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Research Paper

Matrix Equations of the Motion of Multibody Systems with a Tree Structure in Hamiltonian Variables

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Abstract. The paper develops the methods for computer simulation of the dynamics of absolutely rigid body systems with a tree structure. The equations of motion are presented in compact matrix form. The case of holonomic constraints is also considered. Since the independent parameters uniquely determine the positions and velocities of the bodies of the system in space, the generalized coordinates and variables with the dimension of impulses are chosen. A feature of the system of equations is that it is resolved with respect to the derivatives of the generalized momenta and does not contain constraint reactions. The derivation of the proposed form of the equations of motion from the Hamilton principle using the matrix-geometric approach is given. Recursive formulas for determining all kinematic and dynamic variables included in the equations are obtained. As an example of a mechanical system with six degrees of freedom, all stages of preparing primary information and compiling equations of motion in the proposed form are demonstrated. Three algorithms are presented that allow resolving the obtained extended system of equations with respect to the generalized velocities and momenta without direct formation of the Hamilton equations. The classification of the equations of motion of rigid body systems is carried out from their structure point of view. The place of the deduced equations in the general classification is also demonstrated. A comparative analysis of the computational complexity of the considered methods for various classes of mechanical systems is carried out. Some diagrams are constructed that allow choosing the most effective modeling method depending on the characteristics of the mechanical system: the number of bodies, the number of degrees of freedom and the structure of the system.

Keywords: Multibody system, equations of motion, dynamics, mathematical modeling, generalized coordinates, Poisson momenta.

1. Introduction

Computer simulation of the dynamics of mechanical systems is widely used in engineering practice. It allows to reduce the volume of full-scale tests, reduce the time and cost of new developments. Therefore, the development of methods that allow accelerating the process of mathematical modeling of complex technical systems is an urgent task.

The results achieved in the automation of the processes of mathematical modeling of multibody systems are presented in the literature [1–7]. These works describe the main approaches that are currently used for the computer formation of the equations of motion and mathematical modeling of constraint rigid body systems.

First of all, this is an approach based on the Newton-Euler equations of a rigid body and the d'Alembert principle. In this approach, an extended system of differential-algebraic equations (DAE) is constructed with respect to Cartesian accelerations (quasi-accelerations), accelerations of generalized coordinates, and constraint reactions. This closed system of equations of motion includes general equations of rigid body dynamics (Newton-Euler equations), kinematic relations connecting quasi-accelerations and accelerations of generalized coordinates (equations that determine the relationship between relative and absolute motions of a system of bodies), and equations of the principle of ideality of bonds (variational principles of d'Alembert or Jourdain). In the mentioned systems of differential-algebraic equations, either various “sweep” methods are applied to solve it with respect to accelerations and reactions of constraints with subsequent integration of accelerations, or special methods of direct integration are developed.

The Euler-Lagrange method, as another approach, consists in projecting the Newton-Euler equations into tangent and orthogonal subspaces of virtual movements of the bodies of the system. When projecting into a tangent subspace, the equations of motion are obtained in the Lagrange form of the 2nd kind in generalized coordinates. When projecting into an orthogonal subspace, equations are obtained in the Lagrange form of the 1st kind with respect to Lagrange multipliers, quasi-accelerations, and generalized accelerations. The works that consider the methods of this group differ from each other regarding the detail of the algorithms for generating equations and can be applied to various systems of rigid bodies and approaches to resolve the resulting DAEs.

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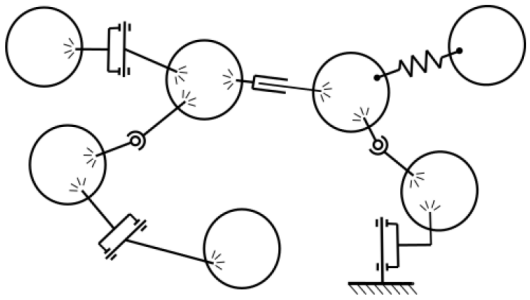

Fig. 1. An example of absolutely rigid body system with a tree structure.

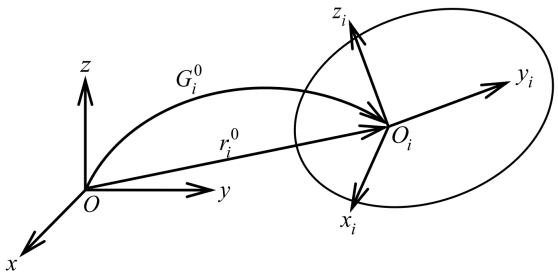

Fig. 2. Geometric parameters that determine the position of the *i*-th body in the absolute coordinate system.

In recent literature, a noteworthy focus has been placed on exploring diverse facets of the aforementioned approaches to derive equations of motion for multibody systems. Notably, several comprehensive studies [8–20] have shed light on this subject matter, offering intricate insights and detailed analyses.

The application of other forms of equations of motion (for example, the Gibbs-Appell equations [21, 22], the Gauss principle [23, 24] or the Hamilton equations [3, 7, 25-27]) for the modeling of systems of rigid bodies is not currently worked out as thoroughly as in the Newton-Euler and Euler-Lagrange approaches. First of all, this concerns the possibility of using Hamilton's equations. The possibility of their use is only indicated in a number of sources [3, 7]. In [25], the recursive algorithm for constructing the Hamiltonian equations for chains of bodies connected by one-stage rotational hinges has been created. In [26, 27], the Hamilton equations were written in Cartesian coordinates, and the constraint equations in joints were taken into account using the classical Lagrange multipliers. However, the algorithmization of the computer formation of these equations for the systems of rigid bodies has not yet been completed.

This work fills the existing gap by deriving the equations of motion for systems of rigid bodies with open kinematic chains (tree structure) using quasi-velocities, generalized coordinates, generalized and Cartesian momenta. It is shown that when projected onto the tangent and orthogonal subspaces of virtual movements, these equations lead to the Hamilton equations. This solves the problem with the algorithmization of the equations of motion of multibody systems in Hamiltonian form. In addition, the algorithms for the “sweep” method are presented that allow (as in the Newton-Euler approach) solving the resulting extended system of equations for generalized velocities without direct formation of the Hamilton equations. The work is theoretical and complements one more stone in the treasury of computer algorithms for modeling of the absolutely rigid body systems.

2. Description of the Mechanical System

We consider a system of absolutely rigid bodies with a tree structure. Mathematical models of, for example, robotic manipulators, walking “centipede” robots, space vehicles, transport multilink trains, cranes on a movable base, and many other technical systems can have such a design scheme. An example of absolutely rigid body system with a tree structure is shown in Fig. 1.

Let us assume that the kinematic constraints implemented in the hinges are holonomic and ideal. Let N be the number of bodies and hinges in the system. With each i-th body, we will associate a Cartesian coordinate system $O_i x_i y_i z_i$ with the origin at an arbitrary point O_i of the i-th body (see Fig. 2).

Let's introduce the notation: r_i^0 to be the radius-vector or matrix-column of point coordinates O_i in the absolute coordinate system $Oxyz$; r_i – vector r_i^0 in projections onto the axes of the i-th coordinate system; G_i^0 is a matrix, the columns of which are the coordinates of the unit basis vectors of the i-th coordinate system in the absolute coordinate system (the matrix of coordinate transformation from the i-th coordinate system to the absolute one or the matrix of direction cosines); G_0^i is the matrix of coordinate transformation from the absolute to the i-th coordinate system. Matrix G_i^0 is orthogonal: $G_0^i G_i^0 = G_i^0 G_0^i = E$ and $(G_0^i)^T = (G_0^i)^{-1} = G_i^0$. The vectors r_i^0 and r_i are related by $r_i = G_0^i r_i^0$.

We introduce the vectors (column matrices) of generalized coordinates in the i-th hinge – $q_i = (q_i^1,...,q_i^{n_i})^T$, n_i is the number of generalized coordinates. Denote by $q = \mathrm{col}(q_1,...,q_N)$ is the vector of all generalized coordinates of the system. Vector r_i^0 and matrix G_i^0 are functions of generalized coordinates and time given by:

$$r_i^0 = r_i^0(q,t), \quad G_i^0 = G_i^0(q,t).$$

Let v_i^0 be the velocity vector of the point O_i in the absolute coordinate system; v_i is a vector v_i^0 in projections onto the axes of the i-th coordinate system; ω_i^0 is the angular velocity vector of the i-th body in the absolute coordinate system; ω_i is a vector ω_i^0 in projections on the axes of the i-th coordinate system.

Let us obtain the formulas relating the vectors of absolute velocities v_i and ω_i with the vector of generalized velocities $\dot{q}$, if the functions $r_i^0(q,t)$ and $G_i^0(q,t)$ are known.

For linear velocity v_i, the following sequence of transformations is valid:

$$v_i = G_0^i v_i^0 = G_0^i \dot{r}_i^0 = G_0^i \frac{\partial r_i^0}{\partial q^T}\dot{q} + G_0^i \frac{\partial r_i^0}{\partial t}. \tag{1}$$

On the other hand,

$$v_i = r_i^* + \omega_i \times r_i = \left(G_0^i r_i^0\right)^* + \tilde{\omega}_i \left(G_0^i r_i^0\right) = \dot{G}_0^i r_i^0 + G_0^i \dot{r}_i^0 + \tilde{\omega}_i G_0^i r_i^0,$$

where the symbol 'asterisk' marks the local derivative in the i-th coordinate system, $\tilde{\omega}_i$ is the skew-symmetric matrix of the vector product as:

$$\tilde{\omega}_i = \begin{pmatrix} 0 & -\omega_{i3} & \omega_{i2} \\ \omega_{i3} & 0 & -\omega_{i1} \\ -\omega_{i2} & \omega_{i1} & 0 \end{pmatrix}.$$

The last formula for the velocity v_i must coincide with formula (1) for any values of r_i^0. Hence, it follows that:

$$\dot{G}_0^i + \tilde{\omega}_i G_0^i = 0, \quad \dot{G}_0^i = -\tilde{\omega}_i G_0^i = \tilde{\omega}_i^T G_0^i \quad \text{and} \quad \tilde{\omega}_i = -\dot{G}_0^i \left(G_0^i\right)^T = G_0^i \left(\dot{G}_0^i\right)^T.$$

We use the last equation to establish a connection between the angular velocity vector ω_i and the vector of generalized velocities $\dot{q}$:

$$\tilde{\omega}_i = G_0^i \left(\dot{G}_0^i\right)^T = G_0^i \left(\frac{\partial G_0^i}{\partial q^T}\dot{q}\right)^T + G_0^i \frac{\partial \left(G_0^i\right)^T}{\partial t} = G_0^i \dot{q}^T \left(\frac{\partial G_0^i}{\partial q^T}\right)^T + G_0^i \frac{\partial \left(G_0^i\right)^T}{\partial t} = G_0^i \frac{\partial \left(G_0^i\right)^T}{\partial q^T}\dot{q} + G_0^i \frac{\partial \left(G_0^i\right)^T}{\partial t}. \tag{2}$$

Let us introduce the notations:

$$B_i^v = \left(b_{i1}^v \; b_{i2}^v \ldots\right) = G_0^i \frac{\partial r_i^0}{\partial q^T}, \quad \varphi_i^v = G_0^i \frac{\partial r_i^0}{\partial t}, \quad \tilde{b}_{il}^{\omega} = G_0^i \frac{\partial \left(G_0^i\right)^T}{\partial q_l}, \quad B_i^{\omega} = \left(b_{i1}^{\omega} \; b_{i2}^{\omega} \ldots\right), \quad \varphi_i^{\omega} = G_0^i \frac{\partial \left(G_0^i\right)^T}{\partial t} = G_0^i \frac{\partial G_i^0}{\partial t}. \tag{3}$$

In accordance with Eq. (2) and Eq. (3), the six-dimensional vector of linear and angular velocities of the i-th body $v_i = \mathrm{col}(v_i, \omega_i)$ will be calculated through the generalized coordinates and velocities of the system according to the formula:

$$v_i = \begin{pmatrix} v_i \\ \omega_i \end{pmatrix} = \begin{pmatrix} B_i^v \\ B_i^{\omega} \end{pmatrix} \dot{q} + \begin{pmatrix} \varphi_i^v \\ \varphi_i^{\omega} \end{pmatrix} = B_i \dot{q} + \varphi_i. \tag{4}$$

Combining formulas Eq. (4) for all bodies of the system, we obtain the formula for the relationship between the vector of absolute velocities $v = \mathrm{col}(v_1,\ldots,v_N)$ and the vector $\dot{q}$:

$$v = B\dot{q} + \varphi, \tag{5}$$

where $B = \mathrm{col}(B_1,\ldots,B_N)$ is the matrix of the tangent local basis of the space of virtual displacements of all bodies, $\varphi = \mathrm{col}(\varphi_1,\ldots,\varphi_N)$ is the vector of the components of the velocity vector v that explicitly depend on time.

In what follows, we will need some properties of the skew-symmetric matrix of the cross product. For arbitrary vectors a and b [2]:

$$\tilde{a}^T = -\tilde{a}, \quad \tilde{a}^T b = -\tilde{b}a, \quad \tilde{a}\tilde{b} = ba^T - a^T b E, \quad \widetilde{a+b} = \tilde{a} + \tilde{b}, \quad \widetilde{\tilde{a}b} = ba^T - ab^T, \quad \tilde{a}\tilde{b} - \widetilde{\tilde{a}b} = \tilde{b}\tilde{a}. \tag{6}$$

Besides,

$$\tilde{\omega}_i r_i = \omega_i \times r_i = G_0^i \left(\omega_i^0 \times G_i^0 r_i\right) = G_0^i \tilde{\omega}_i^0 G_i^0 r_i,$$

in which the skew-symmetric matrix $\tilde{\omega}_i$ transforms upon changing the basis as the matrix of the linear operator:

$$\tilde{\omega}_i = G_0^i \tilde{\omega}_i^0 G_i^0. \tag{7}$$

Let us differentiate matrix B with respect to time. Taking into account properties Eq. (6), the time derivatives of columns b_{il}^v and b_{il}^{ω} of matrices B_i^v and B_i^{ω} can be reduced to the form:

$$\dot{b}_{il}^v = \frac{d}{dt}\left(G_0^i \frac{\partial r_i^0}{\partial q_l}\right) = \dot{G}_0^i \frac{\partial r_i^0}{\partial q_l} + G_0^i \frac{\partial \dot{r}_i^0}{\partial q_l} = -\tilde{\omega}_i G_0^i \frac{\partial r_i^0}{\partial q_l} + G_0^i \frac{\partial G_i^0 v_i}{\partial q_l} = -\tilde{\omega}_i b_{il}^v + G_0^i \frac{\partial G_i^0}{\partial q_l} v_i + G_0^i G_i^0 \frac{\partial v_i}{\partial q_l} = -\tilde{\omega}_i b_{il}^v + \tilde{b}_{il}^{\omega} v_i + \frac{\partial v_i}{\partial q_l} = -\tilde{\omega}_i b_{il}^v - \tilde{v}_i b_{il}^{\omega} + \frac{\partial v_i}{\partial q_l},$$

$$\dot{\tilde{b}}_{il}^{\omega} = \frac{d}{dt}\left(G_0^i \frac{\partial G_i^0}{\partial q_l}\right) = -\tilde{\omega}_i G_0^i \frac{\partial G_i^0}{\partial q_l} + G_0^i \frac{\partial \dot{G}_i^0}{\partial q_l} = -\tilde{\omega}_i \tilde{b}_{il}^{\omega} + G_0^i \frac{\partial G_i^0 \tilde{\omega}_i}{\partial q_l} = -\tilde{\omega}_i \tilde{b}_{il}^{\omega} + G_0^i \frac{\partial G_i^0}{\partial q_l}\tilde{\omega}_i + G_0^i G_i^0 \frac{\partial \tilde{\omega}_i}{\partial q_l} = -\tilde{\omega}_i \tilde{b}_{il}^{\omega} + \tilde{b}_{il}^{\omega}\tilde{\omega}_i + \frac{\partial \tilde{\omega}_i}{\partial q_l} = -\tilde{\omega}_i \widetilde{b}_{il}^{\omega} + \frac{\partial \tilde{\omega}_i}{\partial q_l}.$$

In this way,

$$\dot{B}_i^v = -\tilde{\omega}_i B_i^v - \tilde{v}_i B_i^\omega + \frac{\partial v_i}{\partial q^T}, \quad \dot{B}_i^\omega = -\tilde{\omega}_i B_i^\omega + \frac{\partial \omega_i}{\partial q^T}. \tag{8}$$

In Eq. (8), the partial derivatives with respect to generalized coordinates must be calculated after substituting their expressions from relations (4) instead of the velocities v_i and ω_i.

Let's define matrices:

$$\Omega_i = \begin{pmatrix} \tilde{\omega}_i & 0 \\ \tilde{v}_i & \tilde{\omega}_i \end{pmatrix}, \quad \Omega = \mathrm{diag}(\Omega_1,...,\Omega_N).$$

Combining with Eq. (8), we finally construct formulas for differentiating the matrices B_i and B:

$$\dot{B}_i = \Omega_i^T B_i + \frac{\partial v_i}{\partial q^T}, \quad \dot{B} = \Omega^T B + \frac{\partial v}{\partial q^T}. \tag{9}$$

- Kinematics of the whole system

We number the bodies and hinges so that for each body or hinge in the system graph, the number of the preceding body or hinge is less. In this case, for a complete description of the structure of the interconnections of bodies in the system, one integer array $k = \{k_1,...,k_N\}$ is sufficient, in which the index of the previous body or hinge is located at the i-th place. We associate the following sets with each body of the system: P_i is an ordered set of indices of hinges that make up the path between the zero and i-th bodies; U_i is the set of hinge indices for which the i-th body is the predecessor.

Let us introduce the following notation: ρ_i – matrix-column of coordinates of the origin of the coordinate system associated with the i-th body in the k_i-th coordinate system; G_i^j is a matrix whose columns are the coordinates of the unit basis vectors of the i-th coordinate system in the j-th coordinate system (the matrix of coordinate transformation from the i-th to the j-th coordinate system). Let's assume $G_i = G_{k_i}^i$.

The vector ρ_i and the matrix G_i are functions of the generalized coordinates of only the i-th hinge as:

$$\rho_i = \rho_i(q_i,t), \qquad G_i = G_i(q_i,t).$$

The matrices r_i and G_0^i introduced above are related to the matrices ρ_i and G_i by recursive formulas (see Fig. 3):

$$r_i = G_i\left(r_{k_i} + \rho_i\right), \qquad G_0^i = G_i G_0^{k_i}.$$

Taking the motion of the i-th body as relative, and the motion of the preceding k_i-th body as portable, we can write recurrent formulas for calculating the projections of the linear v_i and angular ω_i velocities of the bodies of the mechanical system on the axis of the i-th coordinate system:

$$v_i = G_i\left(v_{k_i} + \omega_{k_i} \times \rho_i + \frac{\partial \rho_i}{\partial q_i^T}\dot{q}_i + \frac{\partial \rho_i}{\partial t}\right),$$

$$\omega_i = G_i \omega_{k_i} + \omega_i^r,$$

or in six dimensions,

$$v_i = C_i v_{k_i} + A_i \dot{q}_i + v_i^r, \tag{10}$$

where,

$$v_i = \begin{pmatrix} v_i \\ \omega_i \end{pmatrix}, \quad v_i^r = \begin{pmatrix} v_i^r \\ \omega_i^r \end{pmatrix}, \quad A_i = \begin{pmatrix} a_{i1}^v & \ldots & a_{in_i}^v \\ a_{i1}^\omega & \ldots & a_{in_i}^\omega \end{pmatrix}, \quad a_{ij}^v = G_i \frac{\partial \rho_i}{\partial q_i^j}, \quad v_i^r = G_i \frac{\partial \rho_i}{\partial t}, \quad \tilde{a}_{ij}^\omega = G_i \frac{\partial G_i^T}{\partial q_i^j}, \quad j = 1,2,...,n_i, \quad \tilde{\omega}_i^r = G_i \frac{\partial G_i^T}{\partial t}, \quad C_i = \begin{pmatrix} G_i & -G_i\tilde{\rho}_i \\ 0 & G_i \end{pmatrix}.$$

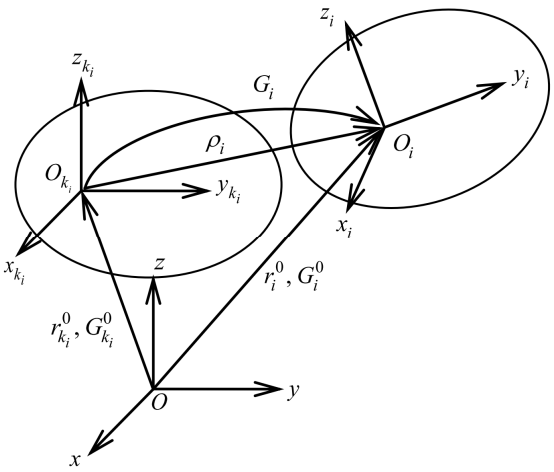

Fig. 3. Geometric parameters that determine the relative and absolute coordinates of the bodies.

Let us introduce a block 6N×6N matrix S with square submatrices of order 6 according to the following formula:

$$S_{ij}=\begin{cases}E_6, & j=i,\\ -C_i, & j=k_i,\\ 0_{6\times6}, & j\neq i\vee k_i,\end{cases}\quad i,j=1,2,\ldots,N. \tag{11}$$

Note that this matrix contains only two non-zero blocks E_6 and $-C_i$ in each row for any kinematic structure. Since the matrix S contains information about both the topological structure of the system and the relative position of the bodies, it is called the matrix of the kinematic structure. Using the matrix S, the equations of the kinematics (10) can be written compactly as follows:

$$Sv=A\dot{q}+v^r, \tag{12}$$

where $A=\mathrm{diag}(A_1,\ldots,A_N)$, $v^r=\mathrm{col}(v_1^r,\ldots,v_1^r)$.

Recurrent formulas (12) can be written as explicit expressions:

$$v=T\left(A\dot{q}+v^r\right), \tag{13}$$

where the matrix $T=S^{-1}$ inverse to S is a block 6N × 6N matrix, the submatrices of which can be calculated using the recursive formulas in accordance with Eq. (10):

$$T_{ij}=\begin{cases}E_6, & j=i,\\ C_iT_{k_i,j}, & j\in P_{k_i},\\ 0_{6\times6}, & j\notin P_i,\end{cases}\quad i,j=1,2,\ldots,N. \tag{14}$$

Comparing formulas (5) and (13), we conclude that:

$$B=TA,\ \varphi=Tv^r. \tag{15}$$

Let us now determine how to calculate the total time derivatives of the matrices C_i, S and T without explicit differentiation using the matrices already introduced. To do this, we will make preliminary calculations. The matrix C_i includes blocks G_i and $G_i\tilde{\rho}_i$. We calculate $\dot{G}_i$ and $d(-G_i\tilde{\rho}_i)/dt$ using the properties of skew-symmetric matrices described above:

$$\dot{G}_i=\frac{d}{dt}\left(G_0^iG_{k_i}^0\right)=\dot{G}_0^iG_{k_i}^0+G_0^i\dot{G}_{k_i}^0=-\tilde{\omega}_iG_0^iG_{k_i}^0+G_0^iG_{k_i}^0\tilde{\omega}_{k_i}=-\tilde{\omega}_iG_i+G_i\tilde{\omega}_{k_i},$$

$$\frac{d}{dt}(-G_i\tilde{\rho}_i)=-\dot{G}_i\tilde{\rho}_i-G_i\tilde{\rho}_i^{\cdot}=\left(\tilde{\omega}_iG_i-G_i\tilde{\omega}_{k_i}\right)\tilde{\rho}_i-G_i\left(\left(\tilde{G_i^Tv_i}\right)-\tilde{v}_{k_i}-\left(\tilde{\omega}_{k_i}\rho_i\right)\right)=\tilde{\omega}_iG_i\tilde{\rho}_i-G_i\tilde{\omega}_{k_i}\tilde{\rho}_i-G_iG_i^T\tilde{v}_iG_i+G_i\tilde{v}_{k_i}+G_i\left(\tilde{\omega}_{k_i}\rho_i\right)=\tilde{\omega}_iG_i\tilde{\rho}_i-\tilde{v}_iG_i-G_i\tilde{\rho}_i\tilde{\omega}_{k_i}+G_i\tilde{v}_{k_i}.$$

Substituting the resulting expressions into the matrix $\dot{C}_i$, we obtain:

$$\dot{C}_i=\Omega_i^TC_i-C_i\Omega_{k_i}^T. \tag{16}$$

Using Eq. (16) it is easy to verify that the derivatives of the matrices S and T are calculated by the formulas:

$$\begin{aligned}\dot{S}&=\Omega^TS-S\Omega^T,\\ \dot{T}&=-T\dot{S}T=\Omega^TT-T\Omega^T.\end{aligned} \tag{17}$$

Equations (17) imply an interesting relationship as:

$$\dot{B}-\Omega^TB=T\left(\dot{A}-\Omega^TA\right),\quad\text{or}\quad S\left(\dot{B}-\Omega^TB\right)=\dot{A}-\Omega^TA. \tag{18}$$

Equation (18) is proved by the following sequence of transformations:

$$\dot{B}=\frac{d}{dt}(TA)=\dot{T}A+T\dot{A}=\Omega^TTA-T\Omega^TA+T\dot{A}=\Omega^TB+T\left(-\Omega^TA+\dot{A}\right).$$

Note that on the left side of equality in Eq. (18), the total time derivative $\dot{B}$ includes partial derivatives with respect to all generalized coordinates, and on the right side, the matrix $\dot{A}$ contains only partial derivatives with respect to the generalized coordinates of the i-th hinge.

Let us apply Eq. (9) and Eq. (18) to differentiate the recurrent relations in Eq. (12) as follows:

$$\frac{\partial\left(A\dot{q}+v^r-Sv\right)^T}{\partial q}=\frac{\partial\left(S\left(B\dot{q}+\varphi-v\right)\right)^T}{\partial q}=\frac{\partial\left(B\dot{q}+\varphi-v\right)^T}{\partial q}S^T+\left(B\dot{q}+\varphi-v\right)^T\frac{\partial S^T}{\partial q}=\left(\dot{B}-\Omega^TB\right)^TS^T=\dot{A}^T-A^T\Omega. \tag{19}$$

3. Equations of Motion of Multibody Systems with a Tree Structure in Hamiltonian Variables

Let m_i be the mass of the i-th body; J_i is the inertia tensor of the i-th body; r_i^c is the radius vector of the center of mass of the i-th body in the coordinate system associated with it; f_i^o and m_i^o are the main vector and the main moment of the active forces i-th body, given in the i-th coordinate system; Q is the column matrix of generalized forces.

Let us introduce the notation:

$$M_i = \begin{pmatrix} m_i E & -m_i \tilde{r}_i^c \\ m_i \tilde{r}_i^c & J_i \end{pmatrix}, \quad M = \mathrm{diag}(M_1,...,M_N), \quad F_i = \mathrm{col}\left(f_i^o, m_i^o\right), \quad F = \mathrm{col}(F_1,...,F_N).$$

The kinetic energy $\mathbb{T}$ and the generalized forces Q are given by the expressions [2]:

$$\mathbb{T}_i = \frac{1}{2} m_i v_i^T v_i + m_i \left(\tilde{v}_i \omega_i\right)^T r_i^c + \frac{1}{2} \omega_i^T J_i \omega_i, \quad F_i^\Sigma = F_i + \sum\nolimits_{j \in U_i} C_j^T F_j^\Sigma, \tag{20a}$$

$$Q_i = A_i^T F_i^\Sigma, \quad \mathbb{T} = \sum_{i=1}^{N} \mathbb{T}_i = \frac{1}{2} v^T M v, \quad Q = col(Q_1,...,Q_N) = A^T T^T F. \tag{20b}$$

We derive the equations of motion from the generalized Hamilton principle [4]. Kinematic relations in Eq. (12) will be taken into account by the Lagrange method. Denote the vector of Lagrange multipliers μ. Let us write out and transform the Hamiltonian variation of the action:

$$\delta H = \int_{t_1}^{t_2} \delta \mathbb{T} + Q^T \delta q + \delta\left(\mu^T \left(A\dot{q} + v^r - Sv\right)\right) dt = \int_{t_1}^{t_2} \frac{1}{2} \delta\left(v^T M v\right) + Q^T \delta q + \delta\mu^T \left(A\dot{q} + v^r - Sv\right) +$$

$$+\delta\left(A\dot{q} + v^r - Sv\right)^T \mu dt = \int_{t_1}^{t_2} \delta v^T \left(Mv\right) + \delta q^T Q + \delta\mu^T \left(A\dot{q} + v^r - Sv\right) + \delta q^T \frac{\partial\left(A\dot{q} + v^r - Sv\right)^T}{\partial q} \mu + \delta\dot{q}^T \left(A^T \mu\right) - \delta v^T \left(S^T \mu\right) dt = 0.$$

We introduce a new vector variable as:

$$p = A^T \mu. \tag{21}$$

We integrate the term $\delta\dot{q}^T p$ by parts to pass from the variation of the velocity vector to the variation of the vector of generalized coordinates. Using Eq. (19), we reduce the variation of δH to the form:

$$\delta H = \int_{t_1}^{t_2} \delta v^T \left(Mv - S^T \mu\right) + \delta\mu^T \left(A\dot{q} + v^r - Sv\right) + \delta q^T \left(Q - \dot{p} + \left(\dot{A}^T - A^T \Omega\right)\mu\right) dt = 0.$$

On a real trajectory, the Hamiltonian variation of δH is zero. The coefficients in front of the variations of the Lagrange multipliers $\delta\mu$ are equal to zero, because are kinematic Eq. (12). The coefficients before the variations of the Cartesian velocities δv can be set to zero by choosing the Lagrange multipliers μ, since their number coincides with the number of the corresponding coefficients. The remaining variations of the generalized coordinates δq are linearly independent, so the variation of δH will be equal to zero when each coefficient in front of δq is equal to zero.

As a result, we obtain the following closed system of differential-algebraic equations describing the dynamics of multibody systems with a tree structure:

$$\begin{cases} Mv - S^T \mu = 0, \\ -Sv + A\dot{q} = -v^r, \\ A^T \mu = p, \end{cases} \tag{22}$$

$$\dot{p} = \left(\dot{A}^T - A^T \Omega\right)\mu + Q. \tag{23}$$

In what follows, we will show that the vector p defined by formula (21) is the vector of generalized impulses, and the vector of Lagrange multipliers μ is the vector of Cartesian impulses of composite bodies with frozen connections.

The derivation of Eq. (22), Eq. (23) from the equations of motion in Euler-Lagrange quasi-velocities was performed in [28]. A feature of Eqs. (22) and (23) is that they are resolved with respect to the derivatives of the generalized impulses $\dot{p}$. The first three equations (22) form a linear system with a symmetric, block tridiagonal sparse matrix of coefficients with respect to the velocities v, $\dot{q}$ and variable μ.

For clarity, we write down the system of Eqs. (22) and (23) in the expanded form over the bodies:

$$\begin{cases} \mu_i = M_i v_i + \sum\limits_{j \in U_i} C_j^T \mu_j, \\ v_i = C_i v_{k_i} + A_i \dot{q}_i + v_i^r, \\ A_i^T \mu_i = p_i, \\ F_i^* = F_i + \sum\limits_{j \in U_i} C_j^T F_j^*, \\ \dot{p}_i = \left(\dot{A}_i^T - A_i^T \Omega_i\right)\mu_i + A_i^T F_i^*, \quad i = 1,2,...,N. \end{cases}$$

This notation of Eq. (22) shows that they have a recurrent structure, consist of elementary matrix blocks, and therefore can be easily written for any system of rigid bodies with a tree structure.

4. Mechanical Features of Equations (22) and (23)

Let us show the relationship between equations (22) and (23) and the Hamilton's equations. According to [3], it is known that the Hamilton equations can be written using the Hamilton function $H = p^T \dot{q} - \mathbb{T} + \Pi$ or the Poisson function $\tilde{\mathbb{T}} = p^T \dot{q} - \mathbb{T}$ as:

$$\begin{cases} \dot{q} = \dfrac{\partial \mathrm{H}}{\partial p} = \dfrac{\partial \tilde{\mathbb{T}}}{\partial p}, \\ \dot{p} = -\dfrac{\partial \mathrm{H}}{\partial q} = -\dfrac{\partial \tilde{\mathbb{T}}}{\partial q} - \dfrac{\partial \Pi}{\partial q} \end{cases} \tag{24}$$

or in general,

$$\dot{p} = -\frac{\partial \mathrm{H}}{\partial q} = -\frac{\partial \tilde{\mathbb{T}}}{\partial q} + \mathrm{Q}, \tag{25}$$

where Π is the potential energy.

It follows from the first equation of system (23) that the vector of Lagrange multipliers $\mu = \mathrm{col}(\mu_1,...,\mu_N)$ is determined by the formula:

$$\mu = T^{\mathrm{T}} M v = T^{\mathrm{T}} \mathrm{M}(B\dot{q} + \varphi). \tag{26}$$

That is, the i-th component of the vector μ is the Cartesian momentum of the composite body containing the i-th body and all its child bodies with frozen connections:

$$\mu_{\mathrm{i}}^{\Sigma} = \mathrm{M}_{\mathrm{i}} v_{\mathrm{i}} + \sum\nolimits_{j \in U_{\mathrm{i}}} \mathrm{C}_{\mathrm{j}}^{T} \mu_{\mathrm{j}}^{\Sigma}.$$

From formulas (21) and (26), it follows that:

$$p = \mathrm{A}^{\mathrm{T}} \mu = B^{\mathrm{T}} \mathrm{M}(B\dot{q} + \varphi) = \mathrm{A}^{\mathrm{T}} T^{\mathrm{T}} \mathrm{M} T \left(\mathrm{A}\dot{q} + v^{\mathrm{r}}\right). \tag{27}$$

We write the expression for the kinetic energy in generalized velocities:

$$\mathbb{T} = \frac{1}{2} v^{\mathrm{T}} M v = \frac{1}{2} (B\dot{q} + \varphi)^{\mathrm{T}} M (B\dot{q} + \varphi). \tag{28}$$

The generalized momentum or the Poisson momentum is a partial derivative of the kinetic energy in Eq. (28) with respect to the generalized velocity:

$$p = \frac{\partial \mathbb{T}}{\partial \dot{q}} = B^{\mathrm{T}} \mathrm{M}(B\dot{q} + \varphi). \tag{29}$$

Comparing expressions in Eqs. (27) and (29), we come to the conclusion that the vector p included in Eq. (22), Eq. (23) is the vector of generalized impulses, and the pair of coordinates $\{q, p\}$ are the Hamilton variables.

Let us now write the explicit matrix form of the equations of motion of multibody systems in the form of Lagrange equations, using formulas (9), (18), (26) and (29) as:

$$\dot{p} = \frac{d}{dt}\frac{\partial \mathbb{T}}{\partial \dot{q}} = \frac{\partial \mathbb{T}}{\partial q} + \mathrm{Q} = \frac{\partial v^{\mathrm{T}}}{\partial q} M v + \mathrm{Q} = \left(\dot{B}^{\mathrm{T}} - \Omega B^{\mathrm{T}}\right) M v + \mathrm{Q} = \left(\dot{\mathrm{A}}^{\mathrm{T}} - \Omega \mathrm{A}^{\mathrm{T}}\right) \mathrm{T}^{\mathrm{T}} M v + \mathrm{Q} = \left(\dot{\mathrm{A}}^{\mathrm{T}} - \Omega \mathrm{A}^{\mathrm{T}}\right) \mu + \mathrm{Q}.$$

Thus, we conclude that Eq. (23) is the Lagrange equation, written with respect to the derivative of the generalized momentum $\dot{p}$ or, in other words, it is the Hamiltonian equation for $\dot{p}$, in which the transition from generalized velocities $\dot{q}$ to impulses p is not performed on the right side.

Let us now show the relationship between Eq. (22) and the first of Hamilton's equations for generalized velocities. Using equality (26), we can collapse Eq. (22) into one equation and express the vector of generalized velocities $\dot{q}$ in terms of the vector p as:

$$\dot{q} = \left(B^{\mathrm{T}} M B\right)^{-1} \left(p - B^{\mathrm{T}} M \varphi\right). \tag{30}$$

On the other hand, we perform the transition in the Poisson function $\tilde{\mathbb{T}}$ from $\dot{q}$ to p and express the Hamilton equation for $\dot{q}$ as follows:

$$\tilde{\mathbb{T}} = p^{\mathrm{T}} \dot{q} - \frac{1}{2} v^{\mathrm{T}} M v = (B\dot{q} + \varphi)^{\mathrm{T}} M B\dot{q} - \frac{1}{2} (B\dot{q} + \varphi)^{\mathrm{T}} M (B\dot{q} + \varphi) = \frac{1}{2} \dot{q}^{\mathrm{T}} B^{\mathrm{T}} M B \dot{q} - \frac{1}{2} \varphi^{\mathrm{T}} M \varphi = -\frac{1}{2} \varphi^{\mathrm{T}} M \varphi + \frac{1}{2} \left(p - B^{\mathrm{T}} M \varphi\right)^{\mathrm{T}} \left(B^{\mathrm{T}} M B\right)^{-1} \left(p - B^{\mathrm{T}} M \varphi\right).$$

In this way,

$$\dot{q} = \frac{\partial \tilde{\mathbb{T}}}{\partial p} = \left(B^{\mathrm{T}} M B\right)^{-1} \left(p - B^{\mathrm{T}} M \varphi\right). \tag{31}$$

It can be seen that the formulas (30) and (31) coincide well. Consequently, equations (22) are an expanded form of the first Hamilton equation for generalized velocities, written with respect to an extended set of variables $\{q, p\}$ (generalized coordinates and impulses) and $\{v, \mu\}$ (Cartesian velocities and impulses).

The introduction of additional variables into the equations of motion is required in computer simulation of the dynamics of multibody systems. In numerical integration, they must be reduced to a normal form, i.e. resolve with respect to higher derivatives. The advantage of the equations of motion in the form Eq. (22), Eq. (23) is as follows. First, equations (23) are immediately resolved with respect to $\dot{p}$. Secondly, the system of equations (22), as a system of linear algebraic equations with respect to an extended set of variables $\{\dot{q}, v, \mu\}$, has a block quasi-tridiagonal structure and can be solved by any of the variants of the "sweep" method [1, 9, 13]. These methods are characterized by a linear increase in the number of arithmetic operations depending on the number of bodies in the mechanical system. From this point of view, with a large number of bodies in the

system, algorithms for solving equations (22) containing an excessive number of variables turn out to be more efficient than algorithms for generating equations of motion in the Lagrange or Hamilton form and their subsequent solution.

Ivanov and Shimanovskiy [29] presented the algorithms for the method of "sweeping" the resolution of the system of equations (22) with respect to the variables $\{\dot{q}, v, \mu\}$, using the asymmetric (Vereshchagin method) and symmetric (Cholesky method) Gaussian elimination.

5. Classification of the Equations of Motion of Rigid Body Systems

In the mathematical modeling of the processes of functioning of newly created technical systems, an engineer needs not only to compose the equations of motion, but also to integrate them. Equations (22) and (23), like most equations of motion for systems of constraint rigid bodies, are implicit systems of differential-algebraic equations, for the solution of which it is necessary to use special methods. However, the vast majority of engineers are not specialists in the field of numerical methods. Therefore, it is desirable that in his research, an engineer could use standard methods for the numerical integration of ordinary differential equations (ODEs) of the Adams or Runge-Kutta type, which are included in the libraries of standard subroutines in all available programming languages.

For this purpose, it is necessary to arm the engineer with algorithms for reducing the equations of motion to the normal form of the ODE, that is, with algorithms for resolving these equations with respect to higher derivatives. All equations of motion of mechanical systems are systems of linear algebraic equations with respect to higher derivatives. Algorithms for solving such systems depend on the structure of the systems matrix. In this section, we present a classification of the equations of motion of rigid body systems from the point of view of their structure. Algorithms for solving the equations of motion with respect to higher derivatives are given in Section 7.

Section 3 constructs an extended system of equations of motion described in Eqs. (22) and (23) in Hamiltonian variables. Equations (22) are a system of linear algebraic equations with respect to the variables $\{\dot{q}, v, \mu\}$ with the systems matrix:

$$M^{\Sigma} = \begin{pmatrix} M & -S^T & 0 \\ -S & 0 & A \\ 0 & A^T & 0 \end{pmatrix}. \tag{32}$$

In the Newton-Euler approach, these equations correspond to an extended system of equations, which includes the equations of rigid body dynamics, kinematic relations between the absolute and relative accelerations of the bodies of the system, and closing equations arising from the principle of ideality of bonds (principles of d'Alembert or Jourdain):

$$\begin{cases} Mw - S^T R = F + F^c, \\ -Sw + A\ddot{q} = -w^r, \\ A^T R = 0, \end{cases} \tag{33}$$

where $w = \mathrm{col}(w_1, \ldots, w_N)$ – is the vector of absolute linear and angular Cartesian accelerations of all bodies of the system, w^r – is the vector of gyroscopic, Coriolis accelerations and accelerations that explicitly depend on time, $R = \mathrm{col}(R_1, \ldots, R_N)$ – is the vector of constraint reactions,

$$F^c = col\left(F_1^c, \ldots, F_N^c\right) = \begin{pmatrix} -M\tilde{\omega}^2 r_c \\ -\tilde{\omega} J \omega \end{pmatrix}.$$

It is easy to see that the Newton-Euler equations are a system of linear algebraic equations in the variables $\{\ddot{q}, w, R\}$ with the same extended matrix of the system (32).

In Section 3, Eqs. (22) were solved with respect to the vector of generalized velocities and the Hamilton equations were written out. Equations (30) and (23) are an explicit form of the Hamiltonian equations for constraint rigid body systems, resolved with respect to the variables $\{\dot{q}, \dot{p}\}$. Equations (30) can be considered as a system of linear algebraic equations with the system matrix:

$$D^{\Sigma} = B^T M B = A^T \left(T^T M T\right) A. \tag{34}$$

In Eq. (34), the brackets indicate the order of multiplication of block matrices in accordance with the conjugate body method [2]. Similarly, if the Newton-Euler relation in Eq. (33) are projected onto the tangent subspace of virtual movements, with a basis whose vectors are the columns of the matrix A (Null Space Method), then we obtain an explicit representation of the Lagrange equations of the 2nd kind for constraint rigid body systems:

$$\left(A^T T^T M T A\right)\ddot{q} = A^T T^T \left(F + F^c - M T w^r\right). \tag{35}$$

Equations (35) are linear equations with respect to generalized accelerations $\ddot{q}$ with the same systems matrix D^{Σ}, which calculated by recursive formulas (34).

Finally, one more variant of bringing the equations of motion to the normal form of an ODE is possible – this is the projection of extended systems of Eq. (22) or Eq. (33) onto an orthogonal subspace (Rank Space Method) of virtual movements. Let the basis of this subspace be determined by the matrix Z, the column vectors of which are orthogonal to the column vectors of the matrix A, that is, $A^T Z = 0$ and $Z^T A = 0$. The matrix Z is easily constructed by differentiating the constraint equations (calculating partial derivatives of the constraint equations with respect to generalized coordinates), or calculated from the known matrix A, by selecting the linearly independent columns of the matrix:

$$Z^* = E - A\left(A^T A\right)^{-1} A^T.$$

It is easy to check that in this case the orthogonality condition for the basis vectors of the tangent and orthogonal subspaces is satisfied. The constraint reaction vectors R belong to the orthogonal subspace due to the principle of their ideality. Therefore,

the vector R can always be represented as a linear combination of column vectors of the matrix Z, that is, it can be represented as $R = Z\lambda$, where the vector of coefficients λ of this linear combination is the vector of Lagrange multipliers. Similarly, the Cartesian momentum vector μ is expanded in terms of the basis vectors of the tangent and orthogonal subspaces as follows:

$$\mu = A\left(A^{T}A\right)^{-1} p + Z\lambda.$$

As a result, when projecting onto an orthogonal subspace, we obtain a system of equations in the form of the Lagrange equations of the 1st kind. In Lagrange variables, this system has the form:

$$\begin{cases} \ddot{q} = \left(A^{T}A\right)^{-1} A^{T}\left(Sw - w^{r}\right), \\ w = M^{-1}\left(S^{T}Z\lambda + F + F^{c}\right), \\ \left(Z^{T}SM^{-1}S^{T}Z\right)\lambda = Z^{T}\left(w^{r} - SM^{-1}\left(F + F^{c}\right)\right). \end{cases} \tag{36}$$

In the Hamilton variables, we obtain similar equations:

$$\begin{cases} \dot{q} = \left(A^{T}A\right)^{-1} A^{T}\left(Sv - v^{r}\right), \\ v = M^{-1}S^{T}Z\lambda, \\ \left(Z^{T}S^{T}M^{-1}SZ\right)\lambda = Z^{T}\left(v^{r} - SM^{-1}S^{T}A\left(A^{T}A\right)^{-1} p\right). \end{cases} \tag{37}$$

Equations (37) are closed by Eq. (23). It can be seen that the systems of Eq. (36) or Eq. (37) can be reduced to the normal form of the ODE if we find the vector of Lagrange multipliers λ from the system of linear algebraic equations included in these equations with the systems matrix:

$$H^{\Sigma} = Z^{T}SM^{-1}S^{T}Z = Z^{T}\left(T^{T}MT\right)^{-1} Z. \tag{38}$$

Thus, for all six considered methods for constructing the equations of motion of constraint rigid body systems, the problem of solving them with respect to the highest derivatives for subsequent integration using standard numerical methods for solving ODEs is reduced to the problem of solving systems of linear algebraic equations with matrices of the system of three types Eqs. (32), (34) and (38).

In Section 7, the algorithms are written for reducing all the considered systems of equations to the normal form of ODEs. The comparative efficiency of the proposed algorithms is given in Section 8.

6. An Example of Compiling Equations of Motion

To illustrate the technique of compiling Eqs. (22) and (23), as well as determining all the parameters included in them, we derive the equations of motion for the mechanical system "Pipelayer", shown in Fig. 4. The design scheme of the "Pipelayer" is shown in Fig. 5.

The system consists of 4 bodies connected in series: chassis, rotating part (RP), lifting part (LP) and retractable boom. First, for all bodies it is necessary to set the mass-inertial characteristics: masses, coordinates of the centers of mass and the inertia tensor: m_i, r_i^c, J_i, $i = 1,\ldots,4$.

Next, we define generalized coordinates. The chassis can move vertically in the absolute coordinate system by the amount z, as well as sequentially rotate around the transverse axis Ox by the angle α and the longitudinal axis Oy by the angle β. The RP rotates around the vertical axis O_2z_2, the angle of rotation is θ. The LP rotates around the O_3x_3 axis by an angle ε. The arrow can move along the O_3y_3 axis by the amount y. Thus, the system has six degrees of freedom and the generalized coordinate vectors are given by:

$$q_1 = (z,\alpha,\beta)^{T}, \quad q_2 = \theta, \quad q_3 = \varepsilon, \quad q_4 = y.$$

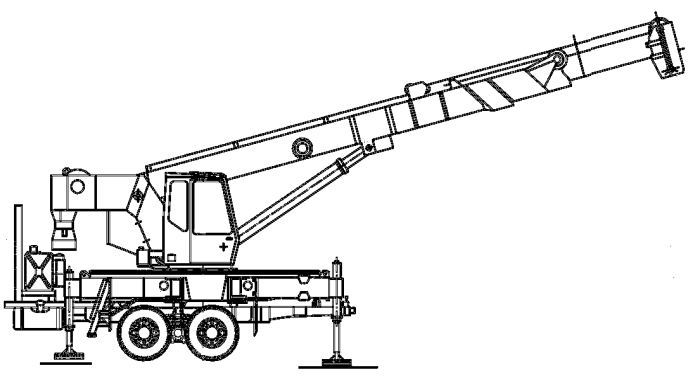

Fig. 4. The multibody system "Pipelayer".

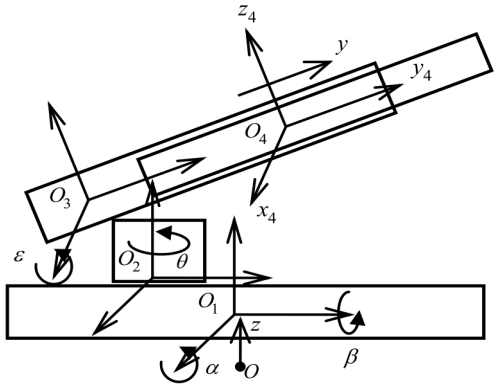

Fig. 5. The design scheme of the "Pipelayer".

The radius vectors $\rho_i = O_{i-1}O_i$ depend on the generalized coordinates as follows:

$$\rho_1 = (0,0,z)^T, \quad \rho_2 = (0,b_{RP},h_{RP})^T, \quad \rho_3 = (0,b_{LP},h_{LP})^T, \quad \rho_4 = (0,y,h)^T.$$

Direction cosine matrices are:

$$G_1 = \begin{pmatrix} \cos\beta & \sin\alpha\sin\beta & -\cos\alpha\sin\beta \\ 0 & \cos\alpha & \sin\alpha \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix}, \quad G_4 = E, \quad G_2 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & \sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon \end{pmatrix}.$$

The last to be set are the matrices A_i that determine the kinematics of the relative motion of the bodies of the system and their derivatives:

$$A_1 = \begin{pmatrix} -\cos\alpha\sin\beta & 0 & 0 \\ \sin\alpha & 0 & 0 \\ \cos\alpha\cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \\ 0 & \sin\beta & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \dot{A}_1 = \begin{pmatrix} \dot\alpha\sin\alpha\sin\beta - \dot\beta\cos\alpha\cos\beta & 0 & 0 \\ \dot\alpha\cos\alpha & 0 & 0 \\ -\dot\alpha\sin\alpha\cos\beta - \dot\beta\cos\alpha\sin\beta & 0 & 0 \\ 0 & -\dot\beta\sin\beta & 0 \\ 0 & 0 & 0 \\ 0 & \dot\beta\cos\beta & 0 \end{pmatrix}, \quad \dot{A}_2 = \dot{A}_3 = \dot{A}_4 = (0)_{6\times1}.$$

To complete the individualization of the mechanical system, it is also necessary to set the active forces and moments acting on the bodies. To understand the algorithm for compiling equations of motion, this step is not essential, and we will omit it. Further steps in the formation of matrix blocks included in Eq. (22) and Eq. (23) are universal for any multibody system and are performed according to the recurrent algorithms described in the previous sections of the work.

The given example illustrates the fact that the amount of information that needs to be prepared to build a mathematical model of a multibody system in the form of Eqs. (22) and (23) is minimal.

7. Recurrent Algorithms for Solving Equations of Motion of Rigid Body Systems with respect to Higher Derivatives

In the numerical simulation of rigid body systems using extended systems of equations of the form Eq. (22), Eq. (23) or Eq. (33), it is necessary to bring them to the normal form of the ODE. To do this, it is necessary to solve equations Eq. (22) with respect to generalized velocities $\dot q$, and Eq. (33) with respect to generalized accelerations $\ddot q$ at each integration step. As emphasized in Section 5, the system of Eq. (22) in the Hamiltonian approach has the same structure as the extended system of equations of motion in Eq. (33) in the Newton-Euler approach. Therefore, when solving them numerically, one can use the "sweeping" methods outlined in the works of *R* according to Featherstone et al. [1, 9] and Shah et al. [13]. These methods are essentially modifications of the Gauss methods for solving systems of linear algebraic equations with a system matrix of the form Eq. (32) having a band structure. They are based on asymmetric *LU* (Vereshchagin method) or symmetric L^TDL (Cholesky method) Gaussian elimination. In these methods, in the forward course, the Lagrange multipliers μ are excluded from the system of Eq. (22) (in the Newton-Euler method – the reactions of constraints). The forward course is performed starting from the last body of the mechanical system. In the reverse course, using explicit formulas, the generalized velocities $\dot q$, Cartesian velocities v and Lagrange multipliers μ (starting from the first body) are calculated sequentially for each body of the system (in the Newton-Euler method – generalized accelerations $\ddot q$, Cartesian accelerations w and constraint reactions *R*).

The derivation of the recurrent formulas of the "sweep" method for Eq. (22) is described in [29]. Here we present the resulting algorithms of this method.

1. Algorithm for diagonalization of Eqs. (22) and (23) using nonsymmetric *LU*-elimination

for $i = N:1$

$$M_i^* = M_i + \sum_{j\in U_i} C_j^T H_j M_j^* C_j, \quad D_i = \left(A_i^T M_i^* A_i\right)^{-1}, \quad H_i = E - M_i^* A_i D_i A_i^T, \quad \phi_i = \sum_{j\in U_i} C_j^T \left[M_j^* A_j D_j p_j + H_j\left(M_j^* v_j^r + \phi_j\right)\right], \quad F_i^* = F_i + \sum_{j\in U_i} C_j^T F_j^*$$

end

for $i = 1:N$

$$\dot q_i = D_i\left[p_i - A_i^T\left(\phi_i + M_i^*\left(C_i v_{k_i} + v_i^r\right)\right)\right], \quad v_i = C_i v_{k_i} + A_i \dot q_i + v_i^r, \quad \mu_i = M_i^* v_i + \phi_i$$

$$\dot p_i = \left(\dot A_i^T - A_i^T \Omega_i\right)\mu_i + A_i^T F_i^*$$

end

The complexity of solving the system of equations (22) and (23) using this algorithm grows linearly depending on the number of bodies in the mechanical system. When implementing this algorithm, only symmetric positive-definite matrices $A_i^T M_i^* A_i$ must be inverted, the order of which is equal to the number of degrees of freedom in the *i*-th hinge, and these matrices are symmetric and positive-definite, and their order is always small (does not exceed six). This is what makes this method so effective.

2. Algorithm for solving Eqs. (22) and (23) using symmetric L^TDL-elimination

for $i = N:1$

$$F_i^* = F_i + \sum_{j\in U_i} C_j^T F_j^*, \quad M_i^* = M_i + \sum_{j\in U_i}\left(C_j^T M_j^* C_j - V_j^T V_j\right), \quad \phi_i = \sum_{j\in U_i}\left[C_j^T\left(M_j^* v_j^r + \phi_j\right) + V_j^T \eta_j\right],$$

Find the factor of the Cholesky L_i matrix $A_i^T M_i^* A_i$: $L_i^T L_i = A_i^T M_i^* A_i$

Solve $L_i^T \eta_i = p_i - A_i^T\left(M_i^* v_i^r + \phi_i\right)$

Solve $L_i^T V_i = A_i^T M_i^* C_i$

end

for $i = 1 : N$

Solve $L_i \dot{q}_i = \eta_i - V_i v_{k_i}$

$v_i = C_i v_{k_i} + A_i \dot{q}_i + v_i^r, \quad \mu_i = M_i^* v_i + \phi_i$

$\dot{p}_i = \left(\dot{A}_i^T - A_i^T \Omega_i\right) \mu_i + A_i^T F_i^*$

end

It is easy to see that in Algorithm 2, the complexity of calculations grows linearly with the number of bodies in the system, as in Algorithm 1. But instead of inverting matrices $A_i^T M_i^* A_i$, their Cholesky factors L_i, which are triangular matrices, are inverted. Computational experiments show that the time spent on solving the equations of motion with respect to generalized velocities by the $L^T DL$- algorithm is lower than by the LU- algorithm for mechanical systems in which bodies are connected by constraints with more than three degrees of freedom (no more than 1.3 – 1.4 times faster). For mechanical systems with fewer degrees of freedom in the hinges, these algorithms show almost the same numerical efficiency. Algorithms for reducing Eq. (33) to the normal form of ODEs can be written by everyone by analogy with algorithms (1) and (2).

To solve the systems of Eq. (30), Eq. (35) and Eq. (36), Eq. (37), it is proposed to use the block method of square roots (the $L^T L$ Cholesky method), since the matrices D^Σ and H^Σ are positive definite. Let us take for all these systems a single notation $\mathbf{M}u = Q$, where $\mathbf{M}$ is the matrix D^Σ or the matrix H^Σ calculated by formulas (34) and (38).

3. L^TL-algorithm for resolving Eq. (30), Eq. (35) and Eq. (36), Eq. (37)

for $i = N : 1$

$L_{ii} = \text{cholesky}[\mathbf{M}_{ii}]$

$L_{ij} = L_{ii}^{-T} \mathbf{M}_{ij}, \qquad j \in P_{k_i}$

$\mathbf{M}_{jk} = \mathbf{M}_{jk} - L_{ij}^T L_{ik}, \qquad k \in P_j$

$h_i = L_{ii}^{-T} Q_i,$

$Q_j = Q_j - L_{ij}^T h_i, \qquad j \in P_{k_i}$

end

for $i = N : 1$

$u_i = L_{ii}^{-1} \left(h_i - \sum_{j \in P_{k_i}} L_{ij} u_j \right)$

end

which have the same order as the dimensions of the tangent and orthogonal subspaces of the individual bodies of the system. Note that the application of this algorithm preserves the sparseness of the structures of the matrices D^Σ and H^Σ, that is, does not lead to their additional filling. In addition, it is known from theory that the Cholesky method is numerically stable.

In the case of a system of bodies having a "simple chain" structure, the matrix D^Σ is completely filled. Therefore, when using the Lagrange equations of the 2nd kind and algorithm 3 in the numerical simulation of such systems, the number of arithmetic operations grows in proportion to the cube of the number of bodies in the mechanical system. On the contrary, if the mechanical system has the "carrier – carried bodies" structure, then the matrix D^Σ in each block row contains only two non-zero blocks. Therefore, in this case, the number of arithmetic operations grows linearly with the number of bodies.

In the case of using the Lagrange equations of the 1st kind for modeling rigid body systems, analyzing the formula in Eq. (38) for calculating the matrix H^Σ and the formulas of the algorithm 3, one can draw opposite conclusions: for a system of bodies with a "simple chain" structure, the computational complexity of solving the system of equations will be proportional to number of bodies, and for a system of bodies with the structure "carrier – carried bodies" – is proportional to the cube of the number of bodies.

In Section 8, some formulas are given that allow one to calculate the number of arithmetic operations spent by the considered algorithms for reducing the equations of motion to the normal form of ODEs for various systems of rigid bodies connected by various types of constraints.

8. Comparison of the Efficiency of Algorithms in Modeling of Rigid Body Systems

In this section, first of all, we will consider theoretical estimates of the number of arithmetic operations of addition, multiplication, extraction of roots (flops) that need to be performed at each step of numerical integration in order to construct the equations of motion and bring them to the normal form of the ODE, for all the algorithms considered in Sections 5 and 7. Based on these estimates, we will analyze and compare the computational efficiency of algorithms for the selected types of rigid body systems (RBS).

Note that the number of flops does not provide a complete estimate of the computational efficiency of the methods, since this approach ignores the time spent of indexing, memory exchanges, and numerous other overheads that arise when a program is executed on a computer. Therefore, the theoretical estimates made in this section have been tested and confirmed by computational experiments. The calculation results are presented in this section.

As already emphasized in Section 5, there are two main approaches to the formation of the equations of motion of systems of rigid bodies – the Newton-Euler + Euler-Lagrange approach and the Hamilton approach. Both of these approaches lead to systems of linear equations identical in form and algorithms for their derivation with matrices of the system of three types: Eq. (32), Eq. (34) and Eq. (38). The main time in numerical modeling is spent precisely on the formation of these matrices and on the resolution of systems of linear equations (inversion of the system matrices) with respect to their groups of unknowns, that is, on bringing systems of differential equations to normal form. When estimating computational costs, we will not take into account the arithmetic operations that must be performed to calculate the free terms of these equations, which contain known force and

kinematic components of the mathematical model. In each specific case, the task of calculating these terms is less time-consuming compared to the selected problems of forming matrices of systems of equations and their inversion.

We will compare the computational efficiency for three variants of algorithms for reducing the equations of motion to the normal form of the ODE:

1. EMH – the construction of an extended system of equations in the form of Eq. (22) and Eq. (23) and its solution with respect to independent variables by the algorithm 2;
2. DAH – direct construction of the Hamilton equations in the form Eq. (23) and Eq. (30) and the solution of Eq. (30) with respect to the generalized velocities by the algorithm 3;
3. HZU – construction of Hamilton equations with Lagrange multipliers in the form Eq. (37) and their reduction to the normal form of ODEs by the algorithm 2.

The conclusions regarding the computational efficiency of these equations naturally extend to the corresponding equations in the Newton-Euler and Euler-Lagrange approaches.

Theoretical estimates of the number of flops will be calculated for systems of rigid bodies with a tree structure, in which each branch can contain "chains" of bodies of various lengths. Such a system in limiting cases is either a simple "chain", or a "carousel", or a "binary tree". In a "chain" system, each body has only one child body. In a "carousel" system, all bodies, starting with the second, are attached to the first body. In a "binary tree" system, each body has two child bodies. Let us introduce generalized parameters of mechanical systems, on which the "computational complexity" of algorithms depends:

$$\eta_i = \sum_{j\in P_i} n_j\,, \quad \tau_i = (6-n_i) + \sum_{j\in U_i}(6-n_j)\,, \quad \vartheta_i = n_i \sum_{j\in P_i} 1\,, \quad i = 1,2,...,N. \tag{39}$$

The parameters η_i depend on the "depth" of the chains connecting each i-th body with the 0-body, the parameters τ_i depend on the "width" of branches in the tree, that is, on the number of child bodies of each i-th body, the parameters ϑ_i set the lengths of the chains between i -th and 0-body. For example, if all hinges are the same, with n degrees of freedom, then for "limit" systems:

1. «chain» – $\eta_i = i\cdot n$, $\tau_i = 2(6-n)$, $\vartheta_i = n\cdot i$, $i = 1,2,...,N-1$, $\tau_N = (6-n)$, $\vartheta_N = n\cdot N$;
2. «carousel» – $\eta_1 = n$, $\tau_1 = (6-n)\cdot N$, $\vartheta_1 = n$, $\eta_i = 2\cdot n$, $\tau_i = (6-n)$, $\vartheta_i = 2n$, $i = 2,3,...,N$;
3. «binary tree» – $\eta_i = \left(\left[\frac{\ln i}{\ln 2}\right]+1\right)\cdot n$, $\vartheta_i = n\left(\left[\frac{\ln i}{\ln 2}\right]+1\right)$, $i = 1,2,...,N$, $\tau_i = 3(6-n)$, if $i \le 2^{\xi-1}-1$, $\tau_i = (6-n)$, if $2^{\xi-1}-1 < i \le N = 2^{\xi}-1$, $\xi = 1,2,3,...$ – floor (level) of branches.

Table 1 gives the formulas that allow calculating the number of flops required at each step of numerical integration, depending on the number of bodies in the mechanical system, its kinematic structure and the number of degrees of freedom in the hinges.

Analysis of the formulas from Table 1 allows us to draw the following conclusions.

In the DAH method, the number of flops grows in proportion to $O(n^3N^3/3)$ for mechanical systems close in structure to the "chain" system and in proportion to $O((427+613n/6+51n^2/2+7n^3/3)N)$ for mechanical systems of the "carousel" type. In contrast, in the HZU method, the number of flops varies in proportion to $O((2779-4501n/6+147n^2/2-7n^3/3)N)$ for chain systems and $O((72-36n+6n^2-n^3/3)N^3)$ – for carousel systems.

Thus, the DAH method is effective for systems with small η_i values, that is, for systems with short kinematic chains and a small number of degrees of freedom in the hinges. On the contrary, the HZU method is effective for systems with a large number of degrees of freedom in the hinges and a small number of branches in the tree structure of the system (the values of τ_i are small). Due to the fact that for a fixed number of bodies in a mechanical system, the condition for minimizing the number of branches means the presence of long chains of bodies, these two methods complement each other.

A negative feature of the DAH and HZU methods is the strong dependence of the amount of computational costs on the structure of the interconnections of bodies in the system. At best, arithmetic complexity of the methods are proportional to the number of bodies in the system; at worst, it depends on the number of bodies according to the cubic law.

The efficiency of the EMH method is completely independent of the kinematic structure of rigid body system, and the number of flops at each integration step for "chain" and "carousel" systems is proportional to $O((864+1105n/6+39n^2/2+n^3/3)N)$, that is, the number of bodies in the system. However, the proportionality coefficient is large, and therefore, with a small number of bodies in the system, the number of arithmetic operations in this method is higher than in the DAH or HZU methods for most structures of mechanical systems.

Theoretical estimates of the effectiveness of algorithms for the synthesis of equations of motion and their resolution with respect to generalized accelerations were confirmed by computational experiments on examples of modeling the dynamics of mechanical systems that differ in the number of bodies, kinematic structure, and the number of degrees of freedom in the hinges. According to Figs. 6 to 8, the graphs show the dependences of the integration time on the number of bodies in the system for two types of interconnection structures ("carousel" and "chain") and three types of hinges (single-stage rotary joints, two-stage gimbal joints, three-stage ball joints). The figures show in which situations the advantages of one or another approach are manifested.

Table 1. Computational efficiency of methods.

Method	Number of flops
EMH	$\sum_{i=1}^{N}\left(\frac{1}{3}n_i^3+\frac{39}{2}n_i^2+\frac{1105}{6}n_i+864\right)-(84n_1+660)$
DAH	$\sum_{i=1}^{N}\left(\frac{1}{3}\eta_i^3+\frac{17}{2}\eta_i^2+\frac{109}{6}\eta_i+42\vartheta_i+222\right)-\sum_{i=2}^{N}\left(\frac{1}{3}\eta_{k_i}^3+\frac{17}{2}\eta_{k_i}^2+\frac{109}{6}\eta_{k_i}-205\right)$
HZU	$\sum_{i=1}^{N}\left(\frac{1}{3}\tau_i^3+\frac{17}{2}\tau_i^2+\frac{361}{6}\tau_i+420\right)-\sum_{i=2}^{N}\left(\frac{1}{3}(6-n_i)^3+\frac{5}{2}(6-n_i)^2+\frac{1}{6}(6-n_i)\right)-48$

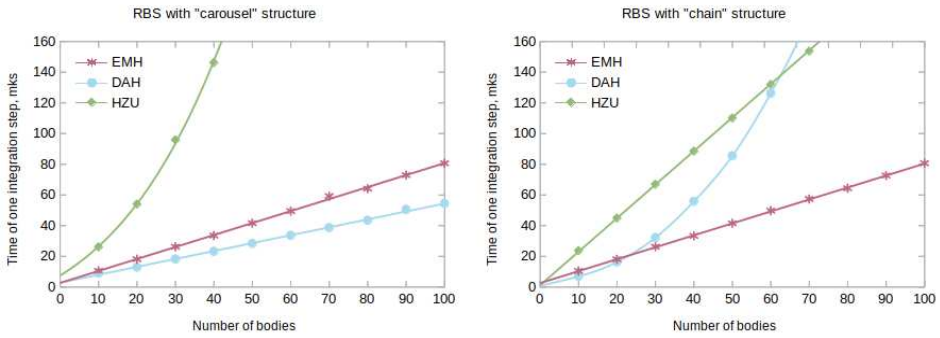

Fig. 6. Dependence of the integration time on the number of bodies for systems with rotational hinges.

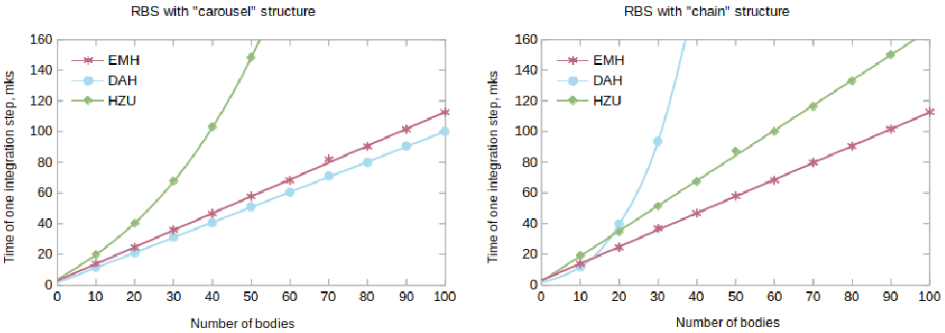

Fig. 7. Dependence of the integration time on the number of bodies for systems with gimbal hinges.

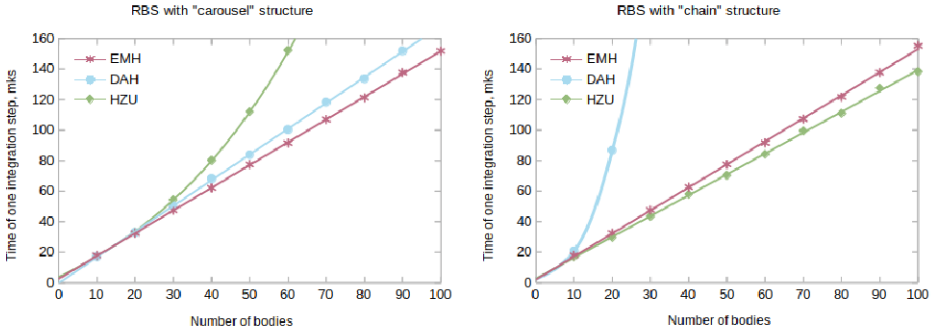

Fig. 8. Dependence of the integration time on the number of bodies for systems with ball hinges.

Figure 9 shows the diagrams for choosing the most efficient modeling method for rigid body systems of various structures. Systems containing chains of bodies of the same length fixed on the base body are considered. On each diagram, the vertical axis shows the number of chains, and the horizontal axis shows their length. The diagrams highlight the areas in the indicated variables, in which the advantage of one or another method is manifested. In addition, each diagram shows the isolines of the number of bodies in the system.

According to the diagrams in Fig. 9, it follows that for mechanical systems with a small number of degrees of freedom in hinges and with short kinematic chains, the DAH method has the least computational costs. As the chain lengths increase, the EMH method begins to show lower costs, and the more degrees of freedom in the hinges, the earlier this happens. The HZU method outperforms these methods only in the case of systems with three-degree and four-degree hinges. At the same time, its effectiveness is manifested with a simultaneous increase in the number and length of branches in a mechanical system.

Similar conclusions can be drawn for the Newton-Euler equations (33), Lagrange 1st kind equations (36) and Lagrange 2nd kind equations (35). Note that such studies have previously been repeatedly performed for partial RBS, for example, in [1, 13, 25], and led to similar conclusions. In the framework of this work, it is not necessary to present the results of these studies. Diagrams for choosing an effective modeling method

The Newton-Euler or Lagrange equations of the 1st kind are usually used in those tasks where information about accelerations and constraint reactions is required. The Lagrange equations of the 2nd kind make it possible to reduce the calculation time when constructing motion trajectories for small systems of bodies with a small number of degrees of freedom, since they contain a minimum set of unknowns. Equations in Hamiltonian variables are used in those problems where momentum information is needed, such as in optimization or control problems.

Determining the directions for further research, we note that the advantage of Hamilton's approach is manifested in the modeling of mechanical systems with closed kinematic constraints (closed loops). This is due to the fact that in Hamilton's approach one can take into account additional constraints at the level of velocities, rather than accelerations, which leads to an increase in the stability of the numerical solution without the use of special procedures for stabilizing additional constraints.

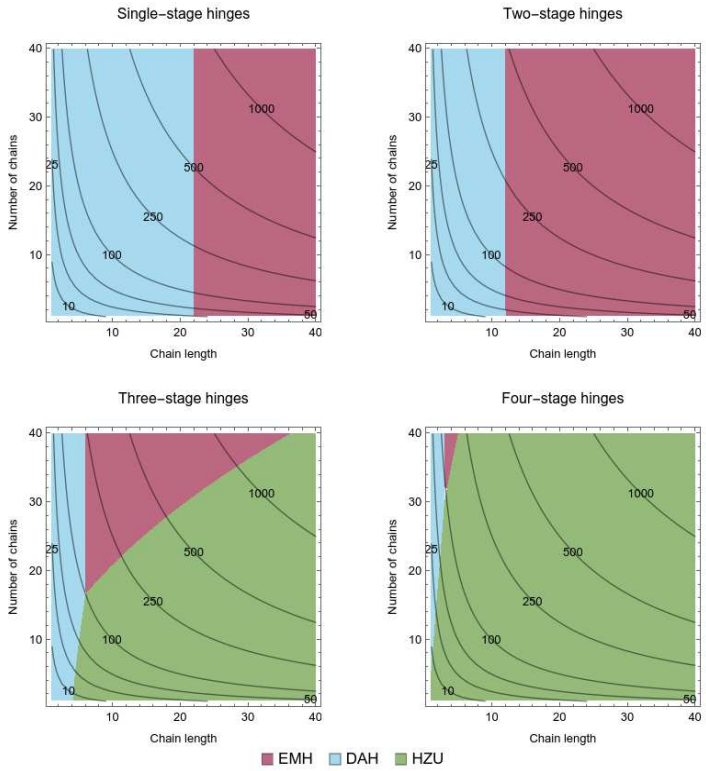

Fig. 9. Diagrams for choosing an effective modeling method.

9. Conclusion

The paper presented the equations of motion of rigid body systems with a tree structure, which were written in a compact matrix form with respect to an extended set of variables. The equations contained Poisson momenta, generalized coordinates, quasi-velocities, and Lagrange multipliers (Cartesian momenta). This form of writing the equations of motion in Hamilton's variables has not yet been encountered in the literature. The connection of the obtained equations with the classical equations of motion of mechanical systems was shown and their place in the general structure of the equations of motion of multibody systems was determined. Recursive formulas were written for computer generation of coefficients of equations from the simplest basic matrices (blocks) that describe the structure, mass-inertial, geometric and kinematic characteristics of individual links (bodies and hinges) of a mechanical system. Recurrent algorithms for solving equations with respect to higher derivatives (reducing ODEs to normal form) were presented. These algorithms were faster for mechanical systems containing long kinematic chains, since they have a linear dependence of the number of arithmetic operations on the number of bodies in a mechanical system, in contrast to multibody systems modeling algorithms based on the equations in the Lagrange form, where a similar dependence is cubic. All stages of generating initial information for computer simulation of one demonstration mechanical system ("Pipelayer") by the described method were given. Classification of the types of equations of motion was carried out in terms of their structure (Newton-Euler, Lagrange equations of the 1st and 2nd kind, Hamilton equations, Hamilton equations with Lagrange multipliers, extended form of Hamilton equations) and their common properties were highlighted. It was shown that two different approaches (Newton-Euler-Lagrange and Hamilton) lead to identical systems of linear equations with system matrices of three types. Formulas were presented that make it possible to calculate the number of arithmetic operations spent by the considered algorithms for reducing the equations of motion to the normal form of ODE for various systems of rigid bodies connected by various types of bonds. The efficiency of various methods for the formation of equations of motion in computer simulation of rigid body systems differing in the number of bodies, the structure of bonds, and the number of degrees of freedom was compared. Some diagrams were constructed to select the most effective method for modeling the systems of rigid bodies of various configurations.

Author Contributions

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