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Research Paper

Exact Solutions for Isobaric Inhomogeneous Couette Flows of a Vertically Swirling Fluid

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Abstract. The paper generalizes the partial class of exact solutions to the Navier–Stokes equations. The proposed exact solution describes an inhomogeneous three-dimensional shear flow in a layer of a viscous incompressible fluid. The solution is studied for the case of the motion of a steady-state isobaric fluid. One of the longitudinal velocity components is represented by an arbitrary-degree polynomial. The other longitudinal velocity vector component is described by the Couette profile. For a particular case (the quadratic dependence of the velocity field on two coordinates), profiles of the obtained exact solution are constructed, which illustrate the existence of counterflows in the fluid layer. The components of the vorticity vector and the tangential stresses are analyzed for this exact solution.

Keywords: Exact solution, isobaric flow, vorticity, counterflow, stagnation point.

1. Introduction

Some of the first exact solutions to the Navier–Stokes equations described isobaric flows of a viscous incompressible fluid. The isobaric unidirectional flows of a viscous incompressible fluid include Couette flow [1-4] and two Stokes problems [2-5].

Plane isobaric motions began to be studied by Berker in [6, 7]. That study discussed many particular exact solutions to the Navier–Stokes equations for fluids moving under constant pressure. The main difficulty one may face when studying plane isobaric flows lies in that the system consisting of the momentum conservation law and the incompressibility equation is overdetermined. Even the pioneering study [6, 7] attempted to establish the condition of the solvability of the overdetermined Navier–Stokes system describing plane isobaric flows of incompressible fluids. This problem was solved in [8-10] and summarized in [11, 12] by Shmyglevsky, who seems to have formulated it independently of Berker [6, 7].

The further studies by Berker and Shmyglevsky were extended to three-dimensional stratified (shear) flows of a viscous incompressible fluid. In this case, the fluid flow is determined by two velocity components, which depend on time and three coordinates [3, 13-15]. It was shown in [16, 17] that there is an exact solution to the overdetermined Navier–Stokes equation system, which belongs to the class of velocities linearly dependent on two horizontal (longitudinal) coordinates and describes a vertical vortex flow ignoring rotation of a fluid as a solid-body.

Generalizations of Couette flow, induced by inhomogeneous velocity distribution or tangential stresses at the upper boundary of a fluid layer under constant pressure, were studied in [16-20]. It was shown that the obtained exact solutions describe amplification of velocities and generation of counterflows for steady-state and nonstationary flows of a viscous incompressible fluid [16-20]. Solutions with these properties could be useful in the description of equatorial counterflows if the effect of the first Coriolis parameter on the structure of the hydrodynamic flow is neglected [16, 18-20]. Thus, it is undoubtedly relevant to find new classes of exact solutions for three dimensional shear flows of viscous incompressible fluids. Note that classes of exact solutions for the Navier–Stokes equations with a velocity field nonlinearly dependent on some coordinates were constructed in [21]. The aim of this paper is to solve a boundary value problem for studying Couette flow, which is described by the velocity field quadratically dependent on the horizontal (longitudinal) coordinates.

2. An Exact Solution for the Velocity Field Varying Nonlinearly with Respect to Some Coordinates

Shear flows of a viscous incompressible fluid are described by the overdetermined system consisting of the Navier–Stokes equations and the continuity equation [16-20]:

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$$\frac{d\mathbf{V}}{dt} = -\nabla P + \nu \Delta \mathbf{V} \tag{1}$$

$$\nabla \cdot \mathbf{V} = 0 \tag{2}$$

Here, $\mathbf{V}(x,y,z,t) = (V_x, V_y, 0)$, is the fluid velocity vector; P is the pressure taken relative to constant fluid density ρ (with absence of external volumetric forces acting in a fluid); ν is the kinematic viscosity of the fluid; $\nabla = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z$ is the Hamilton operator, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator.

In what follows, we study isobaric flows of a viscous incompressible fluid. Thus the system of equations (1) and (2) in a rectangular Cartesian coordinate system is written as:

$$\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_x \frac{\partial V_x}{\partial y} = \nu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)$$

$$\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = \nu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right) \tag{3}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

Equation (3) is overdetermined since two functions, V_x and V_y, are determined from a system of three equations. The case of plane motions for the velocity field $(V_x(x,y,0); V_y(x,y,0))$ was first studied by Berker [6, 7], who presented some particular solutions to an overdetermined (3). Solutions to overdetermined systems describing nonstationary plane flows of a viscous incompressible fluid under constant pressure were also discussed by Shmyglevsky [8, 11, 12].

Strictly speaking, system (3) describes not only isobaric flows, but also incompressible fluid motions for which the potential energy of internal forces of fluid pressure is balanced by the potential energy of external forces:

$$P = G \tag{4}$$

where G is the potential of external forces. The results expounded below will be true when condition (4) is met.

The first nontrivial exact solutions to system (3) were constructed in the Lin–Aristov–Sidorov class in [2, 3, 13, 14, 16-20]:

$$V_x = U(z,t) + u(z,t)y,$$
$$V_y = V(z,t) \tag{5}$$

Solution (5) ensures the solvability of system (3); by the rotation transformation:

$$U_1 = u\cos\theta\sin\theta; \quad U_2 = u\cos^2\theta;$$
$$V_1 = -u\sin^2\theta; \quad V_2 = -u\cos\theta\sin\theta; \tag{6}$$

it can yield:

$$V_x = U_0(z,t) + U_1(z,t)x + U_2(z,t)y;$$
$$V_y = V_0(z,t) + V_1(z,t)x + V_2(z,t)y. \tag{7}$$

Here θ is an arbitrary constant and the function u satisfies the simplest parabolic equation of the dimension (1+1) [16-20]:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}$$

which has the well-known Fourier-type solution [22].

Solution (5) is obtained by substituting the value $\theta = 0$ into expressions (6). Note the following: exact solution (5) does not describe the motion of a fluid as a solid-body, but the vertical vorticity component:

$$\Omega = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = -u(z,t)$$

is nonzero. Thus the asymmetric distortion in Ekman's exact solution [23] also leads to an exact solution describing vertical vortex fluid flow.

For the solution class (7), system (3) is reduced to a simpler system, inheriting the nonlinear properties of system (3). Thus solution (5) can be generalized by writing the velocity field as:

$$V_x = F(y,z,t),$$
$$V_y = V(z,t) \tag{8}$$

Exact solution (8) identically satisfies the continuity equation in (3), and the velocities V_x and V_y are calculated from the following system of parabolic equations:

$$\frac{\partial F}{\partial t} + V\frac{\partial F}{\partial y} = \nu\left(\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\right) \tag{9}$$

$$\frac{\partial V}{\partial t} = \nu\frac{\partial^2 V}{\partial z^2} \tag{10}$$

The system of equations (9), (10) is loosely coupled since the velocity F is computed after the integration of the isolated heat-conduction-type Eq. (10) for the function V. Then after wards, we will not discuss finding a general solution to system (9), (10), but we will demonstrate a new particular solution to the equations, which satisfies system (3).

Since the functions determined by expression (8) are independent of the x-coordinate, it is possible to construct arbitrarily complex polynomial solutions generalizing the linear coordinate dependence of velocities. Consequently, for the convenience of the further discussion, the velocities can be represented as follows:

$$\begin{gathered} V_x = \sum_{n=0}^{N} F_n \frac{y^n}{n!} = F_0 + F_1 y + F_2\frac{y^2}{2} + \sum_{n=3}^{N} F_n\frac{y^n}{n!} \\ V_y = V(z,t) \end{gathered} \tag{11}$$

Here, $F_i = F_i(z,t)$, $(i = \overline{0,N})$. The class of equations (11) generalizes the known family of exact solutions, where velocities linearly depend on some of the coordinates [13]. Using the rotation transformation (6) for the functions forming the velocity field (11), it is possible to *replicate* the class of solutions for the Navier–Stokes equations describing the flow of a viscous incompressible fluid.

3. Stationary Flow of a Viscous Incompressible Fluid with a Velocity Specified by a Quadratic Form along One of the Horizontal Coordinates

Next, we study the steady-state motion of a viscous incompressible fluid. In this case, all the functions V, F_i $(i = \overline{1,N})$ depend only on the z-coordinate. Substituting the stationary case of solution (11) into system (9), (10), we have:

$$\begin{gathered} V\left(F_1 + F_2 y + \sum_{n=3}^{N} F_n \frac{y^{n-1}}{(n-1)!}\right) = \nu\left(F_2 + \sum_{n=3}^{N} F_n\frac{y^{n-2}}{(n-2)!} + \frac{d^2F_0}{dz^2} + y\frac{d^2F_1}{dz^2} + \frac{y^2}{2}\frac{d^2F_2}{dz^2} + \sum_{n=3}^{N}\frac{d^2F_n}{dz^2}\frac{y^n}{n!}\right) \\ \frac{d^2V}{dz^2} = 0. \end{gathered} \tag{12}$$

Note that system of equations (9), (10) is equivalent to system (3) in view of the assumed form of solutions (8). Using the method of undetermined coefficients (the linear independence of the finite basis of the functions $\{1, y, y^2, \ldots, y^N\}$), we write separately the constant terms and the coefficients which can be found at the powers of the horizontal (longitudinal) y-coordinate. Thus, the system of ordinary differential equations is written in the order of integration as:

$$\begin{gathered} \frac{d^2V}{dz^2} = 0 \quad \frac{d^2F_N}{dz^2} = 0 \quad \nu\frac{d^2F_{N-1}}{dz^2} = VF_N \\ \nu\frac{d^2F_{n-1}}{dz^2} = VF_n - \nu F_{n+1} \quad n = \overline{1,N-1}; \end{gathered} \tag{13}$$

System (13) has the general exact polynomial solution, which can be obtained by successive integration of the system equations.

The form of the solution for velocity V and for several functions F_i, coefficients in polynomial (11), which will be further required to illustrate the characteristic properties of fluid flow, are written as follows:

$$\begin{gathered} V = zc_1 + c_2; \quad F_N = zc_3 + c_4; \\ F_{N-1} = \frac{z^4}{12\nu}c_1c_3 + \frac{z^3}{6\nu}(c_1c_4 + c_2c_3) + \frac{z^2}{2\nu}c_2c_4 + zc_5 + c_6 \\ F_{N-2} = \frac{z^7}{504\nu^2}(c_1)^2 c_3 + \frac{z^6}{360\nu^2}\left[c_1c_2c_3 + 2c_1(c_1c_4 + c_2c_3)\right] + \frac{z^5}{120\nu^2}\left[c_2(c_1c_4 + c_2c_3) + 3c_1c_2c_4\right] + \frac{z^4}{24\nu}\left[\frac{1}{\nu}(c_2)^2 c_4 + 2c_1c_5\right] + \\ + \frac{z^3}{6}\left[\frac{1}{\nu}(c_1c_6 + c_2c_5) - c_3\right] + \frac{z^2}{2}\left[\frac{1}{\nu}c_2c_6 - c_4\right] + zc_7 + c_8 \end{gathered} \tag{14}$$

The next functions F_i $(i = N-3,\ldots,0)$, the coefficients of polynomial (8), are computed by the same way.

Note that steady-state flows are rarely implemented in the description of engineering and natural problems. The consideration of a new exact solution (1) and (3) for steady flows is necessary for comparison with the Couette profile [1] and the exact solutions published in [16, 18, 20]. The study of unsteady flows (11) and their hydrodynamic stability is of great interest, but requires writing a separate scientific work.

4. The Boundary Value Problem of the Generalized Inhomogeneous Couette Flow

To illustrate the general solution (3), we consider the boundary value problem describing fluid flow in an infinitely long horizontal layer with a thickness h. We set $N=2$ in the expressions of the assumed form of solution (11); system (13) then becomes:

$$\frac{d^2V}{dz^2}=0 \quad \frac{d^2F_2}{dz^2}=0 \quad \nu\frac{d^2F_1}{dz^2}=VF_2$$
$$\nu\frac{d^2F_0}{dz^2}=VF_1-\nu F_2 \tag{15}$$

The boundary conditions for finding the integration constants c_i, $(i=\overline{1,8})$ are determined as follows. At the lower rigid boundary specified by the plane equation $z=0$, the following no-slip condition is met:

$$V_x=V_y=0. \tag{16}$$

At the upper rigid boundary specified by the plane equation $z=h$, the velocity components are determined as follows:

$$V_x=W\cos\varphi+Ay+B\frac{y^2}{2};$$
$$V_y=W\sin\varphi. \tag{17}$$

Here, the constant W characterizes the background value of velocity on the fluid layer surface; the angle φ is the direction of this velocity vector relative to the chosen coordinate system; the constant A characterizes the background value of spatial acceleration along the Ox -axis; the constant B determines the rate of change of spatial acceleration on the fluid surface along the Oy -axis.

Thus, in view of the assumed form of solutions (11) at $N=2$, the boundary conditions for the required functions of system (13) become:

$$F_0(0)=F_1(0)=F_2(0)=0; \quad V(0)=0;$$
$$F_0(h)=W\cos\varphi;\ F_1(h)=A\ ;F_2(h)=B;\ V=W\sin\varphi. \tag{18}$$

Therefore, the integration constants in the general solution (14) have the form

$$c_1=\frac{W\sin\varphi}{h} \qquad c_3=\frac{B}{h} \qquad c_5=\frac{A}{h}-\frac{WBh\sin\varphi}{12\nu}$$

$$c_7=\frac{W\cos\varphi}{h}+\frac{Bh}{6}-\frac{AWh\sin\varphi}{12\nu}+\frac{5W^2h^3B\sin^2\varphi}{1008\nu^2}$$

$$c_2=c_4=c_6=c_8=0$$

The exact solution of the boundary value problem including system (3) with regard to the assumed form of solutions (11) at $N=2$ and boundary conditions (18) are as follows:

$$V_x=\left(\frac{z}{h}\right)^7\frac{W^2h^4B\sin^2\varphi}{504\nu^2}+\left(\frac{z}{h}\right)^4\left(\frac{AWh^2\sin\varphi}{12\nu}-\frac{W^2Bh^4\sin^2\varphi}{144\nu^2}\right)-\frac{Bh^2}{6}\left(\frac{z}{h}\right)^3+\frac{z}{h}\left(W\cos\varphi+\frac{Bh^2}{6}-\frac{AWh^2\sin\varphi}{12\nu}+\frac{5W^2h^4B\sin^2\varphi}{1008\nu^2}\right)+$$
$$+y\left[\frac{WBh^2\sin\varphi}{12\nu}\left(\frac{z}{h}\right)^4+\left(A-\frac{WBh^2\sin\varphi}{12\nu}\right)\frac{z}{h}\right]+\frac{y^2}{2}B\frac{z}{h}$$
$$V_y=W\sin\varphi\frac{z}{h}. \tag{19}$$

Note some particular cases of the exact solution of the boundary value problem represented by expressions (15) - (17).

Suppose $A=B=0$, then the velocity components have the form $V_y=W\sin\varphi z/h$ and $V_y=W\sin\varphi z/h$, and they describe the classical stratified Couette flow [1], which is reducible to unidirectional flow [16].

When $B=0$, the velocity at the upper boundary describes parabolic wind [16], which is used to solve problems of oceanology and aerophysics. In the fluid flow we obtain the inhomogeneous Couette flow, studied in [16-20].

When $A=0$, the velocity components change only slightly, all the characteristic features of the polynomial behavior of the function being preserved.

If we set $\varphi=0$ in the expressions of boundary conditions (17), the expressions for velocity components (19) at the upper boundary of the layer are set as $V_x=W+Ay+By^2/2$, $V_y=0$, and they describe unidirectional fluid motion. Solution (19) will in this case describe the flow characterized by a cubic profile with respect to the z -coordinate and a quadratic with respect to the y -coordinate:

$$V_x=-\frac{Bh^2}{6}\left(\frac{z}{h}\right)^3+\left(W+\frac{Bh^2}{6}\right)\frac{z}{h}+\frac{y^2}{2}B\frac{z}{h} \tag{20}$$

The velocity V_x from expression (20) can be rewritten as:

$$V_x = \left[\frac{Bh^2}{6}\left\{1-\left(\frac{z}{h}\right)^2\right\} + W + \frac{y^2}{2}B\right]\frac{z}{h} \tag{21}$$

Hence it follows that, when $W = 0$, counterflows may appear in a fluid only if $y \neq 0$ since it is only in this case that the inequality $V_x(0)V_x(h) < 0$ can hold, and this indicates the existence of one root of the function V_x on the interval $z \in (0;h)$. If $W \neq 0$, the number of zeros in the function V_x, and hence stagnation points, can vary from zero to two, depending on the sign of W.

5. Analysis of the Dimensionless Solution

Let us now reduce the obtained solution (19) to a dimensionless form and analyze it:

$$V_X = Z^7\frac{\delta^4 a\text{Ta}\text{Re}\sin^2\varphi}{1008} + Z^4\left(\frac{\delta^2\text{Ta}\sin\varphi}{24} - \frac{\delta^4 a\text{Ta}\text{Re}\sin^2\varphi}{288}\right) - Z^3\frac{a\text{Ta}\delta^2}{12\text{Re}} + Z\left(\cos\varphi + \frac{a\text{Ta}\delta^2}{12\text{Re}} - \frac{\delta^2\text{Ta}\sin\varphi}{24} + \frac{5\delta^4 a\text{Ta}\text{Re}\sin^2\varphi}{2016}\right) +$$
$$+ Y\left[\frac{a\text{Ta}\delta^2\sin\varphi}{24}Z^4 + \left(\frac{\text{Ta}}{2\text{Re}} - \frac{a\text{Ta}\delta^2\sin\varphi}{24}\right)Z\right] + Y^2\frac{a\text{Ta}}{2\text{Re}}Z \tag{22}$$

$$V_Y = Z\sin\varphi.$$

Here, $Z = z/h$, $Y = y/l$ is a dimensionless variable; h and l are typical scales along z and y, respectively; $\delta = h/l$ is the parameter of geometric anisotropy of the layer of a viscous incompressible fluid; $\text{Re} = Wl/\nu$ is the Reynolds number; $\text{Ta} = 2Al^2/\nu$ is the modified Taylor number; $a = Bl/A$.

By virtue of representation (11), the initial flow of a viscous incompressible fluid is a nonlinear superposition of streams with the velocities F_0, $F_1 y$, and $F_2 y^2/2$. To analyze a stream, we study the properties of the polynomials F_0, F_1, and F_2:

$$F_0 = Z^7\frac{\delta^4 a\text{Ta}\text{Re}\sin^2\varphi}{1008} + Z^4\left(\frac{\delta^2\text{Ta}\sin\varphi}{24} - \frac{\delta^4 a\text{Ta}\text{Re}\sin^2\varphi}{288}\right) - Z^3\frac{a\text{Ta}\delta^2}{12\text{Re}} + Z\left(\cos\varphi + \frac{a\text{Ta}\delta^2}{12\text{Re}} - \frac{\delta^2\text{Ta}\sin\varphi}{24} + \frac{5\delta^4 a\text{Ta}\text{Re}\sin^2\varphi}{2016}\right)$$
$$F_1 = \frac{a\text{Ta}\delta^2\sin\varphi}{24}Z^4 + \left(\frac{\text{Ta}}{2\text{Re}} - \frac{a\text{Ta}\delta^2\sin\varphi}{24}\right)Z \quad F_2 = \frac{a\text{Ta}}{\text{Re}}Z \tag{23}$$

The function F_2 is monotonic (increasing or decreasing, depending on the sign of the coefficient $a\text{Ta}/\text{Re}$), passing through the origin in view of the assumed no-slip condition (18).

We write the function F_1 as:

$$F_1 = Z \cdot f_1(Z)$$

where

$$f_1(Z) = \frac{a\text{Ta}\delta^2\sin\varphi}{24}Z^3 + \frac{\text{Ta}}{2\text{Re}} - \frac{a\text{Ta}\delta^2\sin\varphi}{24}.$$

On the interval $Z \in (0;1)$, the function $f_1(Z)$ can have one zero point (stratification) or have none. The condition for the existence of a stagnation point on the interval $(0;1)$ is the fulfilment of the following inequality [16]:

$$0 < \frac{12}{a\text{Re}\delta^2\sin\varphi} < 1.$$

A similar estimate for the case $N = 1$ in formula (11) was made earlier in [16].

When $y = 0$, the velocity component is as follows: $V_x = F_0$. The function F_0 is represented in the multiplicative form:

$$F_0 = \frac{1008}{\delta^4 a\text{Ta}\text{Re}\sin^2\varphi}Z \cdot f_2(Z)$$

where the polynomial $f_2(Z)$ can be represented as

$$f_2(Z) = Z^2\left(Z^4 + \alpha Z - \beta\right) + \gamma = 0 \tag{24}$$

Here, the constant coefficients are computed as follows:

$$\alpha = \frac{42}{a\delta^2\text{Re}\sin\varphi} - \frac{7}{2}; \quad \beta = \frac{84}{\delta^2\text{Re}^2\sin^2\varphi};$$

$$\gamma = \frac{42}{\delta^2\text{Re}\sin\varphi}\left[\frac{24\cos\varphi}{a\delta^2\text{Ta}\sin\varphi} + \frac{2}{\text{Re}\sin\varphi} - \frac{1}{a}\right] + \frac{5}{2}$$

When $\gamma = 0$, the study of the number of roots of this equations on the interval $(0;1)$ is reducible to studying the roots of the polynomial $Z^4 + \alpha Z - \beta$ or the equality of the functions $f_3(Z)$ and $f_4(Z)$, where $f_3(Z) = Z^4$ and $f_4(Z) = \beta - \alpha Z$. This equality can hold at no more than two points on the interval $(0;1)$.

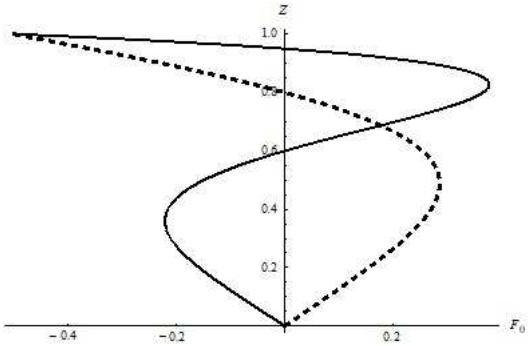

Fig. 1. The profile of the function $F_0(Z)$ (solid curve – 2 roots; dashed curve – 1 root).

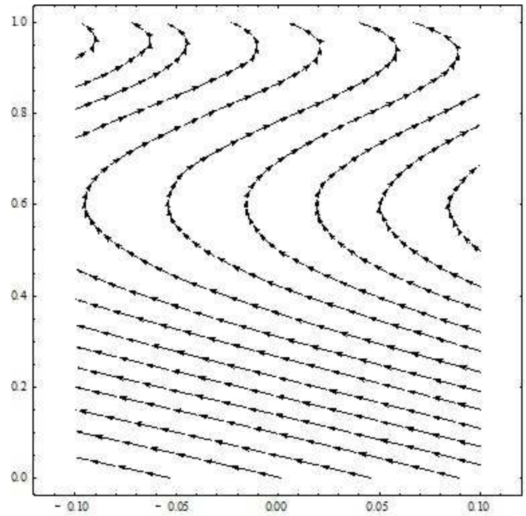

Fig. 2. Streamlines for the cases of two roots of the function F_0 and one root of the function F_1.

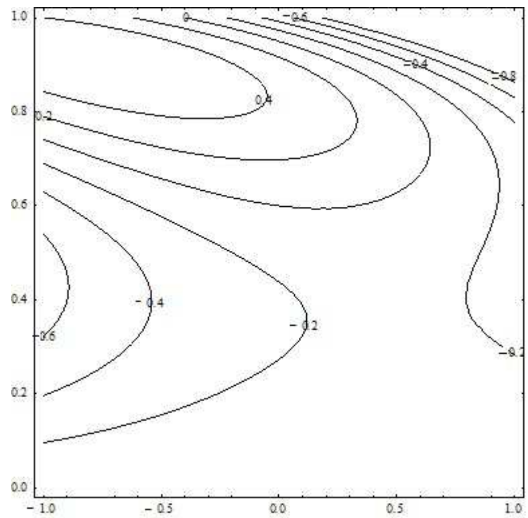

Fig. 3. Isolines of the velocity component V_x for the cases of two roots of the function F_0 and one root of the function F_1.

When $\gamma \neq 0$, the function F_0 can have zero to two real roots on the interval $(0;1)$. Figure 1 shows the profile of the function F_0 for the cases of two and one roots. Consequently, in the bulk of the fluid layer there can exist up to two stationary points, i.e. points with zero velocities.

Figure 2 shows streamlines for the case of two roots of the function F_0 and one root of the function F_1. The counterflows are represented in the figures by a change in the stream direction. Figure 2 depicts the case of fluid inflow to the upper boundary.

6. Analysis of the Vorticity Vector and Tangential Stresses

For the obtained solution (22), we consider the vorticity vector $\Omega = (\Omega_X, \Omega_Y, \Omega_Z)$, where:

$$\Omega_X = -\frac{\partial V_Y}{\partial Z} \quad \Omega_Y = \frac{\partial V_X}{\partial Z} \quad \Omega_Z = \frac{\partial V_Y}{\partial X} - \frac{\partial V_X}{\partial Y} \tag{25}$$

The substitution of velocity components (22) into formulas (25) yields:

$$\Omega_X = \sin\varphi;$$

$$\Omega_Y = Z^6 \frac{\delta^4 a \mathrm{Ta}\,\mathrm{Re}\sin^2\varphi}{144} + Z^3\left(\frac{\delta^2 \mathrm{Ta}\sin\varphi}{6} - \frac{\delta^4 a\mathrm{Ta}\,\mathrm{Re}\sin^2\varphi}{72}\right) - Z^2\frac{a\mathrm{Ta}\delta^2}{4\mathrm{Re}} + \left(\cos\varphi + \frac{a\mathrm{Ta}\delta^2}{12\mathrm{Re}} - \frac{\delta^2\mathrm{Ta}\sin\varphi}{24} + \frac{5\delta^4 a\mathrm{Ta}\,\mathrm{Re}\sin^2\varphi}{2016}\right) + $$
$$+ Y\left[\frac{a\mathrm{Ta}\delta^2\sin\varphi}{6}Z^3 + \left(\frac{\mathrm{Ta}}{2\mathrm{Re}} - \frac{a\mathrm{Ta}\delta^2\sin\varphi}{24}\right)\right] + Y^2\frac{a\mathrm{Ta}}{2\mathrm{Re}} \tag{26}$$

$$\Omega_Z = -\left[\frac{a\mathrm{Ta}\delta^2\sin\varphi}{24}Z^4 + \left(\frac{\mathrm{Ta}}{2\mathrm{Re}} - \frac{a\mathrm{Ta}\delta^2\sin\varphi}{24}\right)Z\right] - \frac{a\mathrm{Ta}}{\mathrm{Re}}ZY$$

The homogeneous component Ω_X assumes positive values at $\varphi \in (0;\pi)$ and negative values at $\varphi \in (\pi;2\pi)$.

The analysis of the longitudinal component Ω_Y of the vortex function is absolutely similar to the analysis of the function $f_2(Z)$ represented by expression (24). Consequently, it can be concluded that the function Ω_Y in the fluid layer $Z \in (0;1)$ can change sign up to three times.

To analyze vertical vorticity, we represent the function Ω_Z as:

$$\Omega_Z = -Z\left(\frac{a\mathrm{Ta}\delta^2\sin\varphi}{24}Z^3 + \frac{\mathrm{Ta}}{2\mathrm{Re}} - \frac{a\mathrm{Ta}\delta^2\sin\varphi}{24} - \frac{a\mathrm{Ta}}{\mathrm{Re}}Y\right)$$

Obviously, the vertical vorticity function Ω_Z can change sign at most at one point of the fluid layer or remain constant.

The vorticity vector components are related to the tangential stress tensor components as:

$$\Omega_X = -\tau_{YZ} \quad \Omega_Y = \tau_{XZ} \tag{27}$$

It can be inferred from expressions (27) and the analysis of the longitudinal component Ω_Y of the vortex function that the tangential stress τ_{XZ} in a fluid layer can change from tensile to compressive up to three times.

7. Conclusion

The paper has presented an exact solution to the Navier–Stokes equation system, which is supplemented by the continuity equation. The solution has been studied for the case of the motion of a steady-state isobaric fluid. One of the longitudinal velocity components is represented by an *N*-degree polynomial. For the particular case N = 2, profiles of the obtained exact solution have been constructed, which illustrate the existence of counterflows in a fluid layer. A possible existence of up to three stagnation points in the bulk of the fluid layer has been demonstrated. Vortex functions and tangential stresses arising in fluid motion have been analyzed for this exact solution (as for specific vortex solutions in case of incompressible ideal fluid flow, one can find them in [24-25], as well as solutions for Marangoni-type of fluid flow in [26-27]).

Author Contributions

In this research, E. Prosviryakov is responsible for the general ansatz and solving procedure, as well as suggested algorithm, simple algebraic calculating, results of the article and also is responsible for the obtaining approximate estimations; S. Ershkov and D. Leshchrnko are responsible for theoretical investigations and deep survey in literature on the problem under consideration. All authors agreed with results and conclusions of each other in Sections 1-6.

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Conflict of Interest

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Nomenclature

V_x, V_y	Velocity components [m/s]	δ	Parameter of geometric anisotropy
ρ	Fluid density	U, U_i	Velocity components (i=1, 2)
P	Pressure taken relative to constant fluid density	V, V_i	Velocity components (i=1, 2)
ν	Kinematic viscosity of the fluid	f_i	Polynomials
G	Potential energy of external forces	θ, φ	Arbitrary constant

c_i	Arbitrary constant (i=1, 2, 3, 4, 5, 6, 7, 8)	x, y, z	Cartesian coordinates
Re	Reynolds number	F, F_i	Velocity components (i=0, 1, … , n)
Ta	Modified Taylor number		

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