A., *On Longitudinal Vibrations of a Round Rod Considering Rotatory Inertia*, Easy Chair preprints, 2022.
[31] Utyashev, I.M., Determination of the Variable Density of the Rod from Natural Frequencies of Longitudinal Vibrations, *Journal of Physics: Conference Series*, 2099, 2021, 012049.
[32] Pala, Y., Kahya, Ç., A Method Based on Riccati Equation for the Vibration Analysis of Rods with Variable Cross-Sections, *International Journal of Structural Stability and Dynamics*, 22(11), 2022, 2250123.
[33] Kadıoğlu, F., Iyidogan, C., Free Vibration of Laminated Composite Curved Beams using Mixed Finite Element Formulation, *Science and Engineering of Composite Materials*,16 (4), 2009, 247-257.
[34] Özütok, A., Madenci, E., Free Vibration Analysis of Cross-ply Laminated Composite Beams by using Mixed Finite Element Formulation, *International Journal of Structural Stability and Dynamics*, 13(2), 2013, 1250056.
[35] Özütok, A., Madenci, E., Kadıoğlu, F., Free Vibration Analysis of Angle-ply Laminate Composite Beams by Mixed Finite Element Formulation using the Gâteaux Differential, *Science and Engineering of Composite Materials*, 21(2), 2014, 257-266.
[36] Özkan, G., Tekin, G., Kadıoğlu, F., An Alternative Method for Analyzing Buckling of Laminated Composite Beams, *Građevinar*, 69(9),2017, 841-850.
[37] Oden, J.T., Reddy, J.N., *Variational Methods in Theoretical Mechanics (2nd ed.)*, Springer-Verlag, Berlin, 1983.
[38] Kadıoğlu, F., Tekin, G., Natural Longitudinal Frequencies of a Uniform Rod with A Tip Mass or A Spring, *Journal of Sound and Vibration*, 371, 2016, 446-447.


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