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Research Paper

A Systematic Computational and Experimental Study of the Principal Data-Driven Identification Procedures. Part II: Numerical Analysis and Experimental Testing

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Abstract. This paper is the second part of a two-part research work intended at realizing a systematic computational and experimental analysis of the principal data-driven identification procedures based on the Observer/Kalman Filter Identification Methods (OKID) and the Numerical Algorithms for Subspace State-Space System Identification (N4SID). More specifically, this second paper treats the presentation of the numerical analysis and the experimental testing carried out in this study. To perform a systematic comparison, the identification methods mentioned before are implemented in a general-purpose computer program developed in the MATLAB computational environment. To this end, a simple two-degrees-of-freedom dynamical model of a vibrating mechanical system is considered first as a demonstrative example. The demonstrative example is used to carry out a numerical analysis of the performance of the computational methods of interest for this investigation. Subsequently, an experimental analysis is carried out focusing on a three-dimensional structure that vibrates under the effect of an external source of impulsive excitation. To perform a thorough analysis, the flexible structure employed as an experimental case study is modeled starting from its CAD geometric model and assuming different levels of complexity, which range from a simple three-degrees-of-freedom lumped parameter model to a relatively more complex linear finite element model. In the paper, the mechanical models of the structural system considered as illustrative examples are principally employed for comparing the results arising from the modal analysis. The computational and experimental analysis of these structural models turned out to be useful to trace guidelines for evaluating the effectiveness and the efficiency of the numerical and experimental results obtained from the identification process. In this study, a numerical and experimental analysis of the OKID algorithms and the N4SID methods is developed. Both classes of techniques enable the time-domain state-space system identification, that is, they construct an estimation of the state, input influence, output influence, and direct transmission matrices which characterize the dynamic properties of a mechanical system. The present investigation demonstrates that, if properly tuned, both the OKID methods and the N4SID algorithms lead to consistent numerical and experimental results, even in the case when the input-output measurements used for performing the identification procedure are affected by a certain degree of noise. The numerical and experimental results found in this second part of the research work confirmed the efficacy of the time-domain system identification methodologies described in the first part of the paper.

Keywords: Applied System Identification, Experimental Modal Analysis, Observer/Kalman Filter Identification Methods (OKID), Numerical Algorithms for Subspace State-Space System Identification (N4SID), Mass, Stiffness, and Damping Matrices Identification (MKR).

1. Introduction

In this section, an outline of the fundamental issues of interest for the second part of this two-part paper is provided. To this end, this introduction section encompasses numerous interesting topics, like the formulation of the specific problems of interest for this research work, a literature review reporting the principal aspects that can be found in the field of reference concerning system identification and structural dynamics, a summary of the scope and the contributions of the present investigation, and the structure employed to organize this manuscript.

1.1 Formulation of the Problem of Interest for this Investigation

In engineering applications, the computational and experimental techniques used to establish and subsequently refine a mathematical model of a mechanical system are typically based on the methodologies of applied system identification [1, 2]. A general definition of the procedure of applied system identification is the numerical-experimental process of formulating mathematical



models of dynamical systems considering input-output experimental data [3, 4]. In general, a mathematical model of a physical system can be used to predict by means of numerical simulations the system dynamic behavior in response to known external excitations [5, 6]. When an approach based on the field of applied system identification is employed, one needs to combine the basic physical laws of classical mechanics with the mathematical methodologies of statistics to devise mathematical models of dynamical systems that are consistent with experimental measurements [7–9]. Furthermore, in the field of applied system identification, the methodologies for the optimal design of experiments and the model reduction techniques that are commonly employed in engineering practice are encountered [10, 11]. In this context, numerical experiments obtained by means of dynamic simulations can be performed by using a reliable mathematical model of a physical system in order to reproduce the input-output relationships observed experimentally.

In applied physics and mechanical engineering, dynamic systems are generally modeled by continuous-time or discrete-time equations [12]. In continuous-time models, the input-output relationships are described by differential equations defined in a continuous domain of time [13]. Discrete-time models, on the other hand, are described by difference equations defined on a discrete-time grid. A good approximation of a continuous-time model can be obtained from a discrete-time model, provided that the sampling frequency is sufficiently high. In the time domain, a discrete linear system can be represented by different forms [14]. Dynamical systems that have the same input-output relationship are equivalent even if written in a different mathematical form [15]. These concepts represent the basic ideas behind the application of the numerical procedures of applied system identification to mechanical engineering problems. In fact, even if the numerical methods founding the base of the applied system identification discipline were originally developed for analyzing mechanical systems, the main object of the system identification process and the dynamical systems to be analyzed can be of various types, such as bio-mechanical systems, industrial processes, electromechanical systems, automotive systems, and chemical reactions processes [16–20].

In general, the methods of system identification contain at least three basic steps [21]. First, a proper class of mathematical models is chosen by the analyst. For instance, one needs to decide if the phenomenon of interest can be described using a linear structure or if it is necessary to consider nonlinear mathematical laws. Second, the best models resulting from the first selection are transformed into the desired form that is more appropriate for the specific engineering application under consideration. For example, the configuration-space form or the state-space form of the mechanical model of the dynamical system to be identified that is more appropriate for the computer implementation of the identification procedure is established [22]. Finally, the third step is to determine, among the previously identified dynamical models, the one that better describes the data observed on the system of interest in terms of a preassigned performance metric. It is, therefore, apparent that the versatile methodologies of system identification are ubiquitous in physics and engineering [23].

1.2 Literature Survey on System Identification and Structural Dynamics

To clarify the importance of the present field of research, a short literature review on the system identification methods related to those of interest for this work, considered in conjunction with the discipline of structural dynamics and the structural control problem in general, is presented herein. In [24], Soares and Serpa analyzed the difficulty of finding more appropriate parameters for setting the ERA method for applied system identification. In [25], Skolnik et al. identified the structural frequencies, the damping ratios, and the first nine mode shapes of a fifteen-story steel structure using the N4SID identification algorithm. Gumussoy et al. investigated whether the models obtained by subspace estimation can be improved by the subsequent re-evaluation of the state-space matrices [26]. Bottura et al. studied a decentralized identification methodology based on subspace identification methods for interconnected serial systems using the N4SID and MOESP (Multivariable Output-Error State sPace) identification algorithms [27]. In [28], Pappalardo et al. developed a computational procedure to identify the mode shapes of linear mechanical systems, based on the ERA-OKID algorithm, which is suitable for automotive systems. Hachicha et al. analyzed the conditioning problem of the subspace identification method, thereby demonstrating that the MOESP algorithm improves the estimation of the parameters [29]. Gustafsson showed how existing MOESP algorithms can be derived from another algorithm by simply changing the tools and the weighting matrices [30]. In [31], Qidwai et al. presented a subspace identification application using the N4SID algorithm for defect detection in metals. In [32], Bathelt et al. compared the orthogonal decomposition subspace identification methods called MOESP, N4SID, and ORT (ORTogonal decomposition) in relation to their ability to cope with disturbances that arise in complex geometries. Pappalardo et al. developed a new computational algorithm for the numerical solution of the adjoint equations necessary for solving the nonlinear optimal control problem [33]. In [34], Ljung analyzed the forecast horizons, the weighting matrices, and the alternative ways to estimate parameters useful for identifying linear multivariable dynamical systems. In [35], Tronci et al. proposed a semiautomatic methodology to extract the mode shapes independently from the chosen parametric system identification technique with minimal user involvement in the parameter selection process. Shen et al. proposed and applied a method for identifying the transmission of onboard micro-vibrations of a satellite using the dynamic identification algorithm called OKID [36]. Tuhta et al. developed a new structural identification tool to identify the modal properties of structures and, in particular, demonstrated its efficiency in identifying the modal data of foundations [37, 38]. In [39], Kim et al. presented a theoretical framework for estimating the mass, stiffness, and damping parameters of a structure from measured experimental data. In [40], Gautier et al. proposed a subspace adaptation method to update, in the time domain, the finite element model of a rotating machine using the subspace identification algorithm called MOESP. In [41], Pappalardo et al. studied the modeling of an aircraft hatch door using the N4SID method with MOESP and CVA (Canonical Variate Analysis) weighting matrices via simulation datasets. Malgaca et al. analyzed the dynamic behavior of a flexible manipulator and identified its continuous-time transfer function using an experimental dataset [42]. In [43], Koc et al. studied the vibration characteristics and the mode shapes of spinning annular disks with several diverse boundary conditions. In [44], Tufekci et al. analyzed the forced vibrations of a rotating disk employing the Galerkin method. In [45], Huang et al. revised the modeling, simulation, and control of parallel robots with lower mobility.

In [46], Zheng et al. proposed an identification and control method for a Stewart configuration platform for realizing micro-vibration excitation. Shokravi et al. tested and compared the accuracy, the noise robustness, and the time efficiency of the weight matrices obtained with the UPC (Unweighted Principle Component) and CVA algorithms [47]. Tran et al. proposed the identification of multiple large-scale bridge structure systems using different identification algorithms such as the OKID and ARX methods [48]. In [49], Ni et al. examined the problems of identifying the modal parameters and the optimally positioning of the sensors. Krollick et al. investigated the state consistency of data-driven parametric reduced-order models (ROMs) obtained with various system identification methods, including the exogenous method (ARX) and the subspace self-regressive identification procedure (N4SID) [50]. Malgaca et al. identified the dynamical model of a five-degree-of-freedom vehicle with the use of the ARX method and performed the active vibration control of this system using the PID control approach [51]. In [52], Musso et al. studied and tested a method based on the Kalman filter theory useful for identifying subspace state-space models with high precision. Fazzi et al. presented a new linear time-invariant identification method in the discrete-time domain which requires prior knowledge of the degree of the common denominator of the matrix of the transfer function of the system and of the degrees of the numerators [53]. Gres et al. developed and applied explicit expressions for the relative covariance of matrices in the state space [54]. In [55], Brownell et al. studied the identification of a time-invariant linear model of a large solar-powered spacecraft using measurements of the sensor already present



on board the vehicle. In [56], Van Vondelen et al. detected the damping of an operational offshore wind turbine using the advanced Kalman filter-based subspace identification. Mao et al. proposed an improved subspace method from the input design perspective, which guarantees consistent and stable identification results with minimal variance [57]. In [58], Tao et al. used measurements of train-induced vibration to develop a data-driven model of the cascaded state space, which can be applied to planned buildings to predict and assess whether train-induced vibrations can adversely affect future occupants of buildings. Carbonari et al. presented a methodology for the identification of the physical parameters of a model that describes the transverse dynamics of soil-foundation-pier systems based on piles [59]. Castro et al. evaluated and experimentally compared the performance of a set of identification algorithms, such as the Peak Picking method that is based on the Fourier transform [60]. In [61], Yu et al. described and studied the identification of blind systems with unknown inputs by introducing three different methods. Jin et al. established a subspace identification scheme to estimate the frequencies of a bridge by processing the dynamic response of a vehicle passing over it [62]. In [63], Shokravi et al. reviewed and analyzed the subspace system identification algorithms to perform the experimental modal analysis of a structure and evaluate its damage. The identification of the proportional damping coefficients of structural systems with a simple least-square numerical procedure using the identified natural frequencies and the identified damping ratios was also performed for the improvement of the damping matrix of mechanical systems in several research studies [64, 65]. In [66], Sadeghi et al. developed a new nonlinear identification method. Li et al. tested the use of the N4SID algorithm to identify the mathematical modeling for the steady-state flow of a wind tunnel [67]. Malgaca and Lök experimentally studied a single-link flexible manipulator and achieved the passive vibration control of this mechanical system by employing a continuous-time system identification method [68]. In [69, 70], an optimized geometry was determined by measuring the error around the feed and the dynamic analysis of a large deployable reflector together with its opening system was carried out.

Prasad and Singh implemented experimental modal analysis methods and forced vibration analysis techniques to identify electrical and mechanical faults of electromechanical systems [71]. Karaaugaccli and Ozguven used a controlled stepped-sine input signal to carry out the experimental modal analysis of a nonlinear beam structure instrumented with several accelerometers placed at different locations along its length [72]. Volkmar et al. proposed a new method that allows for performing a fully automated modal analysis and simultaneously learns an optimal strategy to analyze the data [73]. Schoenherr et al. studied a method to perform experimental modal testing at pyroshock excitation levels by using input forces and measured acceleration responses [74]. Mugnaini et al. presented an algorithm for performing an automatic operational modal analysis based on data-driven machine learning methods applicable to civil and aerospace structures [75]. The numerical procedures of applied system identification are also employed to refine the mathematical modeling of lithium-ion batteries taking into account the time evolutions of the temperature, voltage, and current [76, 77]. For this purpose, Wang et al. performed an experimental analysis based on system identification methods of fractional-order mathematical models for describing lithium-ion batteries and ultra-capacitors [78]. In [79], Peng et al. studied the State of charge (SOC) of lithium-ion batteries and employed an improved Adaptive Dual Unscented Kalman Filter (ADUKF) method to derive the uncertain parameters of the battery mathematical model. In [80], Ren et al. developed an improved recursive least square algorithm for identifying the parameters of the analytical model of a lithium-ion battery. In the literature, several research studies concern system identification methods, the discipline of structural dynamics, and the problem of vibration control, thereby demonstrating the importance of the interconnections between these disciplines.

1.3 Scope and Contributions of this Study

This paper is the second part of a two-part research work aimed at performing a systematic computational and experimental analysis of the principal data-driven identification procedures based on the Observer/Kalman Filter Identification Methods (OKID) and the Numerical Algorithms for Subspace State-Space System Identification (N4SID). In particular, this second paper deals with the presentation of the numerical analysis and the experimental testing carried out in this study.

The OKID and the N4SID identification procedures rely on the numerical methods of the applied system identification discipline that are designed in the time domain. In the paper, these methods and their variants are first tested by means of numerical experiments using a simple two-degree-of-freedom model of a lumped parameter vibrating system. Subsequently, a comparative analysis based on experimental tests carried out on a laboratory case study is performed. For this purpose, a structural system constructed by means of a three-story rigid-flexible mechanical system is employed. In order to define useful guidelines for interpreting the numerical results generated by the experimental analysis, preliminary mechanical models of the rigid-flexible structural system considered as the test problem are derived. In particular, starting from the three-dimensional CAD geometry of the system of interest, a simple lumped parameter model developed using the Euler-Bernoulli elastic beam theory and a more detailed finite element model based on the general theory of continuum mechanics are used for modeling the three-story system and computing the dynamic parameters of interest. To this end, a numerical modal analysis is carried out using the simplified mechanical models of the structural system considered as the case study. This is done in order to establish a reasonable prediction of the modal parameters expected in output from the experimental identification, such as the system natural frequencies, the dimensionless damping ratios, and the geometric form of the mode shapes. By doing so, a systematic comparison of the performance of both the OKID methods and the N4SID algorithms is possible in realistic cases in which the estimation of the dynamic model of an actual structural system needs to be found using force and acceleration measurements subjected to unpredictable noise.

The identification procedures analyzed in this investigation allow for reconstructing continuous-time first-order time-invariant dynamical models of the structural system of interest. Starting from the identified first-order dynamical model represented in the state space, an effective methodology referred to as the MKR method is used in the paper in order to obtain an estimation of a second-order mechanical model represented in the configuration space that is useful for successive dynamics and control applications. A least-square method referred to as the PDC (Proportional Damping Coefficients) method is also employed in the paper for improving the estimation of the damping matrix assuming the hypothesis of proportional damping, which represents a realistic assumption in the case of lightly damped metallic systems such as the case study analyzed in the paper. The system identification approaches considered in this investigation are implemented in general-purpose computer programs developed within the MATLAB simulation environment in order to perform a thorough comparative analysis based on numerical and actual experiments. In general, the comparative analysis based on the numerical and experimental results obtained in this work confirms the effectiveness of both the computational approaches based on the OKID and the N4SID identification techniques.

This study is devoted to the estimation of the physical parameters of mechanical systems employing the numerical procedures of applied system identification. In particular, this second paper presents the numerical and experimental applications, as well as the description of the results found, by using the principal data-driven identification procedures considered in this work in the case of linear dynamical systems. The OKID and N4SID methods are used in the numerical analysis and experimental testing proposed in this second paper to derive first-order state-space models of a two-degree-of-freedom vibrating system as well as of a three-degree-of-freedom frame structure. The goal of this study is, therefore, to carry out the numerical and experimental modal analysis of mechanical systems and to compare the results found. To this end, the MKR approach is subsequently employed to extract the natural frequencies, the damping ratios, and the mode shapes of the structural systems of interest, as well as to construct second-order configuration-space models that are equivalent to the dynamical models identified before. On the other hand, the



simple least-square technique that constitutes the basis of the PDC method is adopted in this work to refine the estimation of the damping parameters of the mechanical systems of interest. Thus, compared to other methodologies, this second paper belonging to the present two-part research work proposes a systematic procedure for identifying and refining the estimation of the physical parameters of structural systems when the interest is on experimental modal analysis.

In future research works, the identification approach considered in this investigation will be employed for the development of mechanical prototypes of dynamical vibration absorbers aimed at reducing the mechanical vibrations induced by external sources of energy applied to the frame structure representing the experimental test rig used in this study.

1.4 Organization of the Manuscript

The subsequent sections of this two-part manuscript are organized as follows. In Section 2., the numerical results obtained in the computational testing of the identification methods considered in this work when applied to a simple benchmark system are reported. Section 3. describes the experimental results found by applying all the methodologies analyzed in this work in the case of an experimental test rig considered as the case study. Considering the numerical and experimental results obtained with the aid of the demonstrative example, as well as by analyzing the case study analyzed in the paper, Section 4. proposes a detailed discussion and a comparative analysis of the performance of the identification procedures investigated in this research work. In Section 5., a summary of the work done, the conclusions of this two-part paper, and some possible directions for future research are provided.

2. Numerical Results

In this section, the dynamic identification techniques based on the OKID method and the N4SID algorithm are implemented and tested through computer programs. The performance of these two methodologies is analyzed by using the numerical data that relates the external exciting force (input) to the acceleration time histories (output), obtained from a known model of a benchmark mechanical system. To this end, a lumped parameter model of a simple mechanical system having two degrees of freedom is considered in the numerical analysis. As mentioned before, once the system dynamical model and the externally applied forces are consistently defined, the corresponding response of the mechanical system of interest can be numerically obtained through dynamical simulations, leading to an input-output dataset suitable for performing numerical experiments on the performance of the system identification procedure considered in the paper. These data are then processed by the OKID and N4SID algorithms, which allow for identifying a first-order space-state model of the system under consideration, from which, under appropriate conditions, it is possible to derive an equivalent second-order model, as well as its natural frequencies, damping ratios, and normal vibration modes. However, actual data acquisitions are always affected by noise and disturbances of various kinds. Consequently, to assess the robustness of the numerical results provided by the OKID and N4SID identification algorithms, the identification procedure was repeated by adding a random high-frequency function to the input and output data, thereby effectively simulating the presence of the noise in an artificial simulation environment. For both cases concerning the OKID and N4SID methods, the comparison between the known a priori model and the identified ones is carried out by comparing the dynamic responses of the two systems, both to the input relative to the identification process and to a different generic input.

2.1 Description of the Benchmark Problem

This subsection describes the mathematical model of a simple mechanical system that serves as a benchmark problem for testing in a virtual environment the performance of the system identification numerical algorithms considered in the paper. The mechanical system considered for this purpose is shown in Figure 1.

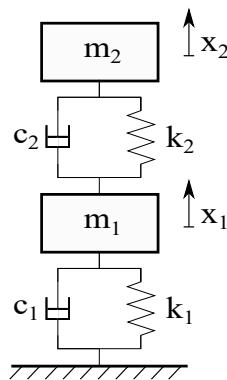


Fig. 1. Two-degree-of-freedom lumped parameter model of a vibrating mechanical system.

As shown in Figure 1, the mechanical model taken into consideration is a simple lumped parameter model of a classic mass-spring-damper system having two degrees of freedom. In particular, the parameters m_1 and m_2 represent the inertial masses of the two particles, the parameters k_1 and k_2 denote the elastic stiffness coefficients of the two springs, and the parameters c_1 and c_2 identify the viscous damping coefficients of the two dashpots that form the benchmark system shown in Figure 1. In Table 1, the numerical values of the physical parameters used for performing numerical experiments in a simulated environment are reported.

Table 1. Numerical values of the physical parameters relative to the benchmark problem used in the numerical experiments.

Symbol	Data (Units)
m_1	1 (kg)
m_2	0.5 (kg)
k_1	500 (N/m)
k_2	300 (N/m)
c_1	0.1 (N × s/m)
c_2	0.01 (N × s/m)



The vibration problem associated with this simple model is assumed as the benchmark problem for performing numerical experiments using the identification procedures analyzed in the paper. Thus, the numerical results obtained in this subsection are useful for testing through dynamical simulations the proposed identification algorithms, thereby allowing for a systematic comparison of their performance considering also additional artificial disturbances. The generalized coordinate vector associated with the mechanical model of the present vibrating system is defined as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (1)$$

where $n_x = 2$ identifies the number of degrees of freedom of the structural system, $\mathbf{x}$ indicates a time-dependent vector of dimensions $n_x \times 1$ representing the system generalized coordinate vector, while x_1 and x_2 respectively denote the time-dependent displacements of the first and second point masses. For the benchmark system at hand, the equations of motion, which mathematically describe the mechanical model of the system, can be readily determined using the basic principles of classical mechanics and can be written in a compact matrix form as:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{R}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \quad (2)$$

where $\mathbf{x}$, $\dot{\mathbf{x}}$, and $\ddot{\mathbf{x}}$ respectively represent time-dependent vectors of dimensions $n_x \times 1$ describing the system generalized displacements, velocities, and accelerations, whereas $\mathbf{K}$, $\mathbf{R}$, and $\mathbf{M}$ respectively identify constant matrices of dimensions $n_x \times n_x$ describing the system stiffness, damping, and mass matrices, while $\mathbf{F}$ denotes a vector of dimensions $n_x \times 1$ containing the external applied forces. It can be easily proved that, for the system under study, the mass, damping, and stiffness matrices can be readily determined using the classical methods of analytical dynamics, leading to the following structural forms:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \quad (3)$$

$$\mathbf{R} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} = \begin{bmatrix} 0.11 & -0.01 \\ -0.01 & 0.01 \end{bmatrix} \quad (4)$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 800 & -300 \\ -300 & 300 \end{bmatrix} \quad (5)$$

Furthermore, for the problem at hand, the generalized force vector denoted with $\mathbf{F}$ can be expressed as follows:

$$\mathbf{F} = \mathbf{B}_a \mathbf{u} \quad (6)$$

where $\mathbf{u}$ is an input vector of dimensions $n_u = 2$ and $\mathbf{B}_a$ is a Boolean matrix of dimensions $n_x \times n_u$ that coincides with the identity matrix $\mathbf{I}$ for the particular problem under consideration. Let $n_z = 2n_x = 4$ be the dimension of the state-space and $\mathbf{z}$ a vector of dimensions $n_z \times 1$ representing the system state vector defined as:

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \quad (7)$$

Knowing the definition of the system state vector and the structural matrices of the benchmark mechanical system, it is possible to obtain the principal continuous-time state-space matrices associated with its continuous-time state-space representation as follows:

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{R} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -800 & 300 & -0.11 & 0.01 \\ 600 & -600 & 0.02 & -0.02 \end{bmatrix} \quad (8)$$

and

$$\mathbf{B}_c = \begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1}\mathbf{B}_a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \quad (9)$$

where $\mathbf{A}_c$ is a square matrix of dimensions $n_z \times n_z$ describing the system state matrix and $\mathbf{B}_c$ is a rectangular matrix of dimensions $n_z \times n_u$ representing the system input influence matrix. By performing an eigendecomposition of the state matrix $\mathbf{A}_c$, one obtains:

$$\boldsymbol{\lambda}_c = \begin{bmatrix} -0.02057 + 16.2515i \\ -0.02057 - 16.2515i \\ -0.0445 + 33.7029i \\ -0.0445 - 33.7029i \end{bmatrix} \quad (10)$$

and

$$\boldsymbol{\Psi}_c = \begin{bmatrix} -0.0001 - 0.0301i & -0.0001 + 0.0301i & & \\ -0.0001 - 0.0536i & -0.0001 + 0.0536i & & \\ 0.4875 - 0.0008i & 0.4875 + 0.0008i & \cdots & \\ 0.8709 & 0.8709 & & \\ & -0.0001 + 0.0197i & -0.0001 - 0.0197i & \\ & -0.0001 - 0.0222i & -0.0001 + 0.0222i & \\ \cdots & -0.6659 - 0.0021i & -0.6659 + 0.0021i & \\ & 0.7455 & 0.7455 & \end{bmatrix} \quad (11)$$



where λ_c is a constant vector having dimensions $n_z \times 1$ whose elements are the system continuous-time state-space eigenvalues and Ψ_c is a constant matrix of dimensions $n_z \times n_z$ containing the system continuous-time state-space eigenvectors stacked by columns. In particular, since the second-order configuration-space model of the mechanical system analyzed as a benchmark problem is only lightly damped, the first-order state-space eigenvalues appear in complex conjugate pairs and assume the following simple form:

$$\lambda_{c,j} = a_{c,j} \pm ib_{c,j}, \quad j = 1, 2, \dots, n_x \quad (12)$$

where $i = \sqrt{-1}$ is the imaginary unit, while $a_{c,j}$ and $b_{c,j}$ are respectively the real and imaginary parts of the continuous-time eigenvalue $\lambda_{c,j}$ labeled with the integer j . Once the system eigenvalues are known, it is possible to determine the system natural angular frequencies denoted with $\omega_{n,j}$, the system natural frequencies denoted with $f_{n,j}$, and the system damping ratios denoted with ξ_j by using the following simple formulas:

$$\begin{cases} \omega_{n,j} = \sqrt{a_{c,j}^2 + b_{c,j}^2} \\ f_{n,j} = \frac{\sqrt{a_{c,j}^2 + b_{c,j}^2}}{2\pi} \\ \xi_j = -\frac{a_{c,j}}{\sqrt{a_{c,j}^2 + b_{c,j}^2}} \end{cases}, \quad j = 1, 2, \dots, n_x \quad (13)$$

By doing so, one can readily determine the following numerical results expressed in a vector form:

$$\omega_n = \begin{bmatrix} 16.2514 \\ 33.7028 \end{bmatrix}, \quad f_n = \begin{bmatrix} 2.5865 \\ 5.3639 \end{bmatrix}, \quad \xi = \begin{bmatrix} 0.0012 \\ 0.0013 \end{bmatrix} \quad (14)$$

where ω_n , f_n , and ξ are constant vectors of dimensions $n_x \times 1$ respectively representing the set of the system natural angular frequencies, the set of the system natural frequencies, and the set of the system damping ratios, which can be readily determined using the fundamental formulas recalled before and given in Equation (13). Furthermore, it is well-known that the system eigenvector matrix can be rewritten using the following matrix partition:

$$\Psi_c = \begin{bmatrix} W_c \\ W_c \Lambda_c \end{bmatrix} \quad (15)$$

being for the benchmark problem at hand:

$$W_c = \begin{bmatrix} -0.0001 - 0.0301i & -0.0001 + 0.0301i & \cdots \\ -0.0001 - 0.0536i & -0.0001 + 0.0536i & \cdots \\ \cdots & -0.0001 + 0.0197i & -0.0001 - 0.0197i \\ \cdots & -0.0001 - 0.0222i & -0.0001 + 0.0222i \end{bmatrix} \quad (16)$$

and

$$\Lambda_c = \begin{bmatrix} -0.0205 + 16.2514i & 0 & & \\ 0 & -0.0205 - 16.2514i & & \\ 0 & 0 & \cdots & \\ 0 & 0 & & \\ & 0 & 0 & \\ \cdots & 0 & 0 & \\ & -0.0446 + 33.703i & 0 & \\ 0 & -0.0446 - 33.703i & & \end{bmatrix} \quad (17)$$

where W_c is a constant matrix of dimensions $n_x \times n_z$ representing the system eigenvector matrix associated with the set of the configuration-space generalized coordinates, whereas Λ_c is a diagonal matrix of dimensions $n_z \times n_z$ whose elements on the diagonal are the system eigenvalues relative to the set of the state-space generalized coordinates. Since the configuration-space eigenvectors appear in complex conjugate pairs following the same pattern of the system state-space eigenvalues, one can extract from the eigenvector matrix W_c the relevant configuration-space eigenvectors denoted with $\varphi_{c,j}$, $j = 1, 2, \dots, n_x$. These eigenvectors form the reduced eigenvector matrix denoted with Φ_c of dimensions $n_x \times n_x$, associated with the configuration-space model, and can be readily recovered by taking only the odd (or the even) columns from the eigenvector matrix W_c as follows:

$$\Phi_c = \begin{bmatrix} \varphi_{c,1} & \varphi_{c,2} \end{bmatrix} = \begin{bmatrix} -0.0001 - 0.0301i & -0.0001 + 0.0197i \\ -0.0001 - 0.0536i & -0.0001 - 0.0222i \end{bmatrix} \quad (18)$$

where $\varphi_{c,1}$ and $\varphi_{c,2}$ are two constant vector of dimensions $n_x \times 1$ representing the system configuration-space eigenvectors. From each of them, one can obtain the relative amplitudes and phase angles of the first and second vibration modes, respectively. To this end, one can conveniently consider the following Cartesian and polar (exponential) representations:

$$\varphi_{c,j} = u_{c,j} + iv_{c,j} = e^{i\Theta_{c,j}} \rho_{c,j}, \quad j = 1, 2, \dots, n_x \quad (19)$$

where $u_{c,j}$ and $v_{c,j}$ are two constant vectors of dimensions $n_x \times 1$ that respectively represent the real and the imaginary parts of the generic configuration-space eigenvector $\varphi_{c,j}$, while $\rho_{c,j}$ denotes a constant vector of dimensions $n_x \times 1$ containing the relative vibration amplitudes of the normal mode labeled with the integer j , and $\Theta_{c,j}$ denotes a constant diagonal matrix of dimensions $n_x \times n_x$ whose diagonal elements form a constant vector denoted with $\theta_{c,j}$ of dimensions $n_x \times 1$ that identifies the vibration phase



angles of the same normal mode. The scalar, vector, and matrix quantities introduced before can be readily determined by employing the following general formulas:

$$\varphi_{c,j} = \begin{bmatrix} \varphi_{c,j,1} \\ \varphi_{c,j,2} \\ \vdots \\ \varphi_{c,j,k} \\ \vdots \\ \varphi_{c,j,n_x} \end{bmatrix}, \quad j = 1, 2, \dots, n_x \quad (20)$$

or

$$\varphi_{c,j} = \begin{bmatrix} u_{c,j,1} + iv_{c,j,1} \\ u_{c,j,2} + iv_{c,j,2} \\ \vdots \\ u_{c,j,k} + iv_{c,j,k} \\ \vdots \\ u_{c,j,n_x} + iv_{c,j,n_x} \end{bmatrix} = \begin{bmatrix} \rho_{c,j,1} e^{i\theta_{c,j,1}} \\ \rho_{c,j,2} e^{i\theta_{c,j,2}} \\ \vdots \\ \rho_{c,j,k} e^{i\theta_{c,j,k}} \\ \vdots \\ \rho_{c,j,n_x} e^{i\theta_{c,j,n_x}} \end{bmatrix}, \quad j = 1, 2, \dots, n_x \quad (21)$$

where:

$$\mathbf{u}_{c,j} = \begin{bmatrix} u_{c,j,1} \\ u_{c,j,2} \\ \vdots \\ u_{c,j,k} \\ \vdots \\ u_{c,j,n_x} \end{bmatrix}, \quad \mathbf{v}_{c,j} = \begin{bmatrix} v_{c,j,1} \\ v_{c,j,2} \\ \vdots \\ v_{c,j,k} \\ \vdots \\ v_{c,j,n_x} \end{bmatrix}, \quad j = 1, 2, \dots, n_x \quad (22)$$

and

$$\rho_{c,j} = \begin{bmatrix} \rho_{c,j,1} \\ \rho_{c,j,2} \\ \vdots \\ \rho_{c,j,k} \\ \vdots \\ \rho_{c,j,n_x} \end{bmatrix}, \quad \theta_{c,j} = \begin{bmatrix} \theta_{c,j,1} \\ \theta_{c,j,2} \\ \vdots \\ \theta_{c,j,k} \\ \vdots \\ \theta_{c,j,n_x} \end{bmatrix}, \quad j = 1, 2, \dots, n_x \quad (23)$$

being:

$$\begin{cases} u_{c,j,k} = \text{Re}(\varphi_{c,j,k}) \\ v_{c,j,k} = \text{Im}(\varphi_{c,j,k}) \end{cases}, \quad j, k = 1, 2, \dots, n_x \quad (24)$$

and

$$\begin{cases} \rho_{c,j,k} = \sqrt{u_{c,j,k}^2 + v_{c,j,k}^2} \\ \theta_{c,j,k} = \tan^{-1}\left(\frac{v_{c,j,k}}{u_{c,j,k}}\right) \end{cases}, \quad j, k = 1, 2, \dots, n_x \quad (25)$$

where $\varphi_{c,j,k}$, $u_{c,j,k}$, $v_{c,j,k}$, $\rho_{c,j,k}$, and $\theta_{c,j,k}$ respectively represent the component k associated with the vibration mode j and relative to the vectors $\varphi_{c,j}$, $\mathbf{u}_{c,j}$, $\mathbf{v}_{c,j}$, $\rho_{c,j}$, and $\theta_{c,j}$. By normalizing the relative amplitudes to the unity and setting the first phase angle to zero, one can readily identify the geometric shapes of the two vibration modes of the benchmark system. By doing so, one obtains the numerical results reported in Table 2.

Table 2. Physical parameters of the first and second normal modes relative to the benchmark problem considered in the numerical analysis.

Symbol	Data (Units)
$f_{n,1}$	2.586 (Hz)
$f_{n,2}$	5.363 (Hz)
ξ_1	0.00125 (—)
ξ_2	0.00130 (—)
ρ_1	$\begin{bmatrix} 1 & 1.7863 \end{bmatrix}^T$ (—)
ρ_2	$\begin{bmatrix} 1 & 1.1196 \end{bmatrix}^T$ (—)
θ_1	$\begin{bmatrix} 0 & -0.0015535 \end{bmatrix}^T$ (rad)
θ_2	$\begin{bmatrix} 0 & -3.1448 \end{bmatrix}^T$ (rad)

Considering the exponential representation of the configuration-space eigenvectors, the geometric shapes of the normal modes can be drawn as shown in Figure 2. For this purpose, in order to provide an intuitive representation of the mode shape, the real and the imaginary parts of normal modes can be represented in the three-dimensional space, as respectively shown in Figures 2a and 2b, which are respectively associated with the first and the second normal modes.



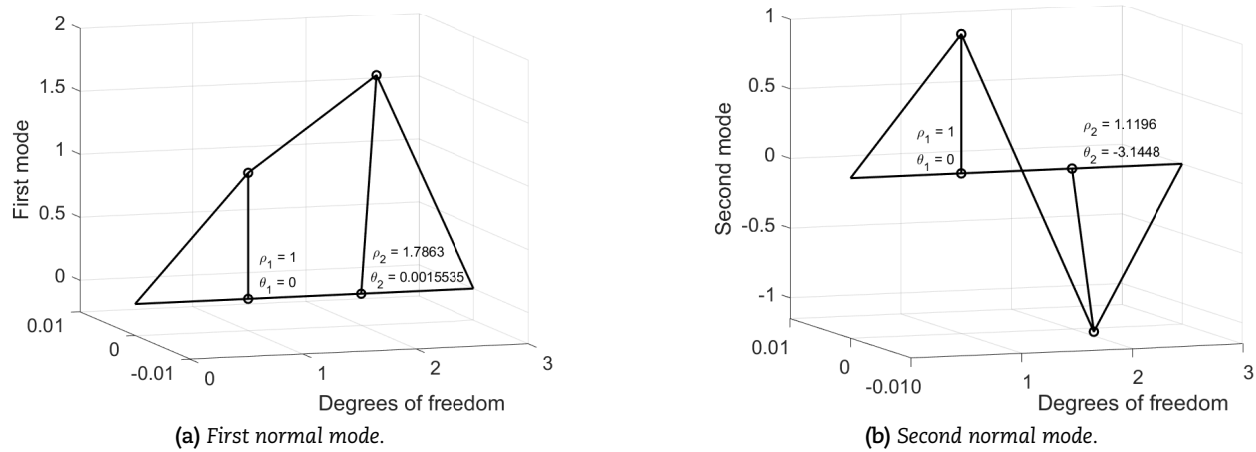


Fig. 2. Geometric shapes of the normal modes of the benchmark system.

In particular, the geometric shape of the first vibration mode is represented in Figure 2a, while the principal physical parameters associated with the same normal mode are reported in Table 2. Similarly, the geometric shape of the second vibration mode is represented in Figure 2b, while the principal physical parameters associated with the same normal mode are reported in Table 2. As shown in Figures 2a and 2b, it is important to note that, since the system is only lightly damped, the normal displacements associated with the first vibration mode are almost in phase, while the normal displacements associated with the second vibration mode are almost out of phase. This physical observation allows for reasonably assuming the proportional damping hypothesis for improving the identification of the system damping matrix. The numerical computations of the fundamental modal parameters that characterize the vibration modes of the mechanical system under study complete the description of the benchmark problem, which is considered as a reference for evaluating the performance of the numerical identification procedure analyzed in the paper in a virtual environment.

2.2 Numerical Analysis of the Benchmark Problem in the Absence of Noise

Considering as an input-output dataset the external exciting forces and the corresponding accelerations obtained from the two-degrees-of-freedom benchmark mechanical model described above, the purpose of the numerical analysis reported in this subsection is to test and compare the outcomes of the OKID and the N4SID system identification procedures. For the benchmark system at hand, one can readily determine the system continuous-time state-space matrices associated with the case of a full acceleration sensing, that is, considering an acceleration sensor for each degree of freedom of the mechanical system. In this scenario, which is quite realistic in the case of vibrating mechanical systems, one can write the output influence matrix denoted with C_c of dimensions $n_y \times n_z$ and the direct transmission matrix denoted with D_c of dimensions $n_y \times n_u$ as follows:

$$C_c = \begin{bmatrix} -C_a M^{-1} K & -C_a M^{-1} R \end{bmatrix} = \begin{bmatrix} -800 & 300 & -0.11 & 0.01 \\ 600 & -600 & 0.2 & -0.2 \end{bmatrix} \quad (26)$$

and

$$D_c = C_a M^{-1} B_a = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad (27)$$

where $n_y = 2$ is the number of the output variables and C_a is a Boolean matrix of dimensions $n_y \times n_x$ that, for the particular problem under consideration, coincides with the identity matrix. Since a set of accelerometers will be used to measure the mechanical vibrations of the experimental test rig during the subsequent experimental phase, the continuous-time state-space matrices C_c and D_c , associated with the benchmark system, are calculated considering that the measured output vector is represented by the generalized accelerations of the vibrating system. In order to allow for numerically performing the modal testing of the benchmark system, the continuous-time state-space matrices A_c and B_c were already computed in Subsection 2.1.

To perform a systematic comparative analysis, a dynamical simulation is carried out considering a time span equal to $T = 60$ (s) with a sampling time equal to $\Delta t = 0.005$ (s). The impulsive input force assumed for performing the system identification of the benchmark model in the absence of noise is represented in Figure 3.

For simplicity, it is assumed that the same impulsive input force represented in Figure 3 is applied to both the first and the second particles at the same time. At this stage, the matrices of the state-space, the input forces, and the initial state of the structural system, which is initially supposed to be stationary, are known. Therefore, one is able to perform dynamical simulations employing the benchmark system under consideration. Thus, the collected data are sufficient to calculate the system response by performing a dynamic simulation in the MATLAB environment. Figures 4 and 5 respectively represent the accelerations of the first and the second concentrated masses corresponding to the two impulsive input forces obtained in the absence of noise.

In particular, Figures 4a and 5a respectively represent the output accelerations of the first and second masses considering the complete time span of the dynamical simulation in the absence of noise, while Figures 4b and 5b respectively represent a zoomed time window of the same output accelerations.

Once the input-output dataset is complete, one can systematically apply the system identification numerical procedures considered in the paper in a virtual environment, namely the OKID method and the N4SID method. Thus, for both the OKID and the N4SID identification algorithms, a singular value decomposition can be performed by considering the input and output measurements described before, and a graphical representation of the singular values can be readily obtained. By doing so, one can readily identify the singular values of the identified discrete-time state-space models arising from the implementation of the identification procedures considered herein. Figure 6 represents the distribution of the singular values related to the computer implementation of the OKID and the N4SID algorithms, both applied to the benchmark problem and obtained in the absence of noise in the input-output dataset.



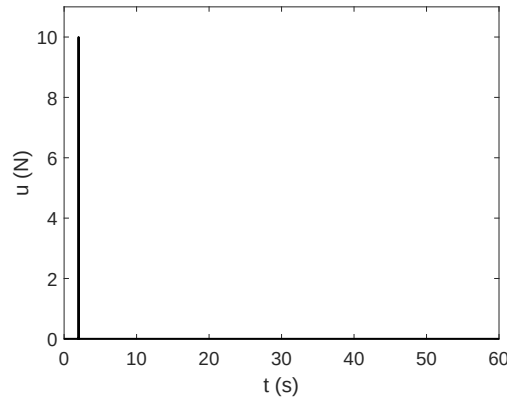
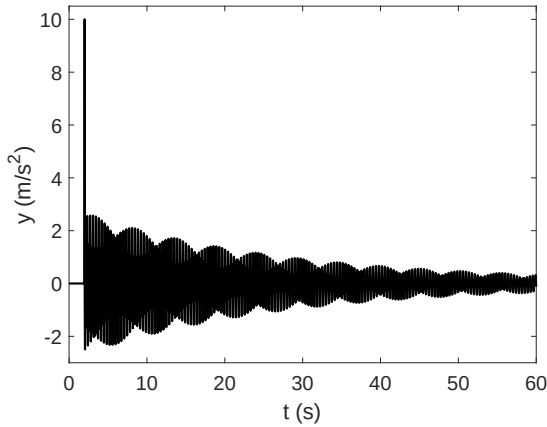
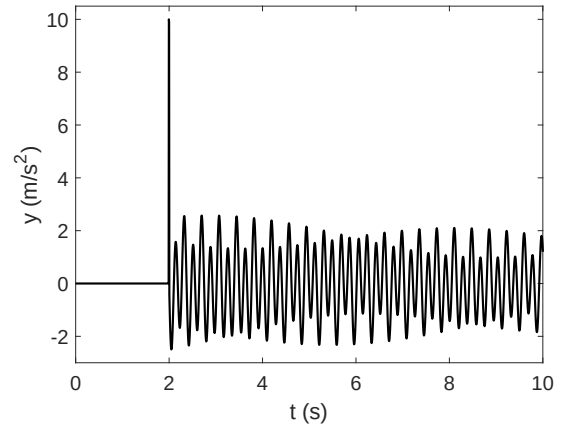


Fig. 3. Impulsive input force applied to the benchmark system in the absence of noise.



(a) Complete time span.



(b) Zoomed time window.

Fig. 4. Impulsive output acceleration of the first mass of the benchmark system in the absence of noise.

The observation of the distribution of the singular values allows for identifying the most relevant degrees of freedom of the identified system state-space model. More specifically, a close inspection of Figures 6a and 6b leads to the identification of two couples of singular values whose magnitudes are significantly higher with respect to the remaining ones. Since both the OKID and the N4SID algorithms allow for properly setting the dimensions of the identified state-space model denoted with $\hat{n}_z = 2\hat{n}_x$, that is, the number of degrees of freedom of the mechanical model to be identified denoted with $\hat{n}_x$, one can deduce from the actual distribution of the identified singular values that the dimension of the identified state-space model is given by the integer $\hat{n}_z = 4$. Consequently, the number of identified degrees of freedom is equal to $\hat{n}_x = 2$. Furthermore, the dimensions of the identified input and output vectors are equal to the original ones and are respectively denoted with the integers $\hat{n}_u = n_u = 2$ and $\hat{n}_y = n_y = 2$.

For the benchmark system under consideration in the absence of artificial noise, by setting $\hat{n}_{z,OKID} = \hat{n}_z = 4$ and $\hat{n}_{z,N4SID} = \hat{n}_z = 4$ in both the OKID and the N4SID numerical procedure, the following set of identified continuous-time state-space matrices can be numerically determined:

$$\hat{\mathbf{A}}_{c,OKID} = \begin{bmatrix} 0.3108 & -33.368 & -3.0135 & -0.1361 \\ 33.4129 & -0.2312 & -0.1376 & 1.2454 \\ 3.4834 & -0.5914 & -0.4493 & 16.2838 \\ -0.4108 & -1.3338 & -16.7805 & 0.2395 \end{bmatrix} \quad (28)$$

$$\hat{\mathbf{B}}_{c,OKID} = \begin{bmatrix} 1.0073 & 0 \\ -1.2267 & -0.0001 \\ -1.3724 & 0 \\ -1.0278 & -0.0001 \end{bmatrix} \quad (29)$$

$$\hat{\mathbf{C}}_{c,OKID} = \begin{bmatrix} -10.0456 & -6.9766 & 1.8122 & -3.8322 \\ 11.9172 & 6.3879 & 5.6726 & -3.5374 \end{bmatrix} \quad (30)$$

$$\hat{\mathbf{D}}_{c,OKID} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad (31)$$

and

$$\hat{\mathbf{A}}_{c,N4SID} = \begin{bmatrix} -0.0373 & -8.0032 & 1.5033 & -14.1148 \\ 8.1065 & -0.0632 & 29.1171 & 1.0375 \\ -1.5381 & -29.0753 & -0.0068 & 16.9246 \\ 14.0522 & -0.9584 & -16.9505 & -0.0229 \end{bmatrix} \quad (32)$$



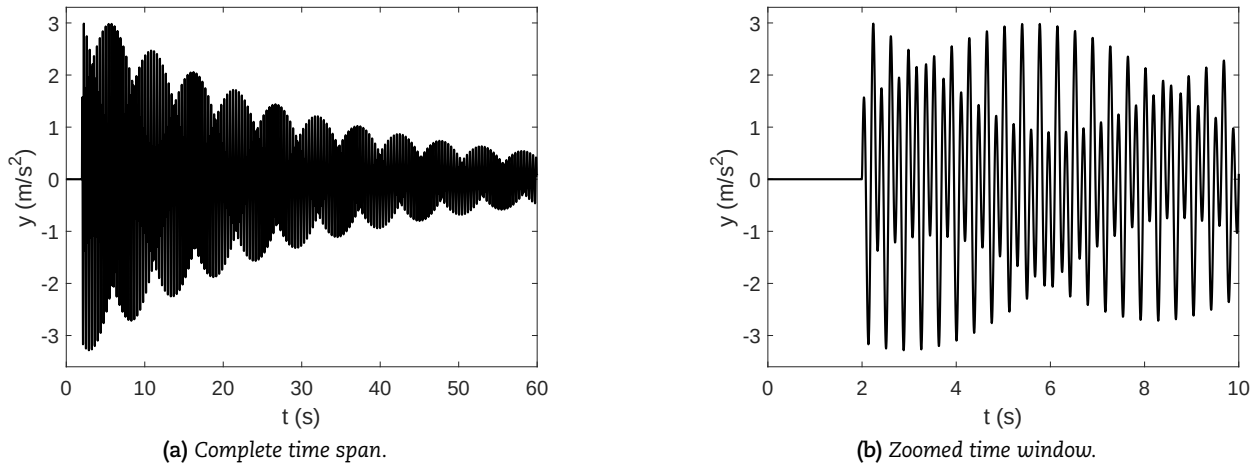


Fig. 5. Impulsive output acceleration of the second mass of the benchmark system in the absence of noise.

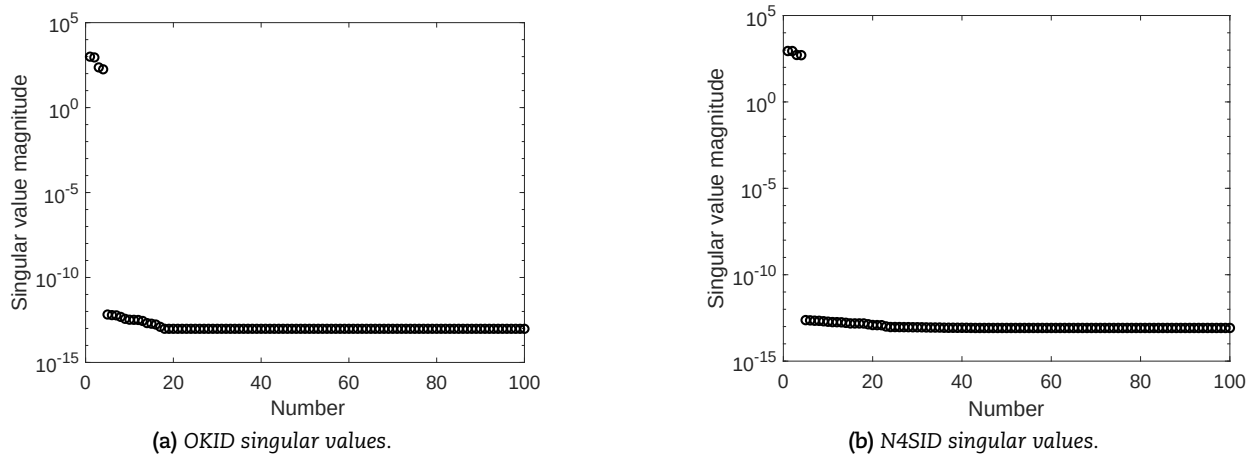


Fig. 6. Distribution of the identified singular values obtained for the benchmark problem by implementing the OKID and the N4SID identification methods in the absence of noise.

$$\hat{B}_{c,N4SID} = \begin{bmatrix} 0.2012 & 0 \\ -0.264 & 0 \\ 0.0593 & -0.0001 \\ 0.1502 & 0 \end{bmatrix} \quad (33)$$

$$\hat{C}_{c,N4SID} = \begin{bmatrix} -1.0244 & -26.4713 & -67.2776 & -19.278 \\ -0.8113 & -13.205 & 76.1602 & -52.0601 \end{bmatrix} \quad (34)$$

$$\hat{D}_{c,N4SID} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad (35)$$

where $\hat{A}_{c,OKID}$, $\hat{B}_{c,OKID}$, $\hat{C}_{c,OKID}$, and $\hat{D}_{c,OKID}$ represent the continuous-time state-space matrices identified by using the OKID numerical procedure, while $\hat{A}_{c,N4SID}$, $\hat{B}_{c,N4SID}$, $\hat{C}_{c,N4SID}$, and $\hat{D}_{c,N4SID}$ represent the continuous-time state-space matrices identified by using the N4SID numerical procedure in the case of the lumped parameter system considered in the paper as the benchmark example. In general, these identified matrices have dimensions $\hat{n}_z \times \hat{n}_z$, $\hat{n}_z \times \hat{n}_u$, $\hat{n}_y \times \hat{n}_z$, and $\hat{n}_y \times \hat{n}_u$, respectively. It is possible to note that the state-space matrices identified using both the numerical procedures considered here are different from those obtained from the original dynamical model employed in the previous modal analysis. However, except for small numerical errors, both the identified state-space representations correspond to the same dynamical system since they identify the same eigenvalues.

The identified eigenvalues are grouped in the vectors $\hat{\lambda}_{c,OKID}$ and $\hat{\lambda}_{c,N4SID}$ of dimensions $\hat{n}_z \times 1$, respectively related to the OKID and the N4SID algorithms, which are given by:

$$\hat{\lambda}_{c,OKID} = \begin{bmatrix} -0.0205 + 16.2514i \\ -0.0205 - 16.2515i \\ -0.0446 + 33.7028i \\ -0.0446 - 33.703i \end{bmatrix} \quad (36)$$

and

$$\hat{\lambda}_{c,N4SID} = \begin{bmatrix} -0.0205 + 16.2514i \\ -0.0205 - 16.2515i \\ -0.0446 + 33.7028i \\ -0.0446 - 33.703i \end{bmatrix} \quad (37)$$



Consequently, the identified natural angular frequencies, the identified natural frequencies, and the identified dimensionless damping ratios are also consistent with the original values found for the benchmark system and are given by:

$$\hat{\omega}_{n,OKID} = \begin{bmatrix} 16.251 \\ 33.702 \end{bmatrix}, \quad \hat{\omega}_{n,N4SID} = \begin{bmatrix} 16.251 \\ 33.702 \end{bmatrix} \quad (38)$$

and

$$\hat{f}_{n,OKID} = \begin{bmatrix} 2.586 \\ 5.363 \end{bmatrix}, \quad \hat{f}_{n,N4SID} = \begin{bmatrix} 2.586 \\ 5.363 \end{bmatrix} \quad (39)$$

and

$$\hat{\xi}_{OKID} = \begin{bmatrix} 0.012 \\ 0.0013 \end{bmatrix}, \quad \hat{\xi}_{N4SID} = \begin{bmatrix} 0.012 \\ 0.0013 \end{bmatrix} \quad (40)$$

where $\hat{\omega}_{n,OKID}$, $\hat{f}_{n,OKID}$, and $\hat{\xi}_{OKID}$ are vectors of dimensions $\hat{n}_x \times 1$ that respectively contain the identified natural angular frequencies, the identified natural frequencies, and the identified damping ratios obtained from the computer implementation of the OKID method, whereas $\hat{\omega}_{n,N4SID}$, $\hat{f}_{n,N4SID}$, and $\hat{\xi}_{N4SID}$ are vectors of the same dimensions representing similar quantities arising from the use of the N4SID technique.

Considering the analytical procedure introduced in the description of the benchmark model, the relative vibration amplitudes and the relative phase angles of each normal mode can be identified as well. By doing so, one can calculate the state-space and the configuration-space eigenvector matrices arising from the implementation of both the OKID and the N4SID procedures, leading to the following set of identified matrices:

$$\hat{W}_{c,OKID} = \begin{bmatrix} 0.0769 - 0.0771i & 0.0769 + 0.0771i & \cdots \\ 0.1376 - 0.1375i & 0.1376 + 0.1375i & \cdots \\ \cdots & -0.0677 + 0.0675i & -0.0677 - 0.0675i \\ \cdots & 0.0754 - 0.0758i & 0.0754 + 0.0757i \end{bmatrix} \quad (41)$$

$$\hat{W}_{c,N4SID} = \begin{bmatrix} -0.077 + 0.0771i & -0.077 - 0.0771i & \cdots \\ -0.1377 + 0.1374i & -0.1377 - 0.1374i & \cdots \\ \cdots & 0.0676 - 0.0674i & 0.0676 + 0.0674i \\ \cdots & -0.0755 + 0.0757i & -0.0755 - 0.0758i \end{bmatrix} \quad (42)$$

and

$$\hat{\Phi}_{c,OKID} = \begin{bmatrix} 0.0769 - 0.0771i & -0.0677 + 0.0674i \\ 0.1376 - 0.1375i & 0.0754 - 0.0758i \end{bmatrix} \quad (43)$$

$$\hat{\Phi}_{c,N4SID} = \begin{bmatrix} -0.077 + 0.0771i & 0.0676 - 0.0675i \\ -0.1377 + 0.1374i & -0.0755 + 0.0757i \end{bmatrix} \quad (44)$$

where $\hat{W}_{c,OKID}$ and $\hat{\Phi}_{c,OKID}$ are the configuration-space eigenvector matrices having respectively dimensions $\hat{n}_x \times \hat{n}_z$ and $\hat{n}_x \times \hat{n}_x$ associated with the state-space model identified using the OKID procedure, whereas $\hat{W}_{c,N4SID}$ and $\hat{\Phi}_{c,N4SID}$ are the configuration-space eigenvector matrices of the same dimensions associated with the state-space model identified using the N4SID algorithm. The identified geometric shapes associated with the two vibration modes of the benchmark system can be determined by normalizing the relative amplitudes and setting the first phase angle to zero. By doing so, one obtains in the absence of noise the numerical results reported in Tables 3 and 4.

Table 3. Physical parameters of the first and second normal modes identified for the benchmark system using the OKID method in the absence of noise.

Symbol	Data (Units)
$\hat{f}_{n,1,OKID}$	2.586 (Hz)
$\hat{f}_{n,2,OKID}$	5.363 (Hz)
$\hat{\xi}_{1,OKID}$	0.00125 (—)
$\hat{\xi}_{2,OKID}$	0.00132 (—)
$\hat{\rho}_{1,OKID}$	$\begin{bmatrix} 1 & 1.7863 \end{bmatrix}^T$ (—)
$\hat{\rho}_{2,OKID}$	$\begin{bmatrix} 1 & 1.1196 \end{bmatrix}^T$ (—)
$\hat{\theta}_{1,OKID}$	$\begin{bmatrix} 0 & -0.0015535 \end{bmatrix}^T$ (rad)
$\hat{\theta}_{2,OKID}$	$\begin{bmatrix} 0 & -3.1448 \end{bmatrix}^T$ (rad)

More specifically, Table 3 contains the physical parameters of the first and the second normal modes identified for the benchmark system, respectively, which were obtained by using the OKID identification method in the absence of noise. Similarly, Table 4 contains the physical parameters of the first and the second normal modes identified for the benchmark system, respectively, which were obtained by using the N4SID identification method in the absence of noise. The numerical values of the eigenvalues and the eigenvectors obtained before completely define the modes of vibration of the benchmark system identified by using the OKID method and the N4SID method.

Considering again the polar representation of the configuration-space eigenvectors, the geometric shapes of the first and second normal modes, identified by using the OKID and the N4SID numerical procedures in the absence of noise, can be drawn as respectively shown in Figures 7 and 8.



Table 4. Physical parameters of the first and second normal modes identified for the benchmark system using the N4SID method in the absence of noise.

Symbol	Data (Units)
$\hat{f}_{n,1,N4SID}$	2.586 (Hz)
$\hat{f}_{n,2,N4SID}$	5.363 (Hz)
$\hat{\xi}_{1,N4SID}$	0.00125 (—)
$\hat{\xi}_{2,N4SID}$	0.00132 (—)
$\hat{\rho}_{1,N4SID}$	$\begin{bmatrix} 1 & 1.7863 \end{bmatrix}^T$ (—)
$\hat{\rho}_{2,N4SID}$	$\begin{bmatrix} 1 & 1.1196 \end{bmatrix}^T$ (—)
$\hat{\theta}_{1,N4SID}$	$\begin{bmatrix} 0 & -0.0015535 \end{bmatrix}^T$ (rad)
$\hat{\theta}_{2,N4SID}$	$\begin{bmatrix} 0 & 3.1384 \end{bmatrix}^T$ (rad)

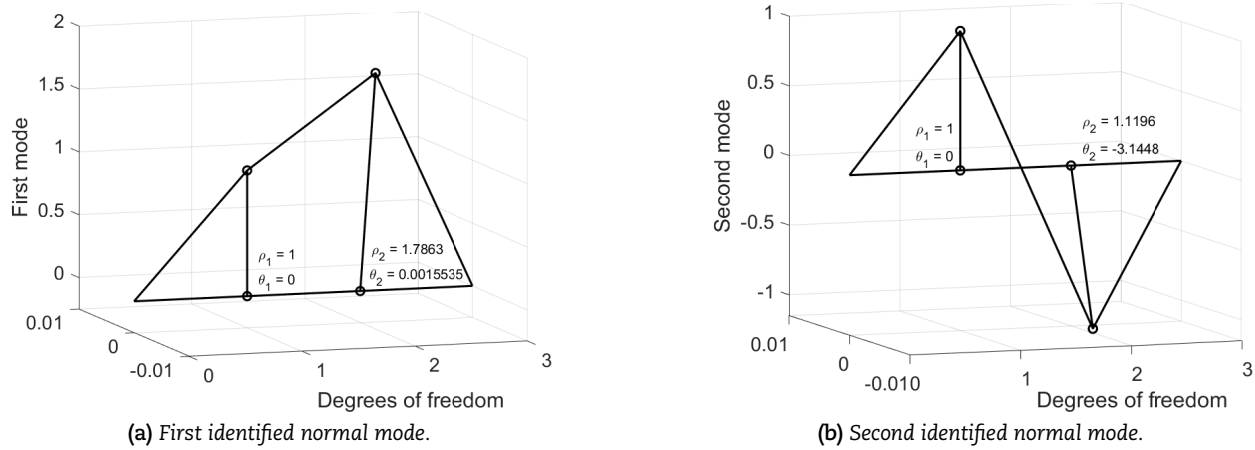


Fig. 7. Geometric shapes of the normal modes identified for the benchmark system using the OKID method in the absence of noise.

In particular, in the case of the use of the OKID identification method in the absence of noise, the geometric shape of the first identified vibration mode is represented in Figure 7a, whereas the identified physical parameters associated with the same normal mode are reported in Table 3. Similarly, in the case of the use of the OKID identification method in the absence of noise, the geometric shape of the second identified vibration mode is represented in Figure 7b, whereas the identified physical parameters associated with the same normal mode are reported in Table 3. On the other hand, in the case of the use of the N4SID identification method in the absence of noise, the geometric shape of the first identified vibration mode is represented in Figure 8a, whereas the identified physical parameters associated with the same normal mode are reported in Table 4. Similarly, in the case of the use of the N4SID identification method in the absence of noise, the geometric shape of the second identified vibration mode is represented in Figure 8b, whereas the identified physical parameters associated with the same normal mode are reported in Table 4.

Since the time response of each degree of freedom is available, it is possible to construct the identified structural matrices associated with the configuration-space dynamical model. In the case of the present benchmark problem and in the absence of noise, one obtains:

$$\hat{M}_{OKID} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \hat{M}_{N4SID} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \quad (45)$$

$$\hat{R}_{OKID} = \begin{bmatrix} 0.11 & -0.01 \\ -0.01 & 0.01 \end{bmatrix}, \quad \hat{R}_{N4SID} = \begin{bmatrix} 0.11 & -0.01 \\ -0.01 & 0.01 \end{bmatrix} \quad (46)$$

$$\hat{K}_{OKID} = \begin{bmatrix} 800 & -300 \\ -300 & 300 \end{bmatrix}, \quad \hat{K}_{N4SID} = \begin{bmatrix} 800 & -300 \\ -300 & 300 \end{bmatrix} \quad (47)$$

where $\hat{M}_{OKID}$, $\hat{R}_{OKID}$, and $\hat{K}_{OKID}$ respectively represent the mass, damping, and stiffness matrices of dimensions $\hat{n}_z \times \hat{n}_z$ identified by using the OKID numerical procedure in the absence of noise, whereas $\hat{M}_{N4SID}$, $\hat{R}_{N4SID}$, and $\hat{K}_{N4SID}$ respectively represent the mass, damping, and stiffness matrices having the same dimensions identified employing the N4SID numerical algorithm in the absence of noise.

To quantitatively appreciate the differences in the dynamic behavior between the identified mechanical system and the original benchmark system, it is possible to reconstruct the time response of the identified mechanical models and then compare them with the responses of the original benchmark model. For this purpose, one can effectively employ a proper fitting function, which serves as a performance metric, based on the Normalized Root-Mean-Square Error (NRMSE) computed between the time responses of the identified mechanical system and the time responses of the benchmark mechanical models. More specifically, in this paper, the formula utilized to quantitatively appreciate the adherence between the two sets of time histories is referred to as the Normalized Root-Mean-Square Fit (NRMSF) function and is defined as follows:

$$NRMSF = 100 \left(1 - \frac{\|y_j^v - \hat{y}_j^v\|}{\|y_j^v - \bar{y}_j^v\|} \right), \quad j = 1, 2, \dots, n_x \quad (48)$$



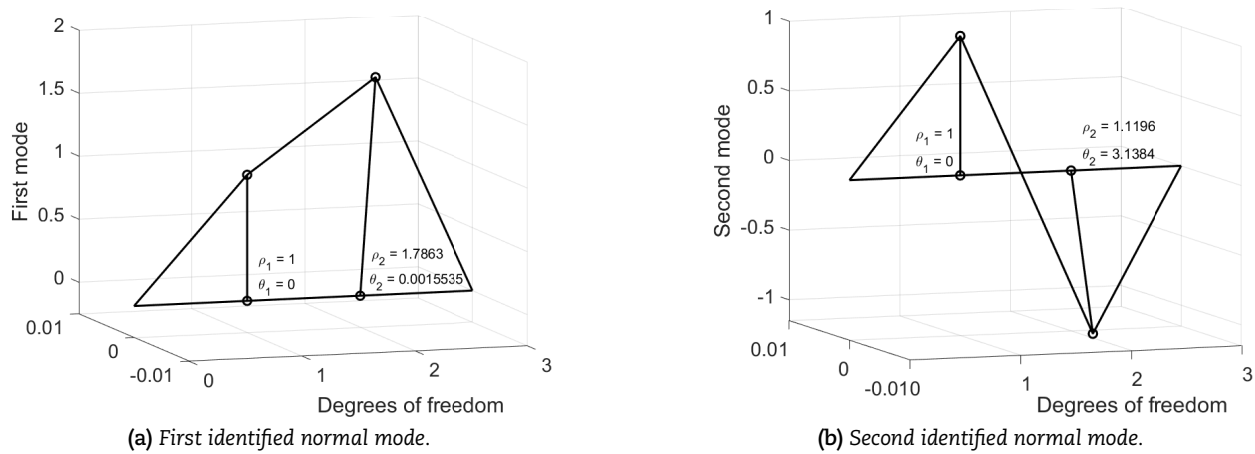


Fig. 8. Geometric shapes of the normal modes identified for the benchmark system using the N4SID method in the absence of noise.

where Δt is the discrete-time step, $N = \text{floor}(T/\Delta t)$ is the number of discretization points excluding the zero, T is the total time interval, $\text{floor}(x)$ is the function that takes as input a real number x and gives as output the greatest integer less than or equal to x , $l = N + 1$ is the length of the discretized arrays associated with the data records, $\mathbf{y}$ is the measurement vector of dimensions $n_y \times 1$ containing the continuous-time response of the benchmark model, $\mathbf{y}_n$ is the measurement vector of dimensions $n_y \times 1$ containing the discrete-time response of the benchmark model, $\mathbf{y}_{n,j}$ is the j component of the measurement vector of dimensions $n_y \times 1$ containing the discrete-time response of the benchmark model, $\mathbf{y}_j^v$ is the reshaped vector associated with the benchmark model of dimensions $l \times 1$ whose entries are the j components of the measurement vector $\mathbf{y}_n$ evaluated at each discrete time n and stacked by rows, $\hat{\mathbf{y}}_j^v$ is the reshaped vector associated with the identified models of dimensions $l \times 1$ whose entries are the j components of the measurement vector $\mathbf{y}_n$ evaluated at each discrete time n and stacked by rows, and $\bar{\mathbf{y}}_j^v$ is the constant vector of dimensions $l \times 1$ containing at each entry the mean value of the time-varying vector $\mathbf{y}_j^v$. For the sake of clarity, the previous vector quantities can be explicitly written as follows:

$$\mathbf{y} = \mathbf{y}(t), \quad \mathbf{y}_n = \mathbf{y}(t_n), \quad \mathbf{y}_{n,j} = y_j(t_n), \quad j = 1, 2, \dots, n_x \quad (49)$$

and

$$\mathbf{y}_j^v = \begin{bmatrix} \mathbf{y}_{0,j} \\ \mathbf{y}_{1,j} \\ \vdots \\ \mathbf{y}_{n,j} \\ \vdots \\ \mathbf{y}_{N-1,j} \\ \mathbf{y}_{N,j} \end{bmatrix} = \begin{bmatrix} y_j(t_0) \\ y_j(t_1) \\ \vdots \\ y_j(t_n) \\ \vdots \\ y_j(t_N) \\ y_j(t_{N-1}) \end{bmatrix}, \quad j = 1, 2, \dots, n_x \quad (50)$$

As shown by its definition given in Equation (48), the function for evaluating the NRMSF of each time response is expressed in percentage terms. Furthermore, this quantitative function is intentionally constructed in such a way that a value close to the 100% corresponds to a total adherence between the identified time responses and those of the benchmark system, whereas a value close to the 0% corresponds to a complete inconsistency between the identified time responses and those of the benchmark system.

Figures 9 and 10 respectively represent the comparison of the time responses of the first and second degrees of freedom, obtained from the mechanical model identified using the OKID method in the absence of noise, with those deriving from the original benchmark lumped parameter model.

In particular, in the case of the use of the OKID identification procedure in the absence of noise, Figures 9a and 10a respectively represent the comparison of the time responses of the first and second degrees of freedom considering the complete time span of the dynamical simulation, whereas Figures 9b and 10b respectively represent the comparison of the time responses of the first and second degrees of freedom considering a zoomed time window of the dynamical simulation. Additionally, Figures 11 and 12 respectively represent the comparison of the time responses of the first and second degrees of freedom, obtained from the mechanical model identified using the N4SID method in the absence of noise, with those deriving from the original benchmark lumped parameter model.

In particular, in the case of the use of the N4SID identification procedure in the absence of noise, Figures 11a and 12a respectively represent the comparison of the time responses of the first and second degrees of freedom considering the complete time span of the dynamical simulation, whereas Figures 11b and 12b respectively represent the comparison of the time responses of the first and second degrees of freedom considering a zoomed time window of the dynamical simulation. As shown in Figures 9, 10, 11, and 12, which were obtained in absence of any artificial noise, the percentage value of the consistency function is close to the maximum possible, thereby demonstrating the complete coherence between the numerical results determined by using the identified mechanical models with those arising from the original benchmark system.

Finally, another important performance assessment of the identification process, which is similar to the response reconstruction carried out before and can still be numerically implemented in a simulation environment, is represented by the robustness analysis. The robustness analysis helps in further assessing the quality of the identification process, as well as the identified dynamical systems, by comparing them again with the benchmark model. It consists in creating a new set of inputs, which should be quite different from the one used to numerically perform the system identification process, and in computing the time responses of both the original benchmark system and the identified mechanical models that are produced in output by the new set of inputs. Figure 13 shows the time history of the new input forcing function used for performing the robustness test.



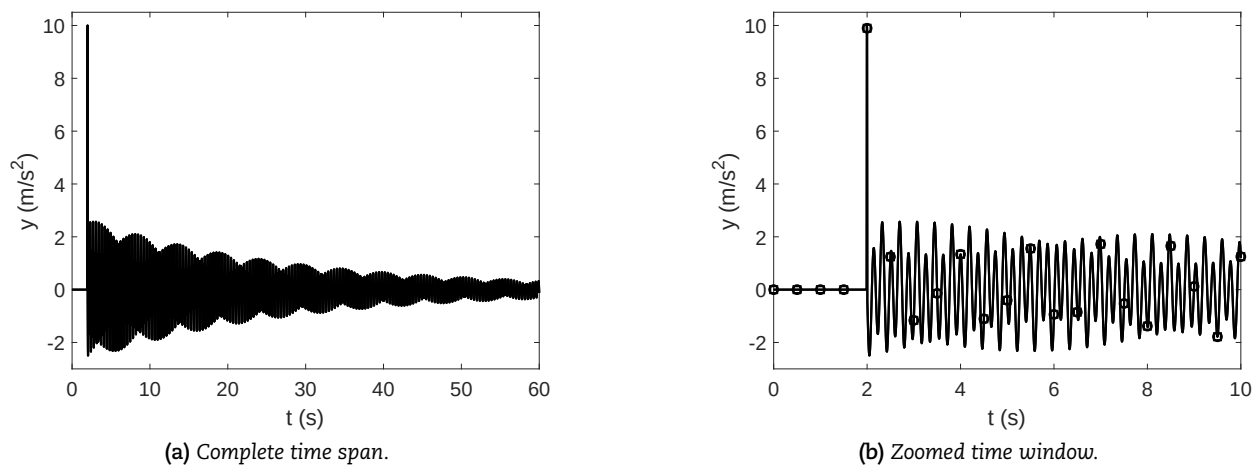


Fig. 9. Comparison between the time response of the first degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the OKID method (circle markers) in the absence of noise. The consistency function arising from the comparison of the time histories of the first degree of freedom is equal to $\text{NRMSF} = 99.999\%$.

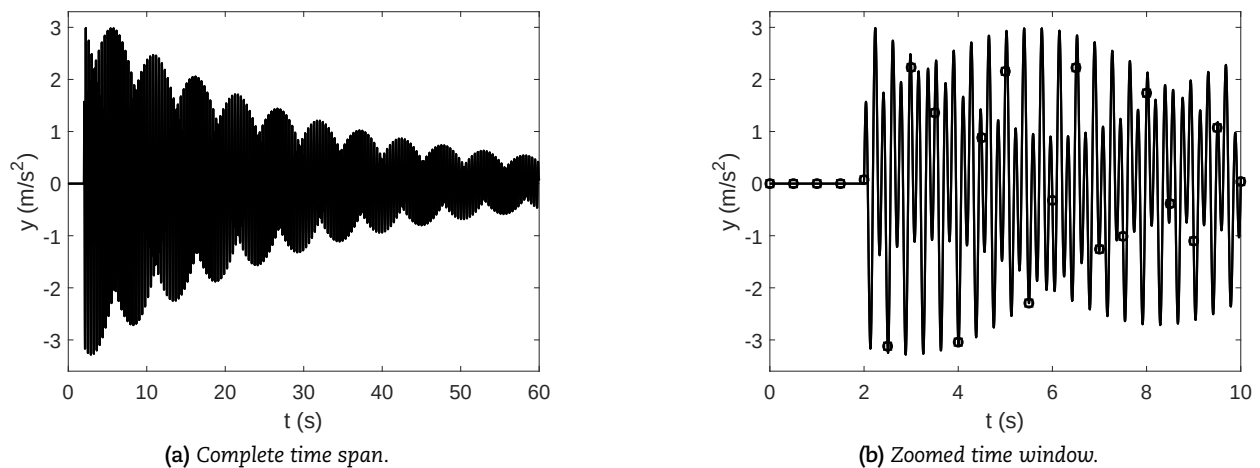


Fig. 10. Comparison between the time response of the second degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the OKID method (circle markers) in the absence of noise. The consistency function arising from the comparison of the time histories of the second degree of freedom is equal to $\text{NRMSF} = 99.999\%$.

As shown in Figure 13, the new input force considered in the robustness test is a function representing a white noise signal having a Gaussian random distribution. As a result, Figures 14 and 15 arise from the robustness test and respectively represent the comparison of the time responses of the first and second degrees of freedom, obtained from the mechanical model identified using the OKID method, with those deriving from the original benchmark lumped parameter model.

In particular, in the case of the use of the OKID identification procedure for performing the robustness analysis, Figures 14a and 15a respectively represent the comparison of the time responses of the first and second degrees of freedom considering the complete time span of the dynamical simulation, whereas Figures 14b and 15b respectively represent the comparison of the time responses of the first and second degrees of freedom considering a zoomed time window of the dynamical simulation. Moreover, Figures 16 and 17 arise from the robustness test and respectively represent the comparison of the time responses of the first and second degrees of freedom, obtained from the mechanical model identified using the N4SID method, with those deriving from the original benchmark lumped parameter model.

In the case of the use of the N4SID identification procedure for performing the robustness analysis, Figures 16a and 17a respectively represent the comparison of the time responses of the first and second degrees of freedom considering the complete time span of the dynamical simulation, whereas Figures 16b and 17b respectively represent the comparison of the time responses of the first and second degrees of freedom considering a zoomed time window of the dynamical simulation. As a final remark, by observing Figures 14, 15, 16, and 17, which were obtained in absence of any artificial noise and considering the robustness test based on an alternative set of forcing functions, one can conclude that the percentage value of the consistency function is close to the maximum possible, thereby demonstrating again the complete coherence between the numerical results calculated by using the identified mechanical models with those obtained from the original benchmark model.

2.3 Numerical Analysis of the Benchmark Problem in the Presence of Noise

The numerical analysis performed so far demonstrates that both the OKID identification method and the N4SID identification technique work optimally in ideal conditions. In real conditions, however, both the measured input and output data are altered by various disturbances. Thus, the presence of noise in real applications makes the identification process more challenging. To simulate this phenomenon in a virtual environment, a random signal of a relatively small entity can be added to the input and output time histories obtained from the dynamical simulations, thereby allowing for assessing the influence of the magnitude of



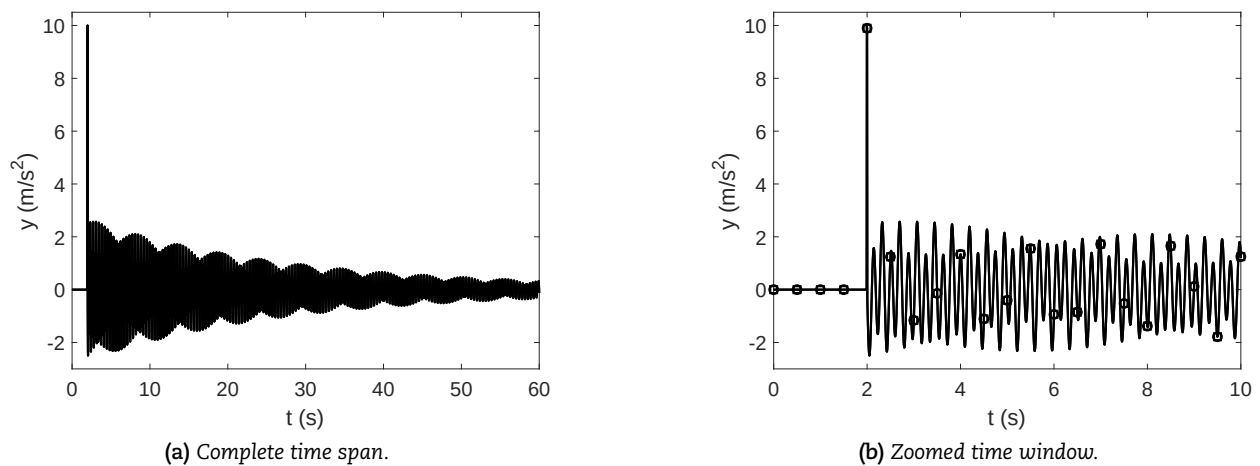


Fig. 11. Comparison between the time response of the first degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the N4SID method (circle markers) in the absence of noise. The consistency function arising from the comparison of the time histories of the first degree of freedom is equal to $\text{NRMSF} = 99.999\%$.

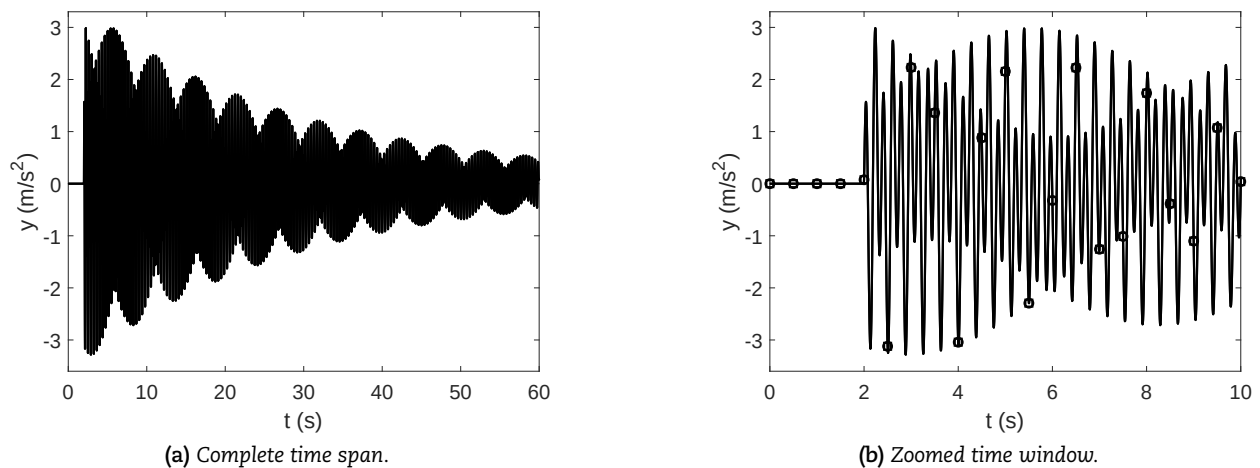


Fig. 12. Comparison between the time response of the second degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the N4SID method (circle markers) in the absence of noise. The consistency function arising from the comparison of the time histories of the second degree of freedom is equal to $\text{NRMSF} = 99.999\%$.

noise on the quality of the identified results and the sensitivity of the identification process to uncontrollable external disturbances. For this purpose, a disturbance signal containing a Gaussian white noise, having a zero mean value denoted with μ and a variable standard deviation value denoted with σ , is added to both the input and the output data. More specifically, to demonstrate the effectiveness of the OKID and the N4SID identification procedures in the presence of noise, the numerical values considered in the virtual environment for simulating the noise are $\mu = 0$ and $\sigma = 10^{1.7} = 50.1187$. Subsequently, the entity of the standard deviation of the noise will be gradually increased starting from the zero value to evaluate the performance of the identification methods considered in the paper.

The impulsive input force assumed for performing the system identification of the benchmark model in the presence of noise is represented in Figure 18.

It is again assumed that the same impulsive input force represented in Figure 18 is applied to both the first and the second particles at the same time. Figures 19 and 20 respectively represent the accelerations of the first and the second concentrated masses corresponding to the two impulsive input forces obtained in the presence of noise.

In particular, Figures 19a and 20a respectively represent the output accelerations of the first and second masses of the benchmark system considering the complete time span of the dynamical simulation in the presence of noise, while Figures 19b and 20b respectively represent a zoomed time window of the same output accelerations.

Figure 21 represents the distribution of the singular values related to the computer implementation of the OKID and the N4SID algorithms, both applied to the benchmark problem and obtained in the presence of noise in the input-output dataset.

Again, a close inspection of Figures 21a and 21b leads to the identification of two couples of singular values whose magnitudes are higher with respect to the remaining ones. Consequently, even in this case, the dimension of the identified state-space model is given by the integer $\hat{n}_z = 4$ and the number of identified degrees of freedom is equal to $\hat{n}_x = 2$. However, in this case in which the presence of noise is taken into account, by observing Figures 21a and 21b representing the distributions of the singular values, one can deduce that the presence of noise made less evident the distinction between the actual system singular values and those induced by the noise, as it is clearly apparent from the smooth decay in the magnitude of the distributions of the singular values. Thus, without having a clear distinction between these singular values, it is more challenging to accurately extract an identified model for real systems affected by noise.

The numerical values of the state-space matrices identified employing the OKID and N4SID methods are not reported here



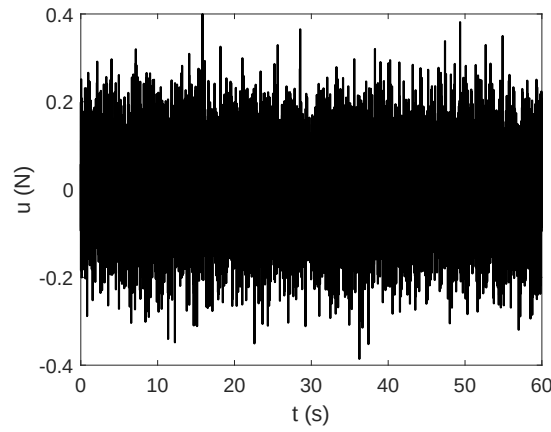


Fig. 13. Alternative input force applied to the benchmark system for performing the robustness analysis in the absence of noise.

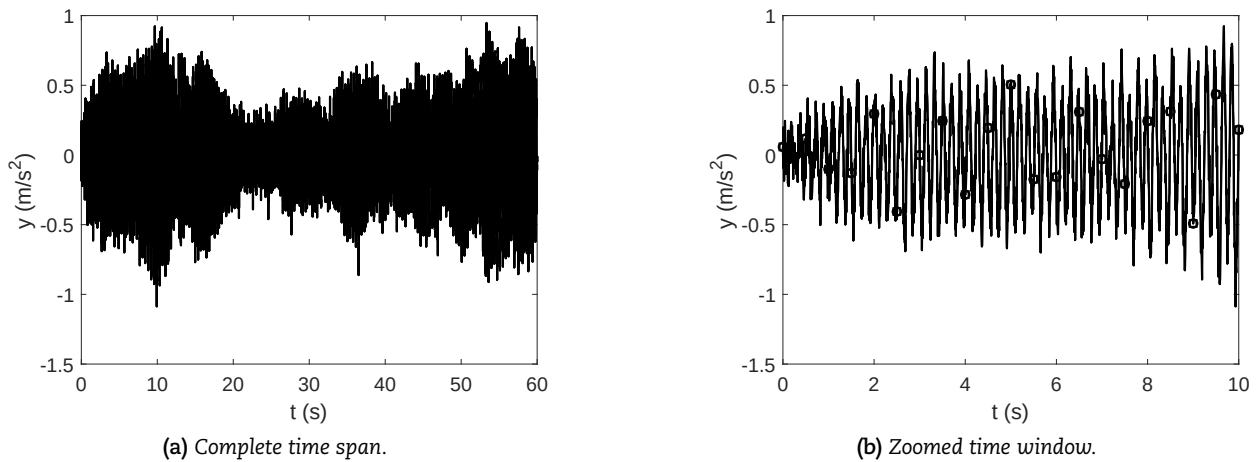


Fig. 14. Robustness comparison between the time response of the first degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the OKID method (circle markers) in the absence of noise. The consistency function arising from the robustness comparison of the time histories of the first degree of freedom is equal to NRMSF = 99.999%.

for saving space. However, in the case of the use of both the OKID and N4SID techniques, a numerical modal analysis can be readily performed using the identified state matrices respectively denoted with $\hat{A}_{c,OKID}$ and $\hat{A}_{c,N4SID}$, leading to the fundamental parameters of interest for the identification process. Once a state-space dynamical model is obtained considering both the OKID and the N4SID identification procedures in the presence of noise, one can readily perform an eigenanalysis of the state matrices found to reconstruct the identified modal parameters. By doing so, one obtains in the presence of noise the numerical results reported in Tables 5 and 6.

More specifically, Table 5 contains the physical parameters of the first and the second normal modes identified for the benchmark system, respectively, which were obtained by using the OKID identification method in the presence of noise. Similarly, Table 6 contains the physical parameters of the first and the second normal modes identified for the benchmark system, respectively, which were obtained by using the N4SID identification method in the presence of noise.

In the presence of noise and considering the benchmark problem, one can construct the identified structural matrices associated with the configuration-space dynamical model when the time response of each degree of freedom is available. Considering both the OKID and the N4SID numerical algorithms, this identification process leads to the following set of matrices:

$$\hat{M}_{OKID} = \begin{bmatrix} 1.01 & -0.004 \\ -0.004 & 0.508 \end{bmatrix}, \quad \hat{M}_{N4SID} = \begin{bmatrix} 1.01 & -0.004 \\ -0.004 & 0.505 \end{bmatrix} \quad (51)$$

$$\hat{R}_{OKID} = \begin{bmatrix} 0.27 & -0.05 \\ -0.05 & 0.14 \end{bmatrix}, \quad \hat{R}_{N4SID} = \begin{bmatrix} 0.16 & -0.022 \\ -0.022 & 0.046 \end{bmatrix} \quad (52)$$

$$\hat{K}_{OKID} = \begin{bmatrix} 815 & -307 \\ -307 & 306 \end{bmatrix}, \quad \hat{K}_{N4SID} = \begin{bmatrix} 813 & -306 \\ -306 & 304 \end{bmatrix} \quad (53)$$

where $\hat{M}_{OKID}$, $\hat{R}_{OKID}$, and $\hat{K}_{OKID}$ respectively represent the mass, damping, and stiffness matrices of dimensions $\hat{n}_z \times \hat{n}_z$ identified by using the OKID numerical procedure in the presence of noise, whereas $\hat{M}_{N4SID}$, $\hat{R}_{N4SID}$, and $\hat{K}_{N4SID}$ respectively represent the mass, damping, and stiffness matrices having the same dimensions identified employing the N4SID numerical algorithm in the presence of noise.

It is apparent that the estimation of the damping matrix in both identification procedures is highly influenced by noise. Therefore, in order to obtain more realistic numerical results, an improved estimation methodology of the identified damping matrix



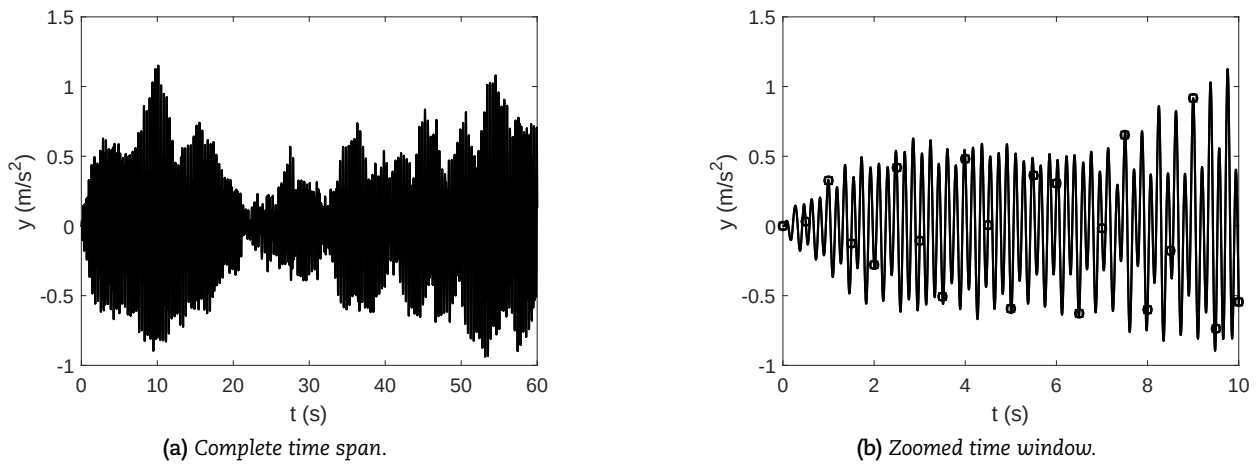


Fig. 15. Robustness comparison between the time response of the second degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the OKID method (circle markers) in the absence of noise. The consistency function arising from the robustness comparison of the time histories of the second degree of freedom is equal to $\text{NRMSF} = 99.999\%$.

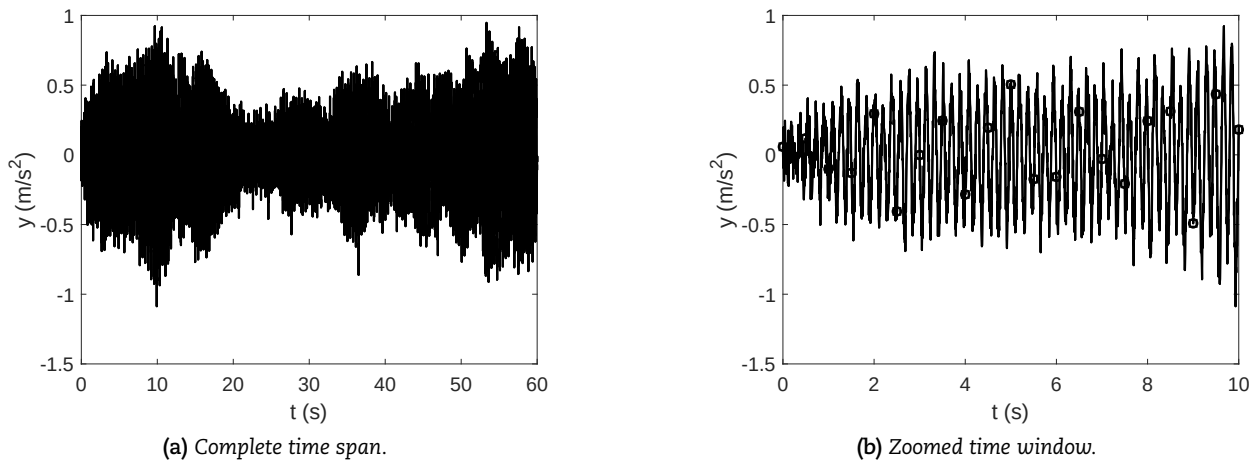


Fig. 16. Robustness comparison between the time response of the first degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the N4SID method (circle markers) in the absence of noise. The consistency function arising from the robustness comparison of the time histories of the first degree of freedom is equal to $\text{NRMSF} = 99.999\%$.

should be adopted. The methodology considered in this paper is a least-square approach for the identification of the proportional damping coefficients, that is, the PDC method based on the hypothesis that the damping matrix is proportional to both the mass matrix and the stiffness matrix. The desired proportionality coefficients resulting from the identification process are denoted with $\hat{\alpha}$ and $\hat{\beta}$, respectively. By implementing the PDC method in the case of the benchmark problem at hand and considering the presence of noise, one obtains:

$$\begin{cases} \hat{\alpha}_{\text{OKID}} = 0.02414 \text{ (1/s)} \\ \hat{\beta}_{\text{OKID}} = 5.303 \text{ (s)} \end{cases} \quad (54)$$

and

$$\begin{cases} \hat{\alpha}_{\text{N4SID}} = 0.02613 \text{ (1/s)} \\ \hat{\beta}_{\text{N4SID}} = 5.413 \text{ (s)} \end{cases} \quad (55)$$

where $\hat{\alpha}_{\text{OKID}}$ and $\hat{\beta}_{\text{OKID}}$ respectively represent the first and the second proportional damping coefficients identified by using the PDC method in the case of the identified model obtained by implementing the OKID identification method in the presence of noise, whereas $\hat{\alpha}_{\text{N4SID}}$ and $\hat{\beta}_{\text{N4SID}}$ respectively represent the first and the second proportional damping coefficients identified by using the PDC method in the case of the identified model obtained by implementing the N4SID identification method in the presence of noise. Thus, two improved estimations of the system damping matrix based on the proportional damping assumption can easily be obtained as follows:

$$\hat{\mathbf{R}}_{\text{OKID}}^* = \hat{\alpha}_{\text{OKID}} \hat{\mathbf{M}}_{\text{OKID}} + \hat{\beta}_{\text{OKID}} \hat{\mathbf{K}}_{\text{OKID}} = \begin{bmatrix} 0.0678 & -0.0165 \\ -0.0165 & 0.0285 \end{bmatrix} \quad (56)$$

and

$$\hat{\mathbf{R}}_{\text{N4SID}}^* = \hat{\alpha}_{\text{N4SID}} \hat{\mathbf{M}}_{\text{N4SID}} + \hat{\beta}_{\text{N4SID}} \hat{\mathbf{K}}_{\text{N4SID}} = \begin{bmatrix} 0.0709 & -0.0168 \\ -0.0168 & 0.0299 \end{bmatrix} \quad (57)$$

where $\hat{\mathbf{R}}_{\text{OKID}}^*$ and $\hat{\mathbf{R}}_{\text{N4SID}}^*$ represent the improved estimations of the damping matrices obtained using the PDC method in the case of the OKID and the N4SID identification methods in the presence of noise, respectively. Considering the use of the PDC method in



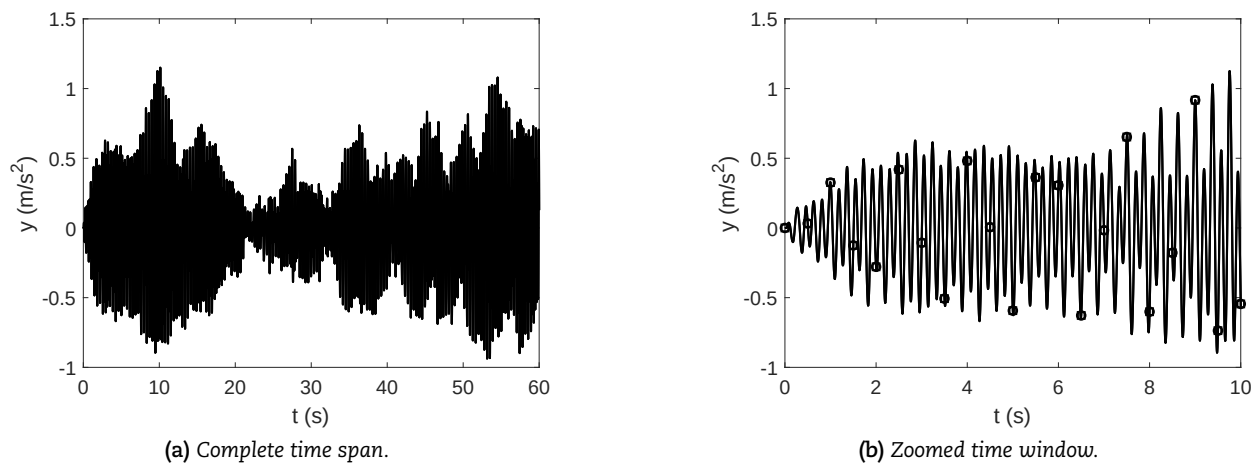


Fig. 17. Robustness comparison between the time response of the second degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the N4SID method (circle markers) in the absence of noise. The consistency function arising from the robustness comparison of the time histories of the second degree of freedom is equal to NRMSF = 99.999%.

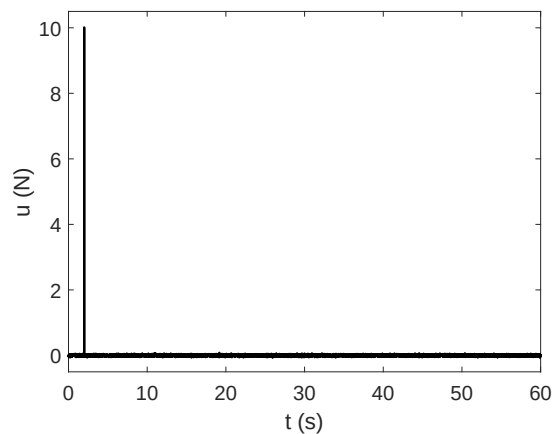


Fig. 18. Impulsive input force applied to the benchmark system in the presence of noise.

practical applications, it can be observed that the smaller is the entity of the damping the better turns out to be the approximation of the damping matrix.

Having identified both the state-space model and the configuration-space model of the benchmark system under consideration, one can again go ahead with the reconstruction of the system time responses in the presence of noise to perform a close comparison between the time histories obtained in this manner. Figures 22 and 23 respectively represent the comparison of the time responses of the first and second degrees of freedom, obtained from the mechanical model identified using the OKID method in the presence of noise, with those deriving from the original benchmark lumped parameter model.

In particular, in the case of the use of the OKID identification procedure in the presence of noise, Figures 22a and 23a respectively represent the comparison of the time responses of the first and second degrees of freedom considering the complete time span of the dynamical simulation, whereas Figures 22b and 23b respectively represent the comparison of the time responses of the first and second degrees of freedom considering a zoomed time window of the dynamical simulation. Furthermore, Figures 24 and 25 respectively represent the comparison of the time responses of the first and second degrees of freedom, obtained from the mechanical model identified using the N4SID method in the presence of noise, with those deriving from the original benchmark lumped parameter model.

In the case of the use of the N4SID identification procedure in the presence of noise, Figures 24a and 25a respectively represent the comparison of the time responses of the first and second degrees of freedom considering the complete time span of the dynamical simulation, whereas Figures 24b and 25b respectively represent the comparison of the time responses of the first and second degrees of freedom considering a zoomed time window of the dynamical simulation. As shown in Figures 22, 23, 24, and 25, which were obtained in presence of artificial noise, the percentage value of the consistency function is still sufficiently high to guarantee the coherence between the two sets of time responses, thereby confirming that the, albeit with some errors, the dynamical model of the original benchmark system was correctly identified.

2.4 Comparison of the System Identification Methods in the Presence of Different Noise Levels

In this subsection, a comparative study of the performance of the system identification methods analyzed in the paper, when applied to the numerical analysis of the benchmark problem considered in this work in the presence of noise having different levels of intensity, is presented. In general, it is not always possible to identify a mechanical model of the dynamic system under study in the presence of noise, especially when the entity of the disturbance is too large and the Signal-to-Noise Ratio (SNR) of the measured input-output data is excessively low. To further clarify the influence of the magnitude of the noise affecting the input and output measurements on the final identified numerical results, a final and systematic analysis is performed herein. The numerical results



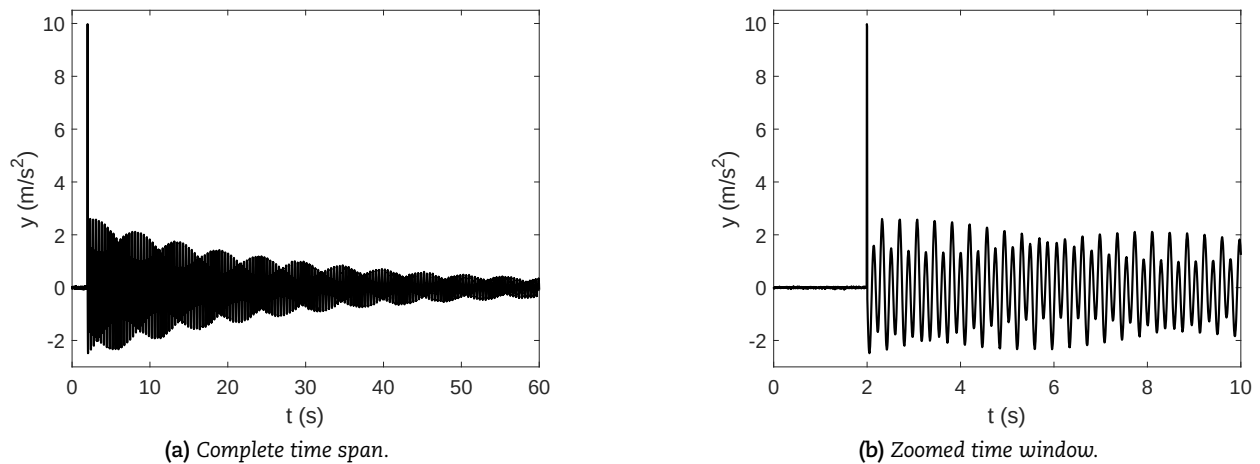


Fig. 19. Impulsive output acceleration of the first mass of the benchmark system in the presence of noise.

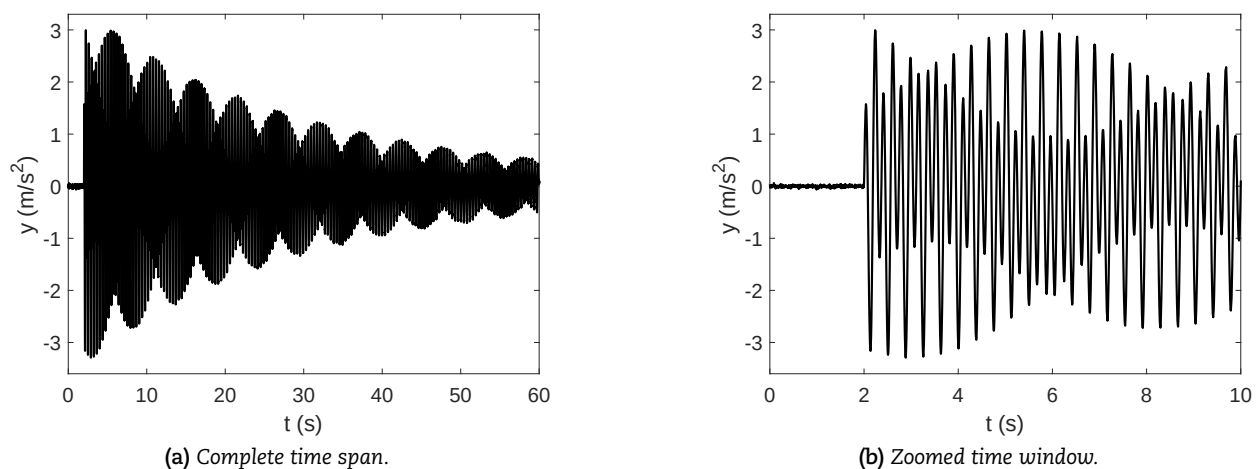


Fig. 20. Impulsive output acceleration of the second mass of the benchmark system in the presence of noise.

concerning the performance of all the identification algorithms considered in the paper are, therefore, compared to achieve this goal. More precisely, this systematic analysis involves the use of the ERA procedure in the case of the OKID identification method and the three variants of the N4SID identification method, which are referred to as the MOESP algorithm, the CVA algorithm, and the SSARX algorithm. This systematic comparison is performed by gradually increasing the standard deviation of the Gaussian white noise added to both the input forces and the output acceleration measurements. The numerical results obtained from the systematic comparison described above are reported in Table 11 provided in Appendix A.

In Table 11, the scalar labeled with the acronym SNR represents the result of a function implemented in the MATLAB environment that returns information relating to the relationship between the signal and the noise, applicable to both the arrays containing the input actions and the output measurements, thereby calculating the signal-to-noise ratios of each time histories. To consistently compare the performance of all the identification algorithms, the calculations of the relative errors of the natural frequencies and the relative errors of the damping are also evaluated and reported in Table 11. By consulting Table 11, one can be observed that, in the absence of noise, that is, when one has $\mu = 0$ and $\sigma = 0$, the SSARX method does not identify the system at all, finding incorrect numerical solutions. Therefore, the ellipsis symbols reported in Table 11 serve to indicate the inaccurate numerical results found. In the numerical experiments reported in Table 11, the noise level is gradually increased from zero to an extreme value, in which the dynamic content of the system response becomes indistinguishable from the random disturbances. In conclusion, the systematic numerical analysis reported in Table 11 demonstrates that, for the benchmark system considered in the presence of an increasing level of noise, the identification procedure that leads to the best set of numerical results is the CVA-N4SID algorithm, immediately followed by the MOESP-N4SID method and, subsequently, by the ERA-OKID technique. The ERA-OKID method and the CVA-N4SID technique are, therefore, the system identification procedures used in the subsequent experimental testing.

3. Experimental Results

The objective of the analysis described in this section is to experimentally identify the dynamical model of a frame structure made of harmonic steel by using the ERA-OKID and the CVA-N4SID identification algorithms. To this end, only the ERA technique is used in the computer implementation of the OKID method, while only the CVA technique is employed in the computer implementation of the N4SID method. A lumped parameter model of the case study is constructed and analyzed first, and then a finite element model is developed in the SOLIDWORKS environment capable of performing a CAD-CAE analysis. The last part of this section deals with the description of the experimental identification process and the corresponding results found by using a pre-processing of the data through signal filters provided to the two identification algorithms mentioned above. Finally, the results of the experimental identification process are compared with those obtained using the lumped parameter model and the finite element model of the vibrating structure considered as the case study.



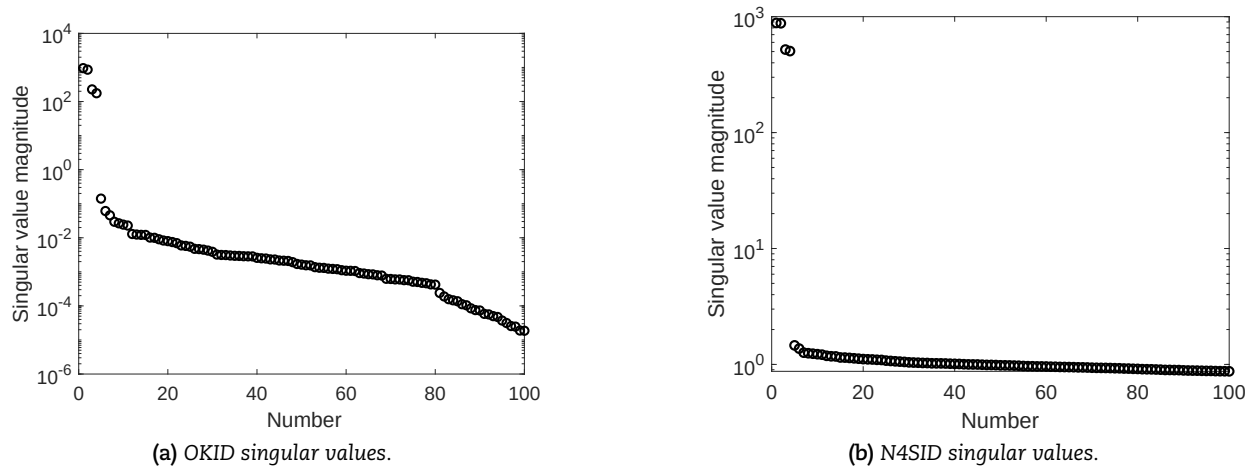


Fig. 21. Distribution of the identified singular values obtained for the benchmark problem by implementing the OKID and the N4SID identification methods in the presence of noise.

Table 5. Physical parameters of the first and second normal modes identified for the benchmark system using the OKID method in the presence of noise.

Symbol	Data (Units)
$\hat{f}_{n,1,OKID}$	2.586 (Hz)
$\hat{f}_{n,2,OKID}$	5.363 (Hz)
$\hat{\xi}_{1,OKID}$	0.00117 (—)
$\hat{\xi}_{2,OKID}$	0.00125 (—)
$\hat{\rho}_{1,OKID}$	$\begin{bmatrix} 1 & 1.7836 \end{bmatrix}^T$ (—)
$\hat{\rho}_{2,OKID}$	$\begin{bmatrix} 1 & 1.1187 \end{bmatrix}^T$ (—)
$\hat{\theta}_{1,OKID}$	$\begin{bmatrix} 0 & 0.00541 \end{bmatrix}^T$ (rad)
$\hat{\theta}_{2,OKID}$	$\begin{bmatrix} 0 & -3.1425 \end{bmatrix}^T$ (rad)

3.1 Description of the Case Study

The case study considered in the experimental analysis presented herein is the flexible structure represented in Figure 26.

The vibrating structure consists of six harmonic steel beams and four aluminum bars rigidly connected through bolted joint elements. The first bar located at the bottom of the structure, which is the longest one shown in Figure 26, is fixed on the test bench through two clamps, thereby making a rigid connection. As discussed in detail below, the case study considered herein is a mechanical system that, in its simplest form, can be schematized as a three-degree-of-freedom dynamical model.

As confirmed by the discussion on the experimental measurements reported below in the subsection describing the experimental acquisitions, the frequency range of interest for the present study is between 0 (Hz) and 15 (Hz). For this frequency range, one can reasonably assume that the flexible beams deform as continuous linear elastic bodies and the bars behave essentially as rigid bodies. The plane taken into consideration is the one on which the structure is deployed, which develops along the main direction of the four bars. Consequently, the structure as a whole can be schematized in a two-dimensional space since the lateral and torsional stiffness coefficients of the flexible structure are considerably greater than the bending stiffness in the plane of interest.

3.2 Lumped Parameter Model

A schematic representation of the lumped parameter model constructed in this study is shown in Figure 27.

On the basis of proper approximations and simplifying assumptions, it is possible to develop a linear model with lumped parameters of the vibrating structure considered as the case study of the experimental analysis, which describes very closely the physical behavior of the real system under study. The capacity of the lumped parameter model to describe the dynamical behavior of the actual frame structure studied in the experimental analysis depends on the extent of the simplifying approximations employed. Thus, the above considerations allow for constructing a mathematical model of the lumped parameter system represented in Figure 27.

The mathematical model developed for the case study is a lumped parameter dynamical model with three degrees of freedom. These are the horizontal displacements of the first, second, and third rigid bars, respectively, which are respectively indicated by x_1 , x_2 , and x_3 . Therefore, the system generalized coordinate vector denoted with x has a dimension $n_x = 3$ and is given by:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (58)$$

In the configuration space, the mathematical model based on lumped parameters of the mechanical system under study can be written in a compact matrix form as follows:

$$M\ddot{x} + Kx = 0 \quad (59)$$

where M and K respectively identify the system mass matrix and the system stiffness matrix, having both dimensions $n_x \times n_x$. Equation (58) defines the vector of generalized coordinates, while Equation (59) represents the matrix form of the equation of motion. Given that the excitations are of not too high intensity and their dynamic content in the frequency spectrum is limited, the behavior



Table 6. Physical parameters of the first and second normal modes identified for the benchmark system using the N4SID method in the presence of noise.

Symbol	Data (Units)
$\hat{f}_{n,1,N4SID}$	2.586 (Hz)
$\hat{f}_{n,2,N4SID}$	5.363 (Hz)
$\hat{\xi}_{1,N4SID}$	0.00125 (—)
$\hat{\xi}_{2,N4SID}$	0.00130 (—)
$\hat{\rho}_{1,N4SID}$	$\begin{bmatrix} 1 & 1.7860 \end{bmatrix}^T$ (—)
$\hat{\rho}_{2,N4SID}$	$\begin{bmatrix} 1 & 1.1190 \end{bmatrix}^T$ (—)
$\hat{\theta}_{1,N4SID}$	$\begin{bmatrix} 0 & 0.00152 \end{bmatrix}^T$ (rad)
$\hat{\theta}_{2,N4SID}$	$\begin{bmatrix} 0 & -3.1439 \end{bmatrix}^T$ (rad)

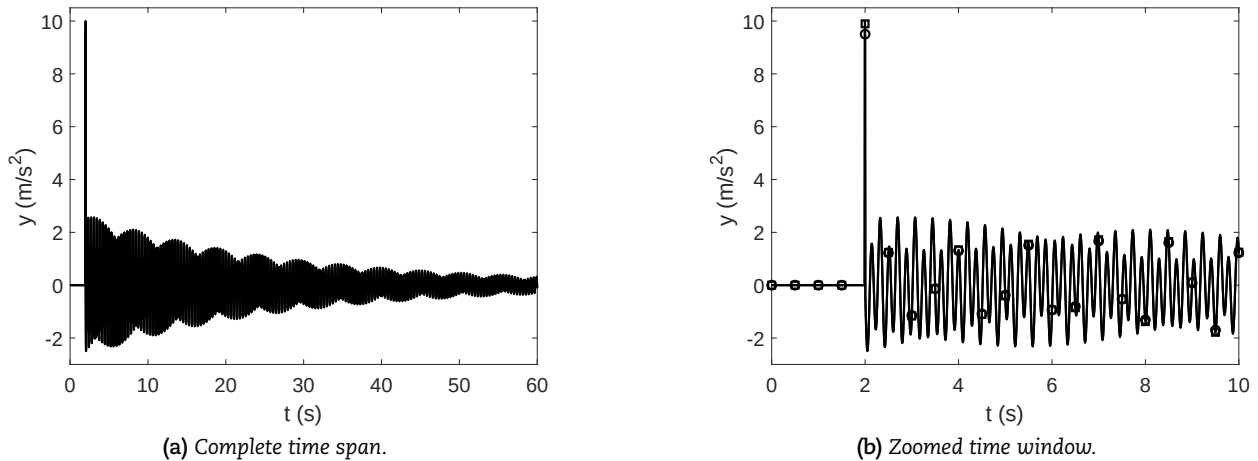


Fig. 22. Comparison between the time response of the first degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the OKID method (circle markers) in the presence of noise. The consistency function arising from the comparison of the time histories of the first degree of freedom is equal to $NRMSF = 86.934\%$.

of the flexible structure is reasonably assumed to be linear with a good approximation. The modal analysis of this mechanical system is necessary to be able to make a comparison with the subsequent experimental identification. Thus, a system modal analysis is performed in this study in a manner similar to the developments described in Subsection 2.1.

In the present study, the damping of the lumped-parameter model is deliberately neglected because it is assumed to be small. However, the value of the damping coefficients will be subsequently estimated during the experimental identification process. On the other hand, by employing a Lagrangian approach, one can readily derive the following mathematical structure of the system mass and stiffness matrices:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \quad (60)$$

where $m_1 = 0.79$ (kg), $m_2 = 0.79$ (kg), and $m_3 = 0.576$ (kg) respectively represent the masses of the rigid bars of the first, second, and third floors, while the stiffness coefficients are calculated considering that each pair of flexible beams can be seen as two horizontal springs in parallel. For this reason, it is possible to calculate the stiffness coefficients considering a clamped-clamped set of boundary conditions of the elastic beam elements as follows:

$$k_1 = \frac{24EJ}{L_1^3}, \quad k_2 = \frac{24EJ}{L_2^3}, \quad k_3 = \frac{24EJ}{L_3^3} \quad (61)$$

where $L_1 = 0.24$ (m), $L_2 = 0.29$ (m), and $L_3 = 0.24$ (m) respectively represent the lengths of the first, second, and third beams that support the floors of the structure, whereas $J = 2.917 \cdot 10^{-12}$ (m⁴) is the area moment of inertia of the flexible beam cross sections since all beams have the same thickness and the same width, while $E = 207 \cdot 10^9$ (N/m²) identifies the elastic modulus of the harmonic steel. The mass and stiffness matrices denoted with $\mathbf{M}$ and $\mathbf{K}$, which contain the numerical values calculated before, are provided below:

$$\mathbf{M} = \begin{bmatrix} 0.79 & 0 & 0 \\ 0 & 0.79 & 0 \\ 0 & 0 & 0.576 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1642 & -595 & 0 \\ -595 & 1642 & -1049 \\ 0 & -1049 & 1048 \end{bmatrix} \quad (62)$$

Following the same steps of the modal analysis described in Subsection 2.1, it is possible to determine the normal modes of the lumped parameter system. In particular, Table 7 contains the fundamental parameters associated with the three normal modes of the lumped parameter model of the frame structure considered as the case study.

In Figure 28, the geometric shapes associated with the three normal modes of the lumped parameter model of the frame structure considered as the case study are represented. More specifically, Figures 28a, 28b, and 28c show a graphical representation of the geometric shapes of the first, second, and third normal modes of the system lumped parameter model.



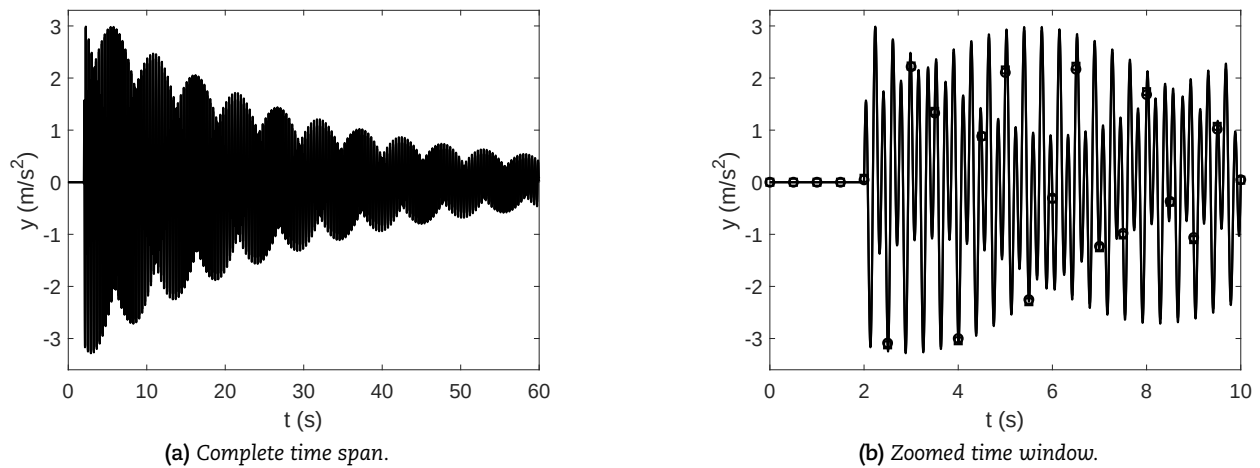


Fig. 23. Comparison between the time response of the second degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the OKID method (circle markers) in the presence of noise. The consistency function arising from the comparison of the time histories of the second degree of freedom is equal to NRMSF = 85.011%.

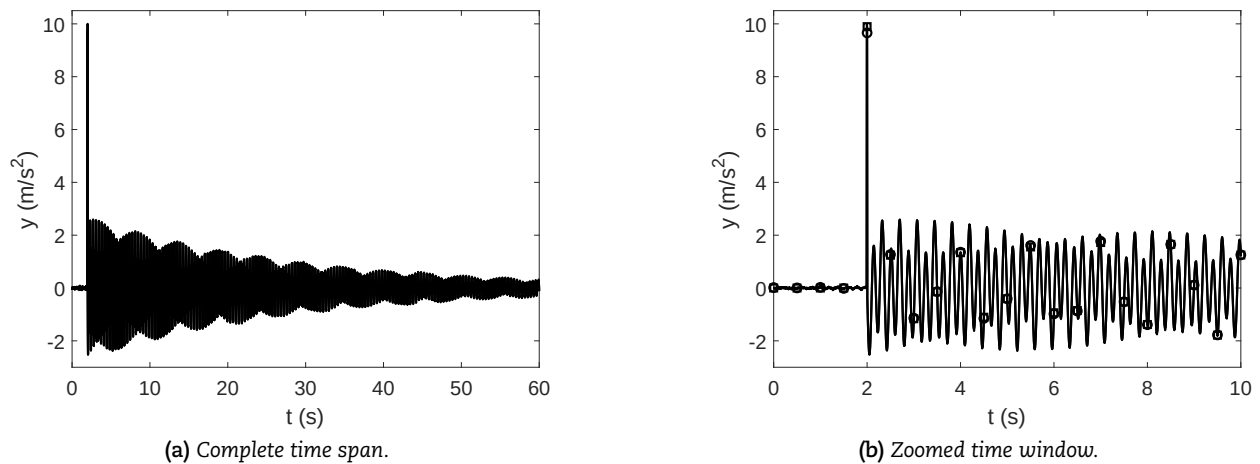


Fig. 24. Comparison between the time response of the first degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the N4SID method (circle markers) in the presence of noise. The consistency function arising from the comparison of the time histories of the first degree of freedom is equal to NRMSF = 90.663%.

As expected, the vibrating modes of the lumped parameter system are three. More precisely, the first vibrating mode has no nodes, the second has only one node, and the third has two nodes. Thus, the modal analysis of the lumped parameter model of the mechanical system considered as the case study of the experimental analysis provides numerical results that are physically consistent.

3.3 Finite Element Model

In this subsection, the numerical results of a finite element analysis carried out considering the frame structure assumed as the case study are described. As expected, the finite element analysis allows for obtaining more accurate results than those obtained with the lumped parameter model. A CAD model of the mechanical system under study in the experimental analysis is shown in Figure 29.

The finite element model of the frame structure was developed in the SOLIDWORKS environment, which is the same environment employed for constructing the CAD model of the mechanical system of interest. More specifically, the frame structure represented in Figure 29 was modeled using the CAD-CAE program implemented in SOLIDWORKS by defining both the geometric dimensions, the inertia matrices, and the material properties of each component. In some specific details, the geometry of the virtual model was simplified to facilitate the subsequent finite element analysis. In this respect, an example is the omission of the connecting screws which were replaced with the imposition of algebraic constraints. The frame structure was fixed to the base and the first three normal modes of the mechanical model were obtained through the use of the SOLIDWORKS simulation section. In particular, Table 8 contains the fundamental parameters associated with the three normal modes of the finite element model of the frame structure considered as the case study.

In Figure 30, the geometric shapes associated with the three normal modes of the finite element model of the frame structure considered as the case study are represented. More specifically, Figures 30a, 30b, and 30c show a graphical representation of the geometric shapes of the first, second, and third normal modes of the system finite element model.

Again, the principal vibrating modes of the finite element model are the first three modes. More precisely, the first vibrating mode has no nodes, the second has only one node, and the third has two nodes. The physical parameters and the geometric shapes of the normal modes found by using the finite element model of the vibrating structure considered as the case study are physically consistent and will serve to corroborate the numerical results arising from the system identification process based on experimental



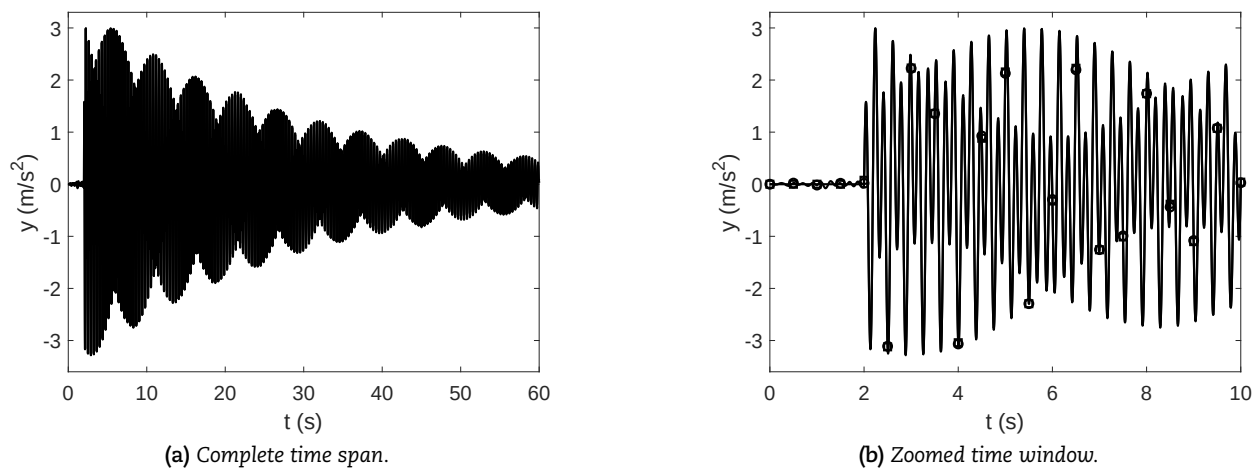


Fig. 25. Comparison between the time response of the second degree of freedom determined by employing the mechanical model of the benchmark system (square markers) and the same time response obtained from the mechanical model identified by using the N4SID method (circle markers) in the presence of noise. The consistency function arising from the comparison of the time histories of the second degree of freedom is equal to $NRMSF = 91.0595\%$.

Table 7. Physical parameters of the first, second, and third normal modes of the lumped parameter model of the frame structure considered as the case study.

Symbol	Data (Units)
$f_{n,1}$	2.4727 (Hz)
$f_{n,2}$	7.1709 (Hz)
$f_{n,3}$	9.6897 (Hz)
ρ_1	$\begin{bmatrix} 1 & 2.4433 & 2.8169 \end{bmatrix}^T (-)$
ρ_2	$\begin{bmatrix} 1 & 0.0651 & 0.5644 \end{bmatrix}^T (-)$
ρ_3	$\begin{bmatrix} 1 & 2.1639 & 2.0873 \end{bmatrix}^T (-)$
θ_1	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T (rad)$
θ_2	$\begin{bmatrix} 0 & 0 & 3.1416 \end{bmatrix}^T (rad)$
θ_3	$\begin{bmatrix} 0 & -3.1416 & 0 \end{bmatrix}^T (rad)$

data described below.

3.4 Experimental Test Rig and Data Acquisition System

The present subsection describes the experimental test rig and the data acquisition system employed for performing the system identification process and the subsequent experimental modal analysis. In Figure 31, the equipment employed for this purpose is shown together with the frame structure to be experimentally identified.

As shown in Figure 31, to ensure only bending vibrations in the reference plane, the flexible structure described in Subsection 3.1 is rigidly connected to the workbench via two clamps. Assuming a schematic representation of the experimental test rig together with the data acquisition system, a frame structure of three floors is endowed with three accelerometers, one on each floor, connected with a signal analyzer for recording the output signals. An impact hammer equipped with a load cell is also connected to the same spectrum analyzer for recording the input signal. By doing so, the acceleration signals measured by the piezoelectric sensors in correspondence to each floor when the frame structure is excited by the impact hammer instrumented with a load cell are simultaneously recorded together with the force signal. The input-output data obtained from the experimental apparatus are, therefore, suitable for the use of the system identification numerical procedures considered in this investigation. More specifically, the instrumentation used for the data acquisition is shown in Figure 32.

The frame structure is excited laterally on one of the three bars by means of the Brüel & Kjær type 8202 impact hammer represented in Figure 32a. The impact hammer is instrumented with a rubber tip, with a metallic counterweight, and with a Brüel & Kjær type 8200 load cell. The load cell used has a sensitivity of $3.8 (pC/N)$. The signal obtained is sent to the Brüel & Kjær type 2692 Nexus charge amplifier shown in Figure 32c, which transforms the signal from $3.8 (pC/N)$ to $100 (mV/N)$, and then send it to the Brüel & Kjær type 2825 signal analyzer represented in Figure 32b.

As shown in Figure 31, the acceleration signals are measured by appropriate transducers positioned laterally to the structure, one on each bar representing the floors of the three-story frame structure. In particular, the three transducers used are uniaxial accelerometers. The first transducer is the Brüel & Kjær type 4507 piezoelectric accelerometer, which has a sensitivity of $10 (mV/(m/s^2))$ and is connected directly to the Brüel & Kjær type 2825 signal analyzer. The second transducer is the Brüel & Kjær type 4371 piezoelectric accelerometer having a sensitivity of $1.01 (pC/(m/s^2))$. Its output signal is sent to the Brüel & Kjær type 2635 amplifier, which transforms the output from $1.01 (pC/(m/s^2))$ to $10 (mV/(m/s^2))$ and then send it to the Brüel & Kjær type 2825 signal analyzer. The third transducer is the ADXL193 accelerometer, which is a MEMS accelerometer, has a sensitivity of $0.8155 (mV/(m/s^2))$, and is connected directly to the Brüel & Kjær type 2825 signal analyzer. The signal of this sensor is more subject to disturbances and noises than those mentioned before. However, by properly filtering the acceleration measurements acquired by using this transducer, it is still possible to obtain a sufficiently high signal-to-noise ratio.

The Brüel & Kjær type 2825 signal analyzer is connected to a computer running the Brüel & Kjær-PULSE Labshop v.4.2 software. This program allows for readily setting the fundamental parameters of the data acquisition system, such as the sensitivity of the inputs to the signal analyzer, the acquisition time T_s , the sampling frequency f_s , and the offset of the measured signal. By doing





Fig. 26. Three-story frame structure analyzed in the experimental study.

so, the acquisition time T_s is set equal to 32 (s), while the sampling frequency f_s is set equal to 50 (Hz).

Before proceeding to the identification process, in order to minimize the distortion effects due to noise and to eliminate parts of the measurements without physical meaning, the data acquired from the measurement system were filtered both in the time domain and in the frequency domain. The time-domain filter consists in imposing equal to zero the values of the measured signals where it is sure that the input/output records must be null. For example, since the experimental test consists of giving a single excitation input to the vibrating structure on one of the three bars, then, before the impact hammer hits the structure, the input and output values of the recorded signals must be null because any information other than zero has no physical meaning. In the frequency domain, on the other hand, a low-pass Butterworth filter was used to eliminate high-frequency noise. In particular, for the data measured using the Brüel & Kjær type 4507 and type 4371 accelerometers, the low-pass Butterworth filter was used with a cut-off frequency of 15 (Hz), thereby reducing the noise at high frequencies. Additionally, in the case of the data measured using the ADXL193 accelerometer, a band-pass Butterworth filter was implemented. In this case, the cut-off frequencies employed were 1.8 (Hz) and 11 (Hz) for the lower and the higher frequency, respectively. These values of the cut-off frequencies were chosen by analyzing the plots of the acceleration measurements transformed in the frequency domain.

3.5 System Identification and Experimental Modal Analysis

The system identification process based on experimental data can be divided into three fundamental steps. In the first step, the instrumentation is set and the data acquisition is carried out. In the second step, the data obtained from the acquisitions are filtered both in the time and in the frequency domains. In the third step, the filtered data is provided to the identification algorithms in order to derive an estimation of a mechanical model of the flexible structure considered as the case study.

The acquisition of the experimental data was performed as described in Section 3.4. More specifically, the three accelerometers were positioned respectively on the first, second, and third floors. The frame structure was excited starting from the condition of static equilibrium by hitting the second floor using the impact hammer, which is instrumented with the load cell to record the external input force applied to the flexible system. Subsequently, the data acquired by the accelerometers were filtered both in the time domain and in the frequency domain.

Figure 33 shows the data acquisition in the time domain. In particular, Figure 33a represents the input force produced by the impact hammer, while Figures 33b, 33c, and 33d respectively represent the corresponding output accelerations of the first, second, and third floors measured in the time domain.

Figure 34 shows the data acquisition in the frequency domain. In particular, Figure 34a represents the input force produced by the impact hammer, while Figures 34b, 34c, and 34d respectively represent the corresponding output accelerations of the first, second, and third floors measured in the frequency domain.

It is important to note that the data acquired from the load cell of the impact hammer was filtered only in the time domain. In Figures 33a and 34a, this data is represented both in the time domain and in the frequency domain, respectively. As discussed above, the frequency-domain filter must be used with appropriately chosen cut-off frequencies. If the frequency range that is filtered also



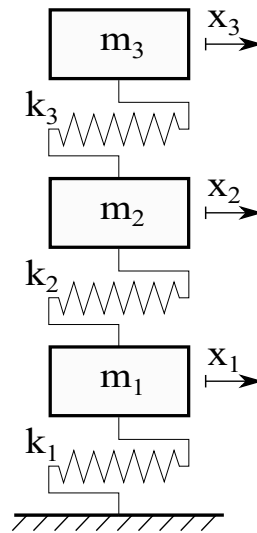


Fig. 27. Three-story frame lumped parameter model.

Table 8. Physical parameters of the first, second, and third normal modes of the finite element model of the frame structure considered as the case study.

Symbol	Data (Units)
$f_{n,1}$	2.3915 (Hz)
$f_{n,2}$	7.1693 (Hz)
$f_{n,3}$	9.8410 (Hz)
ρ_1	$\begin{bmatrix} 1 & 2.8534 & 3.8156 \end{bmatrix}^T (-)$
ρ_2	$\begin{bmatrix} 1 & 0.0752 & 0.5276 \end{bmatrix}^T (-)$
ρ_3	$\begin{bmatrix} 1 & 2.8421 & 3.2400 \end{bmatrix}^T (-)$
θ_1	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T (rad)$
θ_2	$\begin{bmatrix} 0 & 0 & -3.1416 \end{bmatrix}^T (rad)$
θ_3	$\begin{bmatrix} 0 & -3.1416 & 0 \end{bmatrix}^T (rad)$

includes the system vibration frequencies, there is a risk of distorting the data acquisition. For this reason, it was preferred not to use the frequency filter for the forcing signal in order not to corrupt the acquisition of the input measurement, which returned an excellent signal even if it was only filtered with the use of the time-domain filter.

By providing the filtered input-output data to the ERA-OKID method and to the CVA-N4SID algorithm, it is possible to identify a dynamic model of the flexible structure by using both these system identification methods. By doing so, one can readily identify the singular values of the identified discrete-time state-space models arising from the implementation of the identification procedures considered in the paper. Figure 35 represents the distribution of the singular values related to the computer implementation of the ERA-OKID and the CVA-N4SID techniques, both obtained for the case study analyzed in this work.

In analogy to the benchmark problem analyzed in the case of the numerical experiments, the observation of the distribution of the singular values of the case study allows for identifying the most significant degrees of freedom of the identified system state-space model. More specifically, a close inspection of Figures 35a and 35b leads to the identification of three couples of singular values whose magnitudes are significantly higher with respect to the other ones. Since both the ERA-OKID and the CVA-N4SID algorithms allow for properly setting the dimensions of the identified state-space model denoted with $\hat{n}_z = 2\hat{n}_x$, that is, the number of degrees of freedom of the mechanical model to be identified denoted with $\hat{n}_x$, one can infer from the actual distribution of the identified singular values that the dimension of the identified state-space model is given by the integer $\hat{n}_z = 6$. Therefore, the number of identified degrees of freedom is equal to $\hat{n}_x = 3$. Moreover, the dimensions of the identified input and output vectors are equal to the original ones and are respectively denoted with the integers $\hat{n}_u = n_u = 1$ and $\hat{n}_y = n_y = 3$. However, in the case of the experimental analysis presented herein, the distinction between the singular values associated with the proper vibration modes of the dynamical system under study and those arising from the presence of the noise in the experimental input-output measurements is not clear and well-definite such as in the case found for the numerical analysis of the benchmark problem. Nevertheless, for the flexible frame structure under study, by setting $\hat{n}_{z,OKID} = \hat{n}_z = 6$ and $\hat{n}_{z,N4SID} = \hat{n}_z = 6$ in both the ERA-OKID and the CVA-N4SID numerical procedures, the following set of identified continuous-time state-space matrices can be experimentally found:

$$\hat{A}_{c,OKID} = \begin{bmatrix} -0.007 & 61.838 & -0.429 & 0.369 & -0.019 & 0.114 \\ -62.592 & -0.548 & 0.216 & -0.267 & -0.003 & -0.14 \\ 1.136 & 0.133 & 0.175 & 14.853 & 0.032 & 0.077 \\ 0.056 & 0.995 & -15.18 & -0.247 & -0.223 & -0.144 \\ 0.23 & 0.788 & -0.451 & -0.226 & -0.389 & -43.723 \\ -1.262 & -2.003 & 0.801 & 1.53 & 45.401 & -2.355 \end{bmatrix} \quad (63)$$



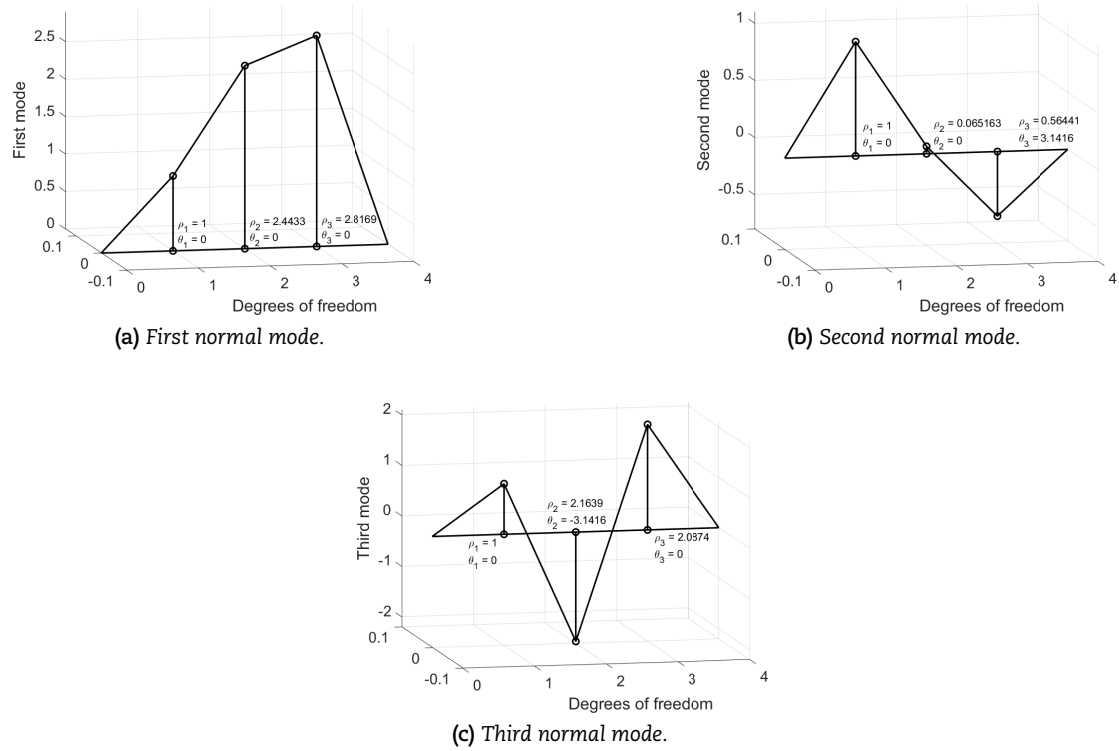


Fig. 28. Geometric shapes of the first, second, and third normal modes of the lumped parameter model of the frame structure considered as the case study.

$$\hat{B}_{c,OKID} = \begin{bmatrix} 0.722 \\ 1.568 \\ -0.851 \\ -1.001 \\ -0.901 \\ 2.83 \end{bmatrix} \quad (64)$$

$$\hat{C}_{c,OKID} = \begin{bmatrix} -62.059 & 30.149 & -3.735 & -20.644 & -1.811 & 6.767 \\ -19.094 & 15.606 & -4.762 & 8.07 & 1.619 & 2.892 \\ 56.476 & -50.801 & 14.099 & -20.682 & 0.481 & 2.447 \end{bmatrix} \quad (65)$$

$$\hat{D}_{c,OKID} = \begin{bmatrix} -0.001 \\ 0.03 \\ -0.019 \end{bmatrix} \quad (66)$$

and

$$\hat{A}_{c,N4SID} = \begin{bmatrix} 0.021 & 15.072 & 0.244 & 0.326 & -0.102 & -0.248 \\ -14.921 & -0.119 & 0.151 & -0.059 & -0.125 & 0.381 \\ 0.005 & 0.016 & -0.634 & -62.284 & 1.07 & -1.06 \\ -0.004 & -0.009 & 61.647 & -0.201 & 0.194 & 0.303 \\ -0.002 & 0 & -0.643 & -0.017 & -0.01 & 44.516 \\ -0.001 & -0.003 & 0.476 & -0.303 & -44.148 & -0.45 \end{bmatrix} \quad (67)$$

$$\hat{B}_{c,N4SID} = \begin{bmatrix} 121.304 \\ 755.80 \\ 365.88 \\ -1584.82 \\ -122.708 \\ -101.39 \end{bmatrix} \quad (68)$$

$$\hat{C}_{c,N4SID} = \begin{bmatrix} -0.015 & 0.037 & -0.079 & -0.001 & 0.049 & -0.049 \\ 0.015 & -0.01 & -0.022 & -0.012 & -0.005 & -0.036 \\ -0.043 & 0.023 & 0.068 & 0.04 & -0.002 & -0.024 \end{bmatrix} \quad (69)$$

$$\hat{D}_{c,N4SID} = \begin{bmatrix} -0.006 \\ 0.021 \\ -0.002 \end{bmatrix} \quad (70)$$



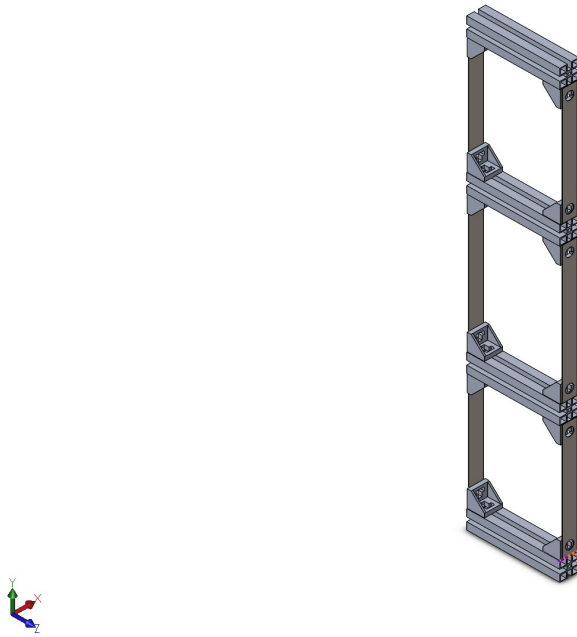


Fig. 29. CAD model of the frame structure considered as the case study of the experimental analysis.

where $\hat{A}_{c,OKID}$, $\hat{B}_{c,OKID}$, $\hat{C}_{c,OKID}$, and $\hat{D}_{c,OKID}$ represent the continuous-time state-space matrices identified by using the ERA-OKID numerical procedure, while $\hat{A}_{c,N4SID}$, $\hat{B}_{c,N4SID}$, $\hat{C}_{c,N4SID}$, and $\hat{D}_{c,N4SID}$ represent the continuous-time state-space matrices identified by using the CVA-N4SID numerical procedure in the case of the flexible structure considered in the paper as the case study.

The identified eigenvalues are grouped in the vectors $\hat{\lambda}_{c,OKID}$ and $\hat{\lambda}_{c,N4SID}$ of dimensions $\hat{n}_z \times 1$, respectively related to the ERA-OKID and the CVA-N4SID algorithms. In the case of the use of the ERA-OKID and the CVA-N4SID algorithms, the identified eigenvalue vectors are respectively found as follows:

$$\hat{\lambda}_{c,OKID} = \begin{bmatrix} -0.034 + 15.001i \\ -0.034 - 15.002i \\ -1.37 + 44.549i \\ -1.37 - 44.55i \\ -0.282 + 62.217i \\ -0.282 - 62.218i \end{bmatrix} \quad (71)$$

and

$$\hat{\lambda}_{c,N4SID} = \begin{bmatrix} -0.049 + 14.995i \\ -0.049 - 14.996i \\ -0.228 + 44.326i \\ -0.228 - 44.327i \\ -0.42 + 61.977i \\ -0.42 - 61.978i \end{bmatrix} \quad (72)$$

Consequently, the identified natural frequencies and the identified dimensionless damping ratios obtained from the use of the ERA-OKID and the CVA-N4SID techniques are given by:

$$\hat{f}_{n,OKID} = \begin{bmatrix} 2.387392 \\ 7.093264 \\ 9.902984 \end{bmatrix}, \quad \hat{f}_{n,N4SID} = \begin{bmatrix} 2.386547 \\ 7.054328 \\ 9.864925 \end{bmatrix} \quad (73)$$

and

$$\hat{\xi}_{OKID} = \begin{bmatrix} 0.002232 \\ 0.030673 \\ 0.004592 \end{bmatrix}, \quad \hat{\xi}_{N4SID} = \begin{bmatrix} 0.003262 \\ 0.005342 \\ 0.006748 \end{bmatrix} \quad (74)$$

where $\hat{f}_{n,OKID}$ and $\hat{\xi}_{OKID}$ are vectors of dimensions $\hat{n}_x \times 1$ that respectively contain the identified natural frequencies and the identified damping ratios obtained from the computer implementation of the ERA-OKID technique, whereas $\hat{f}_{n,N4SID}$ and $\hat{\xi}_{N4SID}$ are vectors of the same dimensions representing analogous quantities deriving from the use of the CVA-N4SID method.

Considering the analytical procedure introduced in the description of the benchmark model, the relative vibration amplitudes



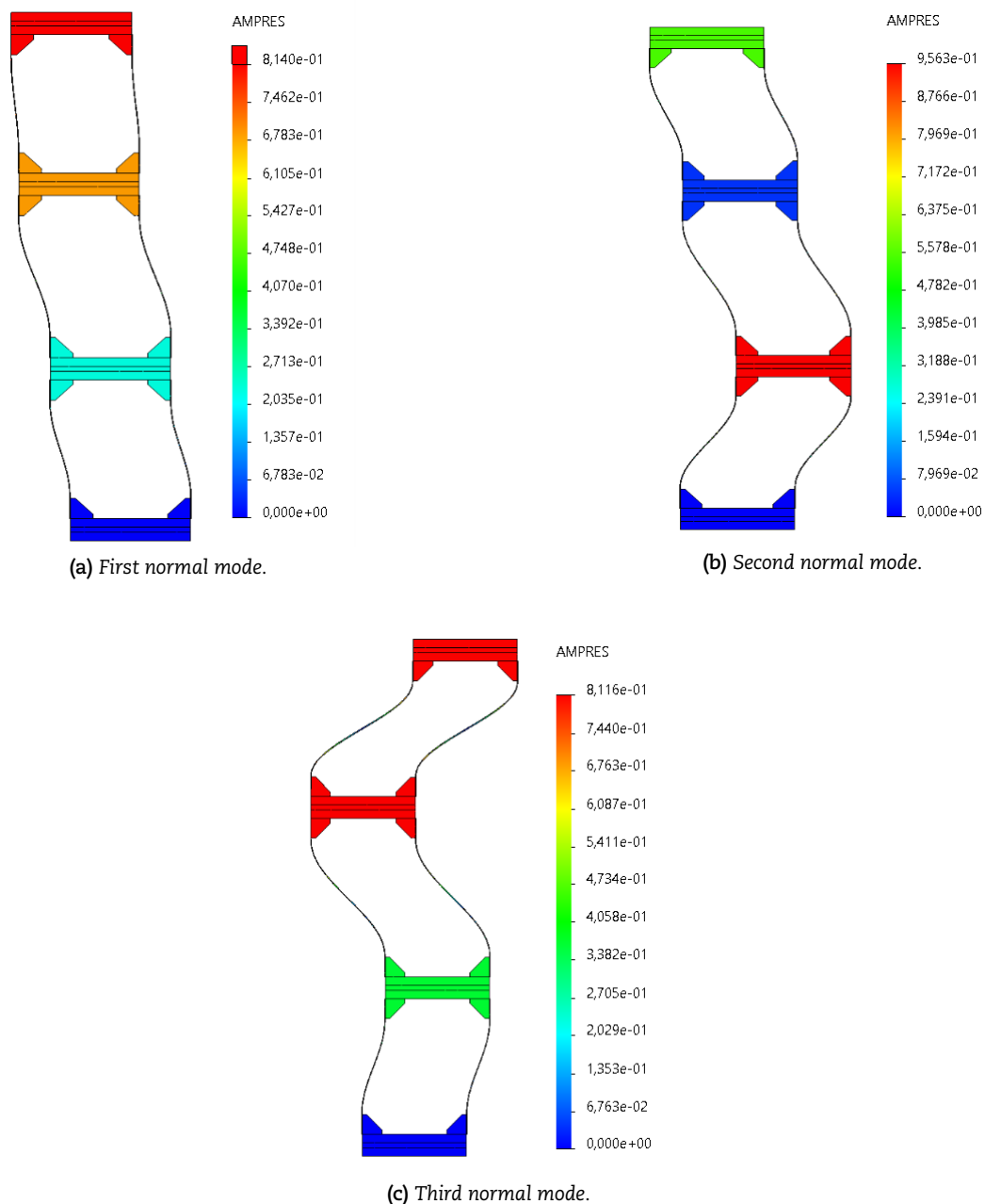


Fig. 30. Geometric shapes of the first, second, and third normal modes of the finite element model of the frame structure considered as the case study.

and the relative phase angles of each normal mode can also be identified in the case of the flexible structure considered in the paper as the case study. The identified geometric shapes associated with the three vibration modes of the case study can be determined by normalizing the relative amplitudes and setting the first phase angle to zero. By doing so, one obtains the experimental results reported in Tables 9 and 10.

More precisely, Table 9 contains the physical parameters of the first, second, and third normal modes, respectively, which were obtained by using the ERA-OKID identification method for the case study. Similarly, Table 10 contains the physical parameters of the first, second, and third normal modes, respectively, which were obtained by using the CVA-N4SID identification method for the case study.

Considering the polar representation of the configuration-space eigenvectors, the geometric shapes of the three normal modes, identified by using the ERA-OKID and the CVA-N4SID numerical procedures for the case study of interest, can be drawn as respectively shown in Figures 36 and 37.

In particular, in the case of the use of the ERA-OKID identification method for the flexible structure considered as the case study, the geometric shapes of the identified first, second, and third vibration modes are represented in Figures 36a, 36b, and 36c, respectively, whereas the identified physical parameters associated with the same normal modes are reported in Table 9. Similarly, in the case of the use of the CVA-N4SID identification method for the flexible structure considered as the case study, the geometric shapes of the identified first, second, and third vibration modes are represented in Figures 37a, 37b, and 37c, respectively, whereas the identified physical parameters associated with the same normal modes are reported in Table 10.

Since the time response of each degree of freedom is available, it is possible to construct the identified structural matrices associated with the configuration-space dynamical model of the frame structure. In the case of the case study of interest, one



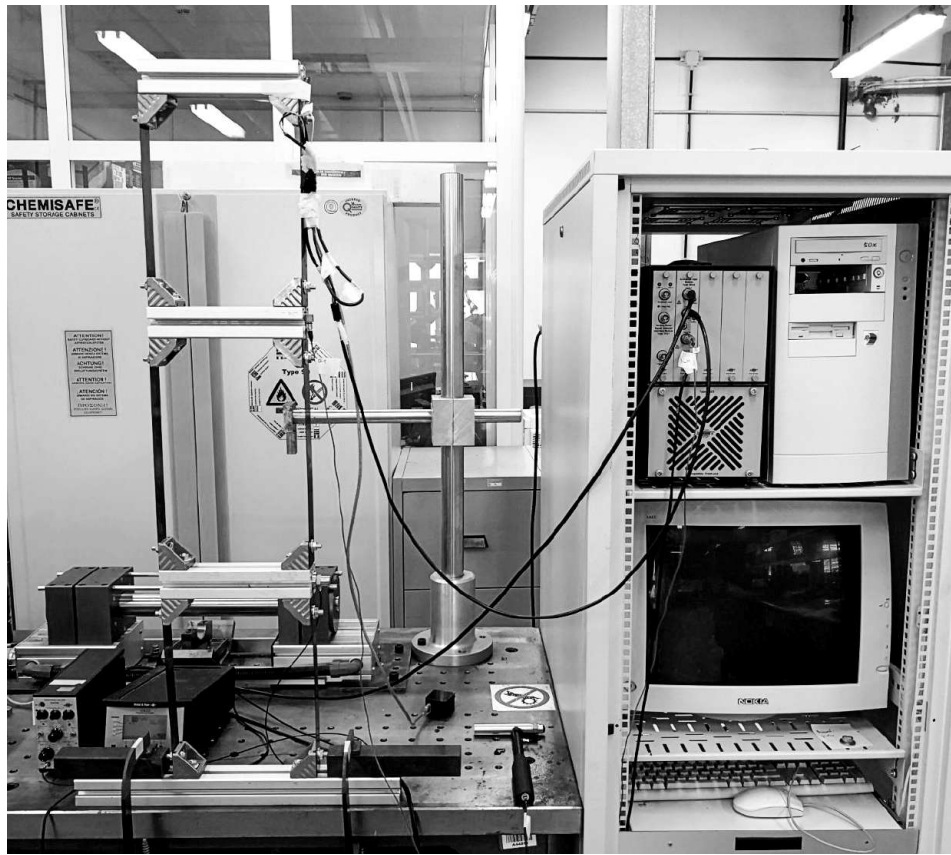


Fig. 31. Experimental apparatus and data measurement system.

obtains:

$$\hat{\mathbf{M}}_{\text{OKID}} = \begin{bmatrix} 0.0014 & -0.0022 & 0.0006 \\ -0.0022 & 0.0067 & 0.0001 \\ 0.0006 & 0.0001 & 0.0007 \end{bmatrix} \quad (75)$$

$$\hat{\mathbf{R}}_{\text{OKID}} = \begin{bmatrix} -0.0121 & -0.0536 & -0.0505 \\ -0.0536 & 0.1954 & 0.0463 \\ -0.0505 & 0.0463 & -0.0451 \end{bmatrix} \quad (76)$$

$$\hat{\mathbf{K}}_{\text{OKID}} = \begin{bmatrix} 0.3611 & -0.2419 & -0.0956 \\ -0.2419 & 6.3838 & 2.1870 \\ -0.0956 & 2.1870 & 0.9815 \end{bmatrix} \quad (77)$$

and

$$\hat{\mathbf{M}}_{\text{N4SID}} = \begin{bmatrix} 0.3678 & -3.7207 & -1.3747 \\ -3.7207 & 207.2415 & 74.6692 \\ -1.3747 & 74.6692 & 27.1365 \end{bmatrix} \quad (78)$$

$$\hat{\mathbf{R}}_{\text{N4SID}} = \begin{bmatrix} 66.3135 & -3697.1546 & -1338.9423 \\ -3697.1546 & 207284.1677 & 74940.2780 \\ -1338.9423 & 74940.2780 & 027097.6091 \end{bmatrix} \quad (79)$$

$$\hat{\mathbf{K}}_{\text{N4SID}} = \begin{bmatrix} 294.9322 & 5909.5229 & 2014.3836 \\ 5909.5229 & 48302.5066 & 13788.2410 \\ 2014.3836 & 13788.2410 & 3760.5909 \end{bmatrix} \quad (80)$$

where $\hat{\mathbf{M}}_{\text{OKID}}$, $\hat{\mathbf{R}}_{\text{OKID}}$, and $\hat{\mathbf{K}}_{\text{OKID}}$ respectively represent the mass, damping, and stiffness matrices of dimensions $\hat{n}_z \times \hat{n}_z$ identified by using the ERA-OKID numerical procedure, whereas $\hat{\mathbf{M}}_{\text{N4SID}}$, $\hat{\mathbf{R}}_{\text{N4SID}}$, and $\hat{\mathbf{K}}_{\text{N4SID}}$ respectively represent the mass, damping, and stiffness matrices having the same dimensions identified employing the CVA-N4SID numerical algorithm in the analysis of the flexible structure assumed as the case study. Thus, it is possible to experimentally identify the system mass, damping, and stiffness matrices and, unlike the case of the benchmark problem, since the flexible system under study is composed of continuous bodies instead of discrete components, the resulting structural matrices turned out to be full matrices instead of diagonal or banded matrices. From the analysis carried out previously, it is known that in practical experiments the matrix that is most affected by the presence of noise is the damping matrix. Consequently, this matrix needs to be recalculated by using the PDC method discussed in the paper.





(a) Brüel & Kjær type 8202 impact hammer.



(b) Brüel & Kjær type 2825 signal analyzer.



(c) Brüel & Kjær type 2692 Nexus charge amplifier.

Fig. 32. Brüel & Kjær data acquisition instrumentation.

Therefore, the damping matrices denoted with $\hat{\mathbf{R}}_{\text{OKID}}^*$ and $\hat{\mathbf{R}}_{\text{N4SID}}^*$ that are given below respectively represent the optimized damping matrices for both the ERA-OKID and CVA-N4SID algorithms:

$$\begin{aligned}\hat{\mathbf{R}}_{\text{OKID}}^* &= \hat{\alpha}_{\text{OKID}} \hat{\mathbf{M}}_{\text{OKID}} + \hat{\beta}_{\text{OKID}} \hat{\mathbf{K}}_{\text{OKID}} \\ &= \begin{bmatrix} 0.0003 & -0.0002 & 0 \\ -0.0002 & 0.0037 & 0.0012 \\ 0 & 0.0012 & 0.0006 \end{bmatrix}\end{aligned}\quad (81)$$

and

$$\begin{aligned}\hat{\mathbf{R}}_{\text{N4SID}}^* &= \hat{\alpha}_{\text{N4SID}} \hat{\mathbf{M}}_{\text{N4SID}} + \hat{\beta}_{\text{N4SID}} \hat{\mathbf{K}}_{\text{N4SID}} \\ &= \begin{bmatrix} 0.0796 & 1.0194 & 0.3420 \\ 1.0194 & 20.6609 & 6.7024 \\ 0.3420 & 6.7024 & 2.1793 \end{bmatrix}\end{aligned}\quad (82)$$

being:

$$\begin{cases} \hat{\alpha}_{\text{OKID}} = 4.3651 \cdot 10^{-2} \text{ (1/s)} \\ \hat{\beta}_{\text{OKID}} = 5.3267 \cdot 10^{-4} \text{ (s)} \end{cases}\quad (83)$$

and

$$\begin{cases} \hat{\alpha}_{\text{N4SID}} = 5.1877 \cdot 10^{-2} \text{ (1/s)} \\ \hat{\beta}_{\text{N4SID}} = 2.0516 \cdot 10^{-4} \text{ (s)} \end{cases}\quad (84)$$

where the two proportionality constants are respectively denoted as $\hat{\alpha}_{\text{OKID}}$ and $\hat{\beta}_{\text{OKID}}$ for the ERA-OKID method, whereas the same parameters are respectively denoted as $\hat{\alpha}_{\text{N4SID}}$ and $\hat{\beta}_{\text{N4SID}}$ for the CVA-N4SID method.

Figures 38, 39, and 40 respectively represent the comparison of the time responses of the first, second, and third degrees of



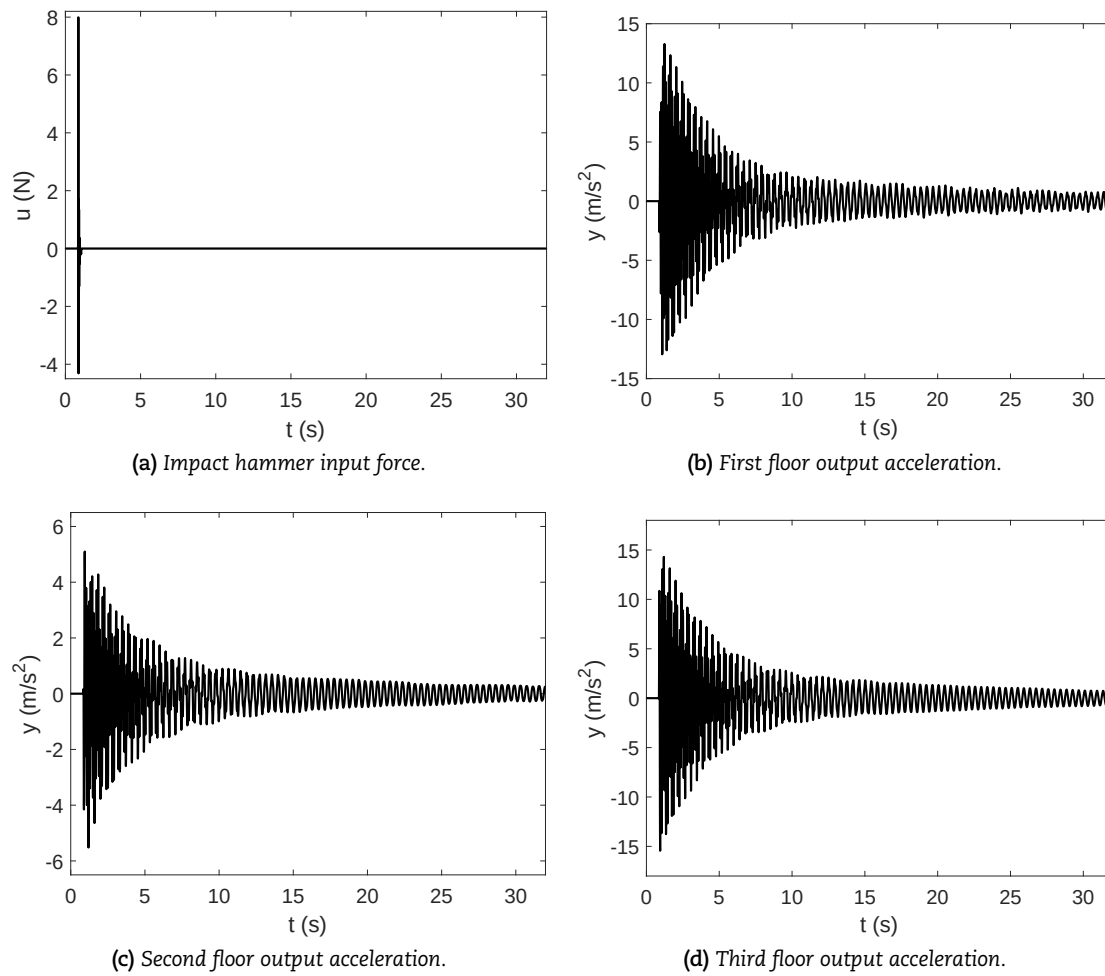


Fig. 33. Time domain data acquisition for the flexible structure considered as the case study.

freedom, obtained from the mechanical model identified using the ERA-OKID method in the experimental analysis of the case study, with those deriving from the original input-output measurements obtained by using the experimental apparatus.

More specifically, Figures 38a, 39a, and 40a respectively represent the comparison of the time responses of the first, second, and third degrees of freedom considering the complete time span of the data record, while Figures 38b, 39b, and 40b respectively represent the comparison of the time responses of the first, second, and third degrees of freedom considering a zoomed time window of the data record in the case of the use of the ERA-OKID identification method.

Figures 41, 42, and 43 respectively represent the comparison of the time responses of the first, second, and third degrees of freedom, obtained from the mechanical model identified using the CVA-N4SID method in the experimental analysis of the case study, with those deriving from the original input-output measurements obtained by using the experimental apparatus.

More specifically, Figures 41a, 42a, and 43a respectively represent the comparison of the time responses of the first, second, and third degrees of freedom considering the complete time span of the data record, while Figures 41b, 42b, and 43b respectively represent the comparison of the time responses of the first, second, and third degrees of freedom considering a zoomed time window of the data record in the case of the use of the CVA-N4SID identification method.

As a conclusive remark, by observing Figures 38, 39, 40, 41, 42, and 43, one can conclude that the percentage value of the consistency function is quite high for each degree of freedom, thereby demonstrating the consistency between the identified results calculated by using the ERA-OKID and CVA-N4SID methods, applied to the case study, with those measured by using the experimental test rig, assembled for the case study of interest for this investigation.

4. Comparative Analysis and General Discussion

In the following discussion, a summary of the main aspects and the principal difficulties encountered in this research paper is provided, and the main conclusions drawn in this detailed investigation are reported. In this work, the OKID and N4SID algorithms were tested in a simulated environment and by using an experimental apparatus for the data acquisition from a flexible structure. In the case of the OKID method, at the beginning of the present investigation, the ERA technique or the ERA/DC technique were tested and used. Subsequently, the ERA approach was chosen in lieu of the ERA/DC approach. On the other hand, the three types of variants of the N4SID algorithms implemented in MATLAB were also taken into account in the present study, namely the CVA, MOESP, and SSARX algorithms. All the system identification methods mentioned before were tested employing dynamical simulations as well as experimental data.

In this comparative study, several numerical simulations were carried out and numerous data acquisitions were performed. This comparative study was carried out first in a virtual environment by analyzing the forced vibrations of a two-degree-of-freedom mechanical system that served as a simple benchmark problem. By doing so, the ERA method associated with the OKID approach was compared with the MOESP, CVA, and SSARX techniques corresponding to the N4SID algorithm in the presence of noise and in the case of the analysis of experimental data obtained from a vibrating structure. The numerical results obtained in the comparative



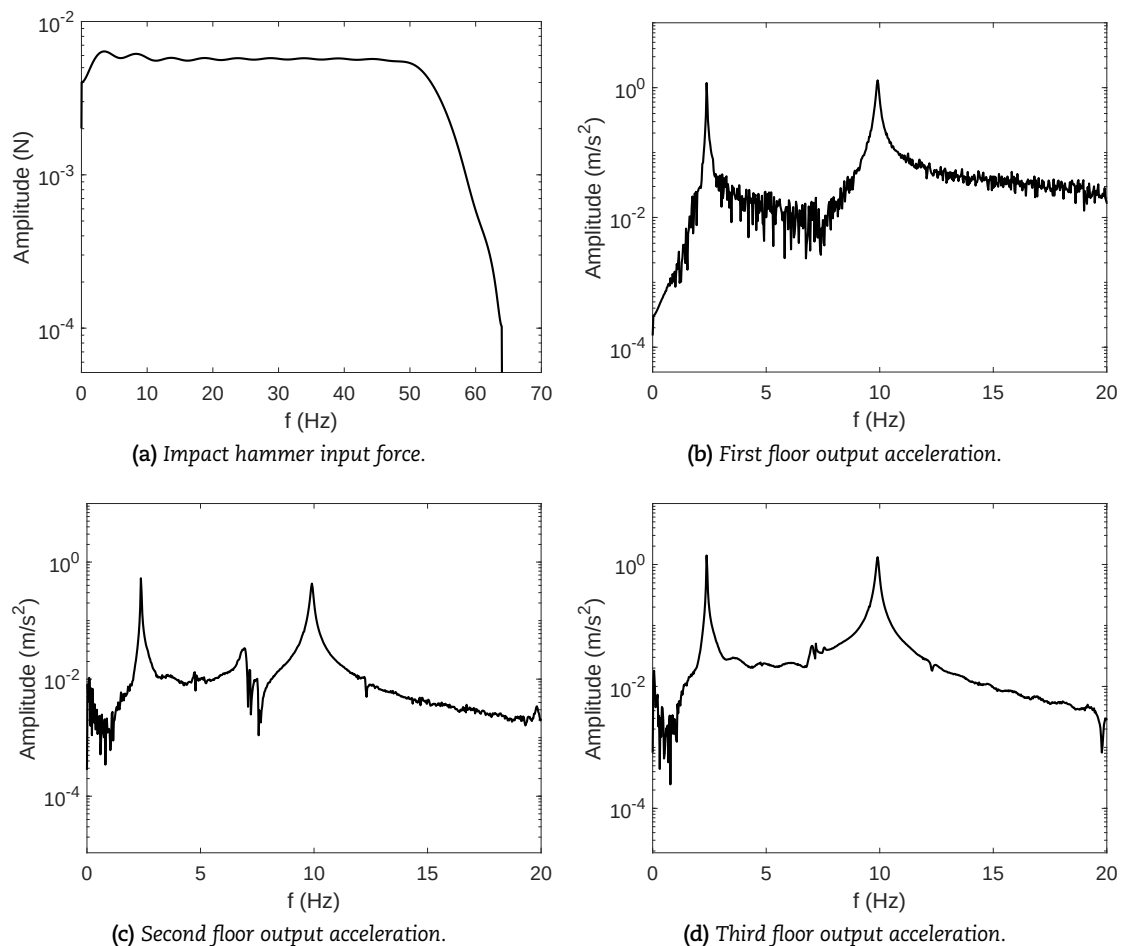


Fig. 34. Frequency domain data acquisition for the flexible structure considered as the case study.

analysis were examined by observing the natural frequencies, the damping ratios, the vibration modes, the reconstruction of the measured outputs using the measured inputs resulting from the identified dynamical models, and the robustness of the identified models in correspondence of a different set of inputs. It was found that the CVA-N4SID identification algorithm provided the best numerical results. Additionally, the results of the MOESP-N4SID and ERA-OKID approaches followed immediately after in the quality ranking, while the SSARX-N4SID technique performed worst than the other three identification methods previously mentioned. In synthesis, Table 11 shows the performance and the numerical results found by using the identification methodologies discussed before for obtaining state-space dynamical models of the benchmark system considered in the paper.

In the subsequent developments of this study, the identification algorithms considered in this work were also experimentally tested on a vibrating frame structure featuring three degrees of freedom, which can be simply modeled as a mechanical system having a lumped distribution of the inertial and elastic parameters. The experimental identification process necessary for the comparative analysis was carried out considering two cases. In the first case, an impact hammer equipped with a load cell was used in combination with two piezoelectric accelerometers having a high sensitivity. In the second case, three piezoelectric accelerometers were employed in conjunction with the impact hammer. In this second case, however, the third accelerometer used had a low sensitivity. More specifically, the experimental identification carried out with the two sensitive accelerometers, when compared with the finite element model constructed from the CAD geometry, was excellent. In this case, however, one cannot determine all the modes of vibration from the output data available. In this particular case, the best identification results were obtained by employing the CVA-N4SID technique. On the other hand, in the second experimental campaign of data acquisition, in which one less sensitive accelerometer was used in addition to the two sensitive piezoelectric transducers, leading to a full set of sensors for each degree of freedom of the flexible structure, the numerical algorithm that presented to the best performance was the ERA-OKID method.

Subsequently, starting from the experimental identifications of the fundamental matrices in the state-space, a second-order mechanical model was then obtained, thereby estimating the matrices of mass, stiffness, and damping by means of the MKR approach. Since the dissipative effects are small for the test rig considered in this work, for a better estimation of the damping matrices, proportional damping is hypothesized. This hypothesis allowed for implementing a simple least-squares technique to estimate the proportionality factors by using the PDC method, from which an estimation of the damping matrix closest to reality can be identified. The dynamical model of the structural system considered as the case study was thus experimentally identified with the two approaches mentioned before, namely the CVA-N4SID technique and the ERA-OKID method, and the results obtained were compared with a third analysis made using a finite element model of the case study, as well as with the simple model of the vibrating structure having concentrated parameters. A good agreement was found in general between the modal parameters pertaining to the preliminary dynamical models constructed for the flexible frame structure and those arising from the state-space models identified by using the ERA-OKID method and the CVA-N4SID method.



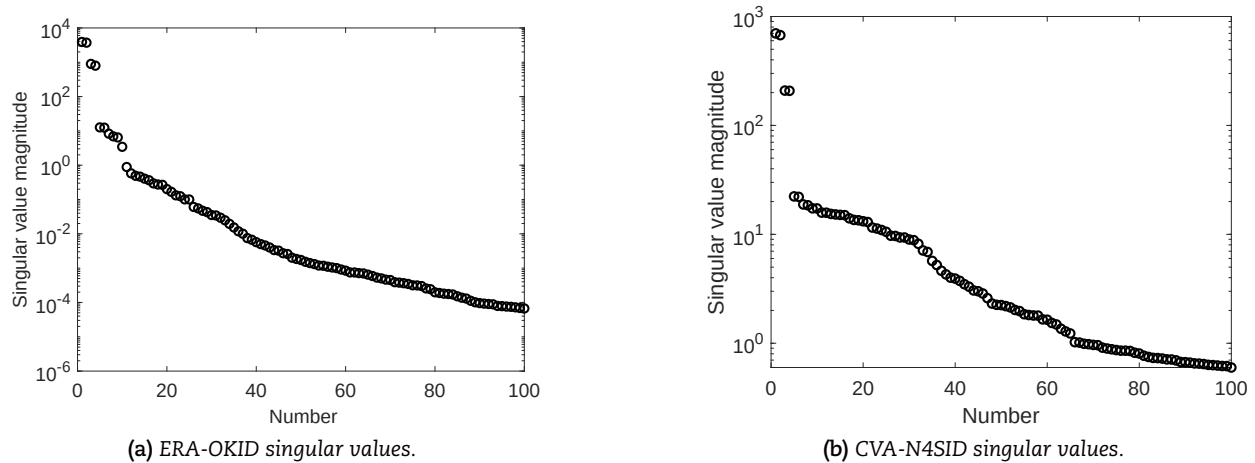


Fig. 35. Distribution of the identified singular values obtained by implementing the ERA-OKID and the CVA-N4SID identification methods for the flexible structure considered as the case study.

Table 9. Physical parameters of the first, second, and third normal modes identified using the ERA-OKID method for the flexible structure considered as the case study.

Symbol	Data (Units)
$\hat{f}_{n,1,OKID}$	2.3875 (Hz)
$\hat{f}_{n,2,OKID}$	7.0933 (Hz)
$\hat{f}_{n,3,OKID}$	9.9023 (Hz)
$\hat{\xi}_{1,OKID}$	0.0022 (—)
$\hat{\xi}_{2,OKID}$	0.0302 (—)
$\hat{\xi}_{3,OKID}$	0.0045 (—)
$\hat{\rho}_{1,OKID}$	$\begin{bmatrix} 1 & 0.4525 & 0.3567 \end{bmatrix}^T$ (—)
$\hat{\rho}_{2,OKID}$	$\begin{bmatrix} 1 & 0.3535 & 1.0842 \end{bmatrix}^T$ (—)
$\hat{\rho}_{3,OKID}$	$\begin{bmatrix} 1 & 0.4542 & 1.2118 \end{bmatrix}^T$ (—)
$\hat{\theta}_{1,OKID}$	$\begin{bmatrix} 0 & 0.7761 & 0.5468 \end{bmatrix}^T$ (rad)
$\hat{\theta}_{2,OKID}$	$\begin{bmatrix} 0 & -0.2362 & 2.8518 \end{bmatrix}^T$ (rad)
$\hat{\theta}_{3,OKID}$	$\begin{bmatrix} 0 & 3.8493 & 0.7544 \end{bmatrix}^T$ (rad)

5. Summary, Conclusions, and Future Work

The principal objective of this work, which represents the second part of a two-part investigation, is to carry out a systematic computational and experimental analysis of the principal data-driven identification procedures based on the Observer/Kalman Filter Identification Methods (OKID) and the Numerical Algorithms for Subspace State-Space System Identification (N4SID). More precisely, all the system identification numerical procedures considered in this two-part research work are applicable to the experimental modal analysis problem associated with mechanical systems based on input-output measurements. For this purpose, this second paper proposes the presentation of the numerical analysis and the experimental testing carried out in this two-part research work.

In this investigation, a systematic comparison of the principal time-domain numerical procedures used for identifying dynamical models of mechanical systems is performed. The comprehensive comparison carried out in this work is based on numerical and experimental data obtained by means of numerical simulations, as well as by applying the identification techniques considered in the paper to the experimental measurements generated employing a laboratory test rig. All the methods considered in this paper worked properly and were firstly tested in a virtual environment by using a dynamical model of a two-degrees-of-freedom vibrating mechanical system. In order to demonstrate these facts, the influence of the noise in the input and output measurements on the algorithm behaviors was investigated first. Subsequently, an experimental modal analysis of a three-degrees-of-freedom flexible frame structure was employed for performing laboratory experiments aimed at assessing the actual performance of the methods of interest in the case of an actual engineering application. The identified models of the three-dimensional structure analyzed in the case study were also in accordance with both a lumped parameter model and a finite element model of the vibrating structure, which were constructed starting from its CAD geometry. These additional mechanical models were developed in order to have unique and independent references for the numerical and experimental results.

On the basis of a comprehensive set of computer simulations and experimental testing carried out in this investigation, it was found that, if correctly formulated and considering appropriate conditioning of the data entry, all the computational methods considered in this work provided identified models that are consistent with each other. More precisely, from the tests conducted, it was possible to understand that, if the experimental measurements are available with a high signal-to-noise ratio, the CVA-N4SID approach is the best method for identifying the desired dynamical model. If, on the other hand, the input-output data present a high noise, the best identification method was found to be one between the MOESP-N4SID, CVA-N4SID, and ERA-OKID techniques. However, in principle, if properly tuned, all the identification algorithms considered in this study lead to good numerical results, can correctly find an estimation of a state-space dynamical model of the system of interest, and are capable of reproducing the input-output set of data employed in the applied system identification process.

As mentioned before, one of the most important aspects of the present investigation is the identification of the damping coefficients of a given mechanical model describing a dynamical system. This is a key point since the presence of dissipative effects in a



Table 10. Physical parameters of the first, second, and third normal modes identified using the CVA-N4SID method for the flexible structure considered as the case study.

Symbol	Data (Units)
$\hat{f}_{n,1,N4SID}$	2.3862 (Hz)
$\hat{f}_{n,2,N4SID}$	7.0544 (Hz)
$\hat{f}_{n,3,N4SID}$	9.8646 (Hz)
$\hat{\xi}_{1,N4SID}$	0.0033 (—)
$\hat{\xi}_{2,N4SID}$	0.0051 (—)
$\hat{\xi}_{3,N4SID}$	0.0067 (—)
$\hat{\rho}_{1,N4SID}$	$\begin{bmatrix} 1 & 0.5212 & 0.3443 \end{bmatrix}^T$ (—)
$\hat{\rho}_{2,N4SID}$	$\begin{bmatrix} 1 & 0.3142 & 1.0153 \end{bmatrix}^T$ (—)
$\hat{\rho}_{3,N4SID}$	$\begin{bmatrix} 1 & 0.4465 & 1.1932 \end{bmatrix}^T$ (—)
$\hat{\theta}_{1,N4SID}$	$\begin{bmatrix} 0 & -0.9248 & -0.7335 \end{bmatrix}^T$ (rad)
$\hat{\theta}_{2,N4SID}$	$\begin{bmatrix} 0 & -0.4671 & 2.6312 \end{bmatrix}^T$ (rad)
$\hat{\theta}_{3,N4SID}$	$\begin{bmatrix} 0 & 3.7675 & 0.6833 \end{bmatrix}^T$ (rad)

dynamical system modeled considering the linear damping assumption leads to more realistic numerical results that are consistent with the experiments. More specifically, the present investigation discusses the use of a simple least-square numerical method for improving the estimation of the damping coefficients of an identified model. This method is simply referred to as the identification procedures for the Proportional Damping Coefficients (PDC) and was applied to the second-order mathematical models obtained by using an algorithm for the identification of the Mass, Stiffness, and Damping (MKR) matrices. The second-order mechanical model so found were, in turn, extracted from the identified first-order state-space models constructed by employing the Observer/Kalman Filter Identification Methods (OKID) and the Numerical Algorithms for Subspace State-Space System Identification (N4SID). This process was carried out for both the benchmark system considered in the numerical experiments and considering the experimental data arising from the laboratory apparatus used as a simple test rig of a vibrating structure. In both cases, it was found that, as expected, the identified damping matrices obtained from the use of the MKR need to be improved since the magnitudes of their coefficients were too large and/or not consistent with the problem at hand. The use of the PDC method, on the other hand, helped in improving the estimation of the damping coefficients in the analysis of both the benchmark system and the case study investigated in this work. In addition, the system identification procedures were systematically compared and analyzed with the MKR method and PDC technique considered in this paper. However, the estimation of the damping coefficients is a crucial problem that deserves further investigation since the results found in this study are good but, in the authors' opinion, could be improved further.

In future works, there are several directions that can be followed. As far as the implementation of the families of the OKID and N4SID techniques is concerned, the recently developed variants of these approaches should be investigated. For this purpose, for example, the OKID method could be combined with the Principal Hankel Component Algorithm (PHCA), as well as with other least-squares regression techniques which lead to the identification of dynamical models expressed in the observable canonical form or in the controllable canonical form. Additionally, the implementation difficulties and the unexpected inconsistencies in the numerical results associated with the SSARX technique associated with the family of the N4SID algorithms should be further investigated. More importantly, it would be challenging but promising to apply all these system identification numerical procedures in the case of the experimental modal analysis problem associated with input-output measurements generated using a three-dimensional mechanical system in which the bending vibration modes are coupled with the torsional modes of vibration, such as, for example, a rotating beam structure.

Author Contributions

This research paper was principally devised and developed by the first author (Carmine Maria Pappalardo). Great support in the development of the paper was provided by the second author (Filippo Califano) and by the third author (Sefika Ipek Lok). The detailed review carried out by the fourth author (Domenico Guida) considerably improved the quality of the work. The manuscript was written with the contribution of all authors. All authors discussed the results, reviewed the methodology, and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.



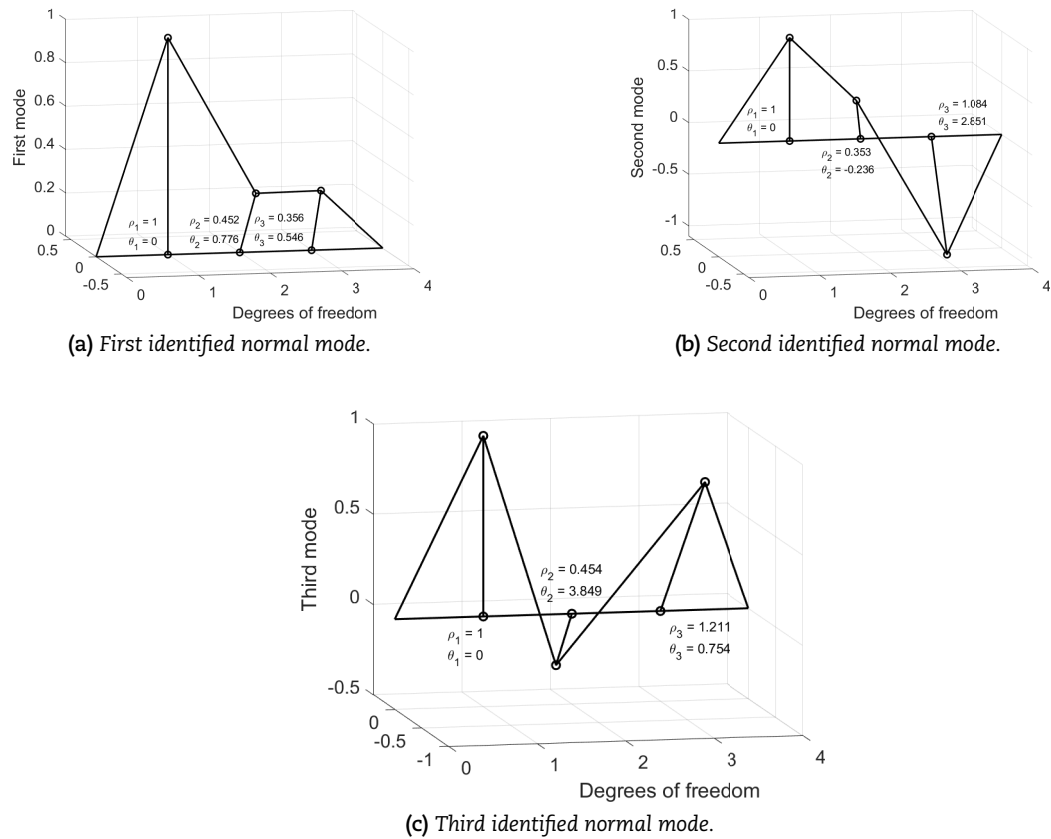


Fig. 36. Geometric shapes of the normal modes identified using the ERA-OKID method for the flexible structure considered as the case study.

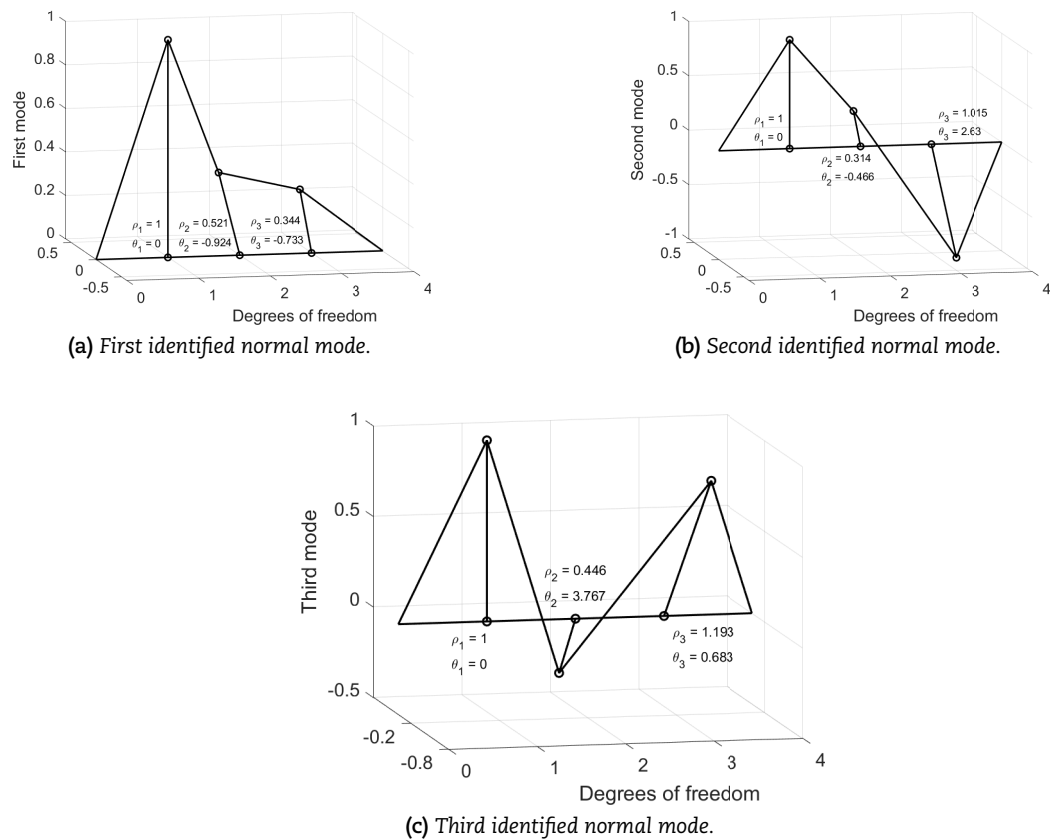


Fig. 37. Geometric shapes of the normal modes identified using the CVA-N4SID method for the flexible structure considered as the case study.



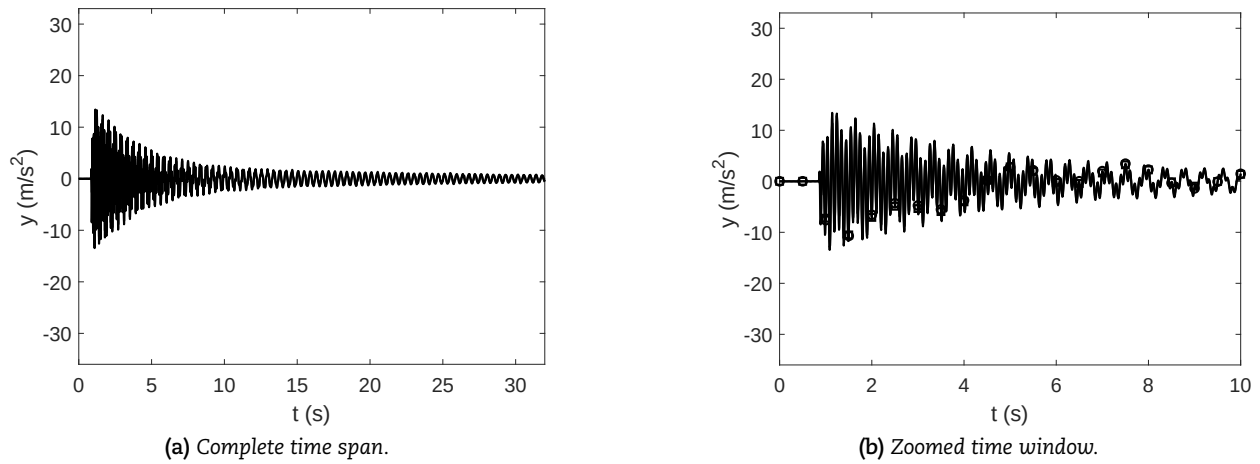


Fig. 38. Comparison between the time response of the first degree of freedom determined by employing the experimental test rig assembled for the case study (square markers) and the same time response obtained from the mechanical model identified by using the ERA-OKID method (circle markers). The consistency function arising from the comparison of the time histories of the first degree of freedom is equal to NRMSF = 80.248%.

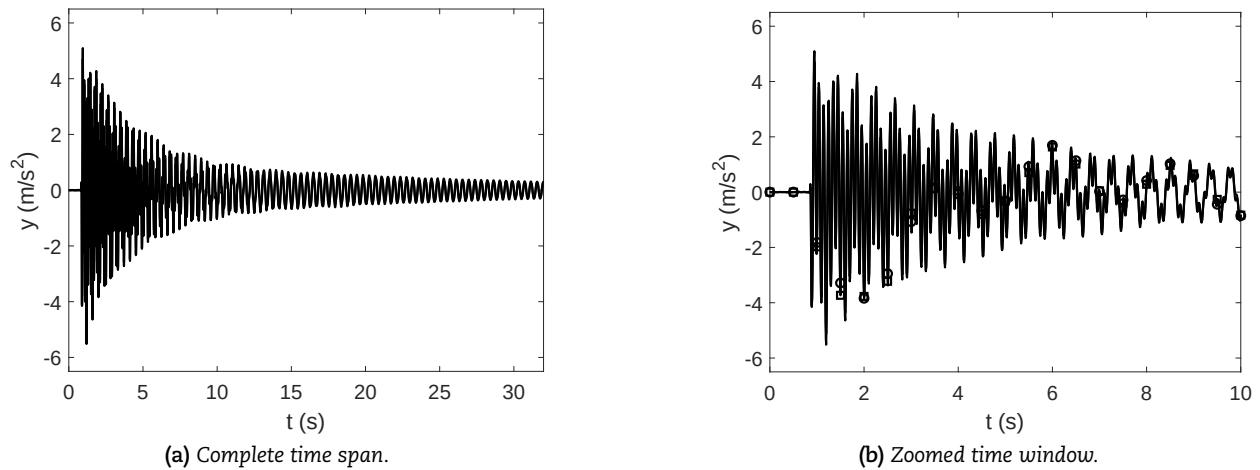


Fig. 39. Comparison between the time response of the second degree of freedom determined by employing the experimental test rig assembled for the case study (square markers) and the same time response obtained from the mechanical model identified by using the ERA-OKID method (circle markers). The consistency function arising from the comparison of the time histories of the second degree of freedom is equal to NRMSF = 87.024%.

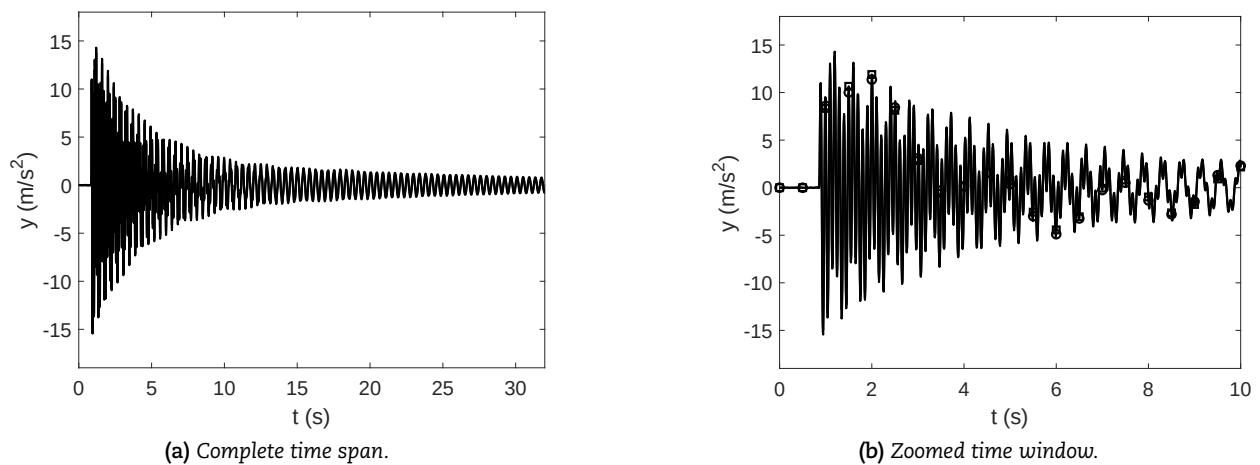


Fig. 40. Comparison between the time response of the third degree of freedom determined by employing the experimental test rig assembled for the case study (square markers) and the same time response obtained from the mechanical model identified by using the ERA-OKID method (circle markers). The consistency function arising from the comparison of the time histories of the third degree of freedom is equal to NRMSF = 89.361%.



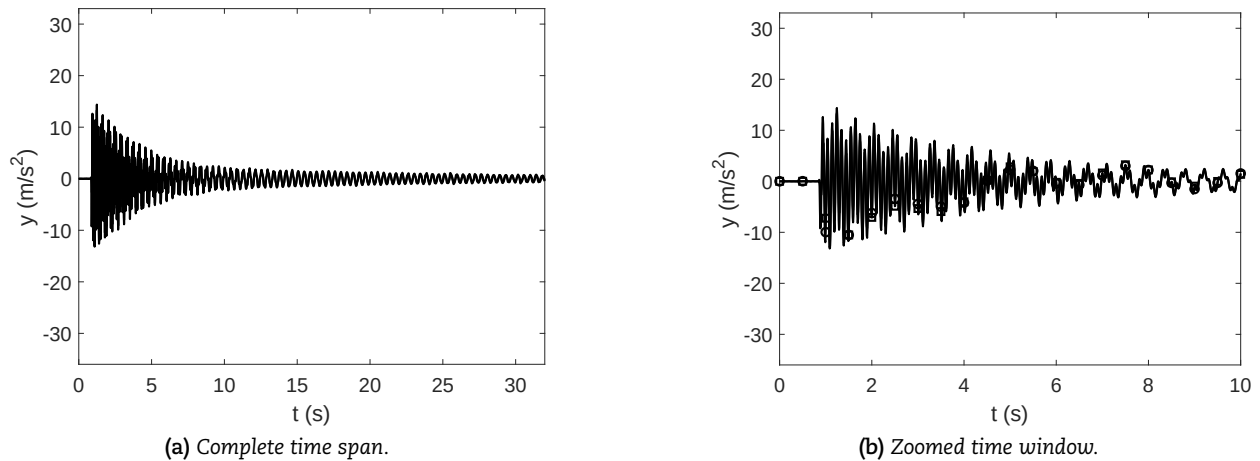


Fig. 41. Comparison between the time response of the first degree of freedom determined by employing the experimental test rig assembled for the case study (square markers) and the same time response obtained from the mechanical model identified by using the CVA-N4SID method (circle markers). The consistency function arising from the comparison of the time histories of the first degree of freedom is equal to NRMSF = 80.962%.

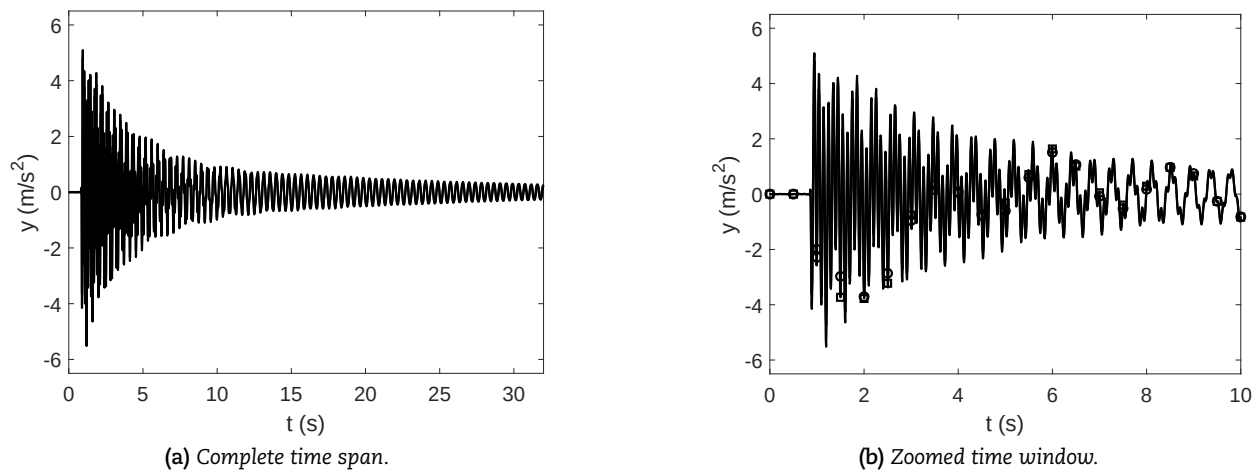


Fig. 42. Comparison between the time response of the second degree of freedom determined by employing the experimental test rig assembled for the case study (square markers) and the same time response obtained from the mechanical model identified by using the CVA-N4SID method (circle markers). The consistency function arising from the comparison of the time histories of the second degree of freedom is equal to NRMSF = 87.063%.

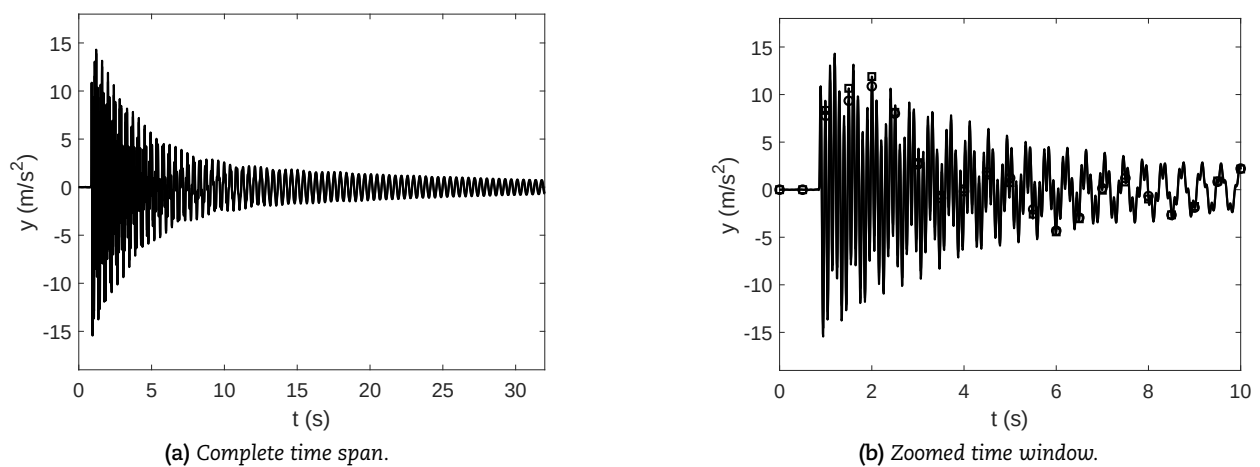


Fig. 43. Comparison between the time response of the third degree of freedom determined by employing the experimental test rig assembled for the case study (square markers) and the same time response obtained from the mechanical model identified by using the CVA-N4SID method (circle markers). The consistency function arising from the comparison of the time histories of the third degree of freedom is equal to NRMSF = 89.714%.



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Appendix A. Performance Comparison of the Applied System Identification Methods

In this appendix, the numerical results obtained from the systematic comparison of the ERA-OKID, MOESP-N4SID, CVA-N4SID, and SSARX-N4SID identification approaches are provided in detail. The performance comparison of these methods was implemented in the case of the two-degree-of-freedom mechanical system considered in the paper as a benchmark problem and the corresponding numerical results found are reported in Table 11.



Table 11. Comparison of the performance of the ERA-OKID, MOESP-N4SID, CVA-N4SID, and SSARX-N4SID identification procedures applied to the benchmark problem in the presence of noise having an increasing level of intensity.

NOISE STANDARD DEVIATION	INPUT FORCE SNR	I DOF OUTPUT SNR	II DOF OUTPUT SNR	IDENTIFICATION ALGORITHM	ERROR $f_{n,1}$	ERROR $f_{n,2}$	ERROR ξ_1	ERROR ξ_2	I DOF RECONSTRUCTION NRMSF	II DOF RECONSTRUCTION NRMSF	I DOF ROBUSTNESS NRMSF	II DOF ROBUSTNESS NRMSF
0	INF	INF	INF	ERA-OKID	3.44E-13	3.79E-14	8.54E-11	2.01E-11	99.99%	99.99%	99.99%	99.99%
				CVA-N4SID	2.91E-15	1.65E+16	4.07E+12	1.27E-12	99.99%	99.99%	99.99%	99.99%
				MOESP-N4SID	1.40E-15	9.90E-16	6.20E-12	1.20E-12	99.99%	99.99%	99.99%	99.99%
				SSARX-N4SID								
1.00E-04	62.33	77	78.51	ERA-OKID	1.06E-07	2.44E-07	5.62E-07	4.88E-04	99.97%	99.97%	99.97%	99.97%
				CVA-N4SID	4.11E-09	2.41E-08	5.94E-06	2.12E-05	99.97%	99.97%	99.99%	99.99%
				MOESP-N4SID	3.45E-08	8.55E-09	1.29E-04	1.54E-04	99.95%	99.95%	99.99%	99.99%
				SSARX-N4SID	1.60E-06	4.30E-08	3.37E-04	1.95E-04	99.96%	99.97%	99.98%	99.98%
1.00E-03	42.2	56.93	58.6	ERA-OKID	1.02E-05	4.09E-06	5.50E-03	3.90E-03	99.65%	99.62%	99.71%	99.69%
				CVA-N4SID	3.40E-07	2.00E-06	8.70E-04	2.40E-04	99.50%	99.54%	99.93%	99.94%
				MOESP-N4SID	2.54E-06	9.29E-07	7.20E-04	6.87E-04	99.59%	99.56%	99.93%	99.92%
				SSARX-N4SID	1.90E-05	3.50E-06	5.40E-03	5.40E-03	99.54%	99.59%	99.65%	99.63%
1.00E-02	22.23	36.88	38.6	ERA-OKID	1.67E-04	3.34E-06	9.27E-02	3.27E-02	96.41%	95.32%	97.05%	96.35%
				CVA-N4SID	6.27E-06	5.95E-06	3.70E-03	5.58E-03	95.13%	95.66%	99.23%	99.28%
				MOESP-N4SID	1.66E-05	1.11E-05	6.02E-03	5.10E-03	96.76%	96.92%	99.53%	99.58%
				SSARX-N4SID	1.13E-04	8.91E-06	8.86E-02	4.08E-02	95.69%	95.53%	96.89%	96.64%
1.00E-01	4.36	17.1	18.56	ERA-OKID	1.15E-03	2.35E-05	4.99E-01	9.69E-01	67.67%	67.14%	68.09%	72.45%
				CVA-N4SID	1.13E-05	1.49E-05	6.16E-02	1.15E-01	41.02%	40.72%	71.29%	74.37%
				MOESP-N4SID	3.50E-04	2.57E-05	6.15E-02	4.41E-02	60.52%	60.63%	68.33%	70.32%
				SSARX-N4SID	4.82E-04	2.84E-04	1.64E-01	5.09E-01	60.61%	61.79%	72.23%	71.02%
1.00E+00	0.07	1.75	2.34	ERA-OKID	5.64E-03	2.94E-03	1.39E+00	1.51E+01	0.32%	0.18%	0.51%	0.29%
				CVA-N4SID	6.96E-04	4.39E-05	1.79E+00	3.58E-01	5.37%	3.97%	1.67%	2.34%
				MOESP-N4SID	1.22E-04	5.95E-05	7.94E-01	6.26E-03	0.65%	1.74%	1.97%	1.85%
				SSARX-N4SID	3.68E-04	2.97E-04	8.46E-01	5.73E-01	4.30%	4.56%	2.80%	3.31%

