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Research Paper

# Impact of Memory-dependent Response of a Thermoelastic Thick Solid Cylinder

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**Abstract.** An internal heat source is assumed to act on a cylindrical body with radiation-like boundary conditions to explore the memory-dependent thermoelastic response of a solid object. The top and bottom surfaces of the solid cylinder are subjected to additional heating conditions. To obtain the thermal behaviour of the considered medium, the integral transform method is used, while the inversion solution of the heat transfer equation, the thermoelastic displacement and stress functions are presented in the Laplace domain due to the complexity of the calculation. To understand the numerical calculations, the material properties of aluminium metal are taken into account, and all the obtained results are presented graphically.

**Keywords:** Memory-dependent derivatives, solid circular cylinder, temperature, displacement, integral transform.

## 1. Introduction

In recent decades, fractional calculus has been applied in a variety of fields, including astronomy, chemistry, control engineering, signal processing, quantum field theory, and electromagnetics. The influential works of Oldham and Spanier [1] and Samko et al. [2] provided details on many mathematical elements of fractional calculus. Fractal differential equations have been the focus of the works of Miller and Ross [3], Podlubny [4], and Diethelm [5], as well as a recent detailed paper by Kilbas et al. [6]. There are also numerous analytical studies of thermoelastic problems in the context of fractional order theory, which are presented in the following parts. Povstenko [7-12] successfully investigated the fractional responses in various solids and studied their corresponding thermal behaviour. Lamba et al. [13-16] investigated the significant influence of fractional order theory on cylindrical objects such as circular plates and cylinders using the integral transform method.

Many scientists have begun some investigations into memory-based derivatives in the last decade. By overcoming the fractional derivative in 2011, Wang and Li [17] developed the concept of MDD (memory-dependent derivatives). Due to its ability to reflect the memory-dependent reactions in a variety of physical processes, MDD has currently emerged as a new aspect of fractional calculus which is continuously growing. To study the MDD problem for a Cartesian half-space body, El-Karamany and Ezzat [18] developed a generalized thermoelastic theory with time delay. Based on the assumption that the temperature gradually increases after heat transfer, Sun and Wang [19] rebuilt the storage-dependent heat model. Using the integral transform technique and a heat conduction model with memory-dependent derivatives, Xue et al. [20] investigated the thermoelastic properties of hollow cylinders with internal and surface cracks based on the generalized nonlocal thermoelastic theory and MDD theory. The dynamic behaviour of an infinite hollow cylinder under thermal shock was studied by Ma and Gao [21]. Marin [26] made a contribution on the special properties of thermo-elasto-dynamics for void-filled bodies. Moreover, Marin [27] established an equation for the time-evolving elasticity of micropolar bodies with cavities. Some other authors also contributed to the study of thermal response by considering cylindrical bodies under different boundary conditions [28-30].

Abouelregal et al. [31] developed a novel generalized thermoelastic heat conduction model using the idea of memory-dependent derivative with time delay. Based on the nonlocal thermoelasticity theory and the effect of thermal conductivity on the dynamics, Abouelregal et al. [32] developed the analytical solution of the deflection, thermal bending moment and temperature function for a rotating nanobeam. Abouelregal et al. [33] created a theoretical framework for the analysis of nonlocal thermoelastic model, which includes a general kernel function for memory-based derivatives. Abouelregal et al. [34] proposed the general equations for characterizing functionally graded thermo-piezoelectric materials based on the model of Lord and Shulman by incorporating memory-dependent derivatives to improve the conventional theory of coupled thermoelasticity. In the framework of the partial elastic thermal diffusion theory based on the Atangana-Baleanu operator and a nonlocal single core, Atta [35] studied a spherical cavity within a thermoelastic material.

According to the available literature, it appears that limited works have been done to consider the memory-related derivative (MDD) in heat transfer, although MDD is the extension part of fractional-order derivative, MDD is considered to be more suitable and convenient for time transformations. The duration of memory effect is represented by the time delay and the kernel function which gives the MDD-dependent weight.

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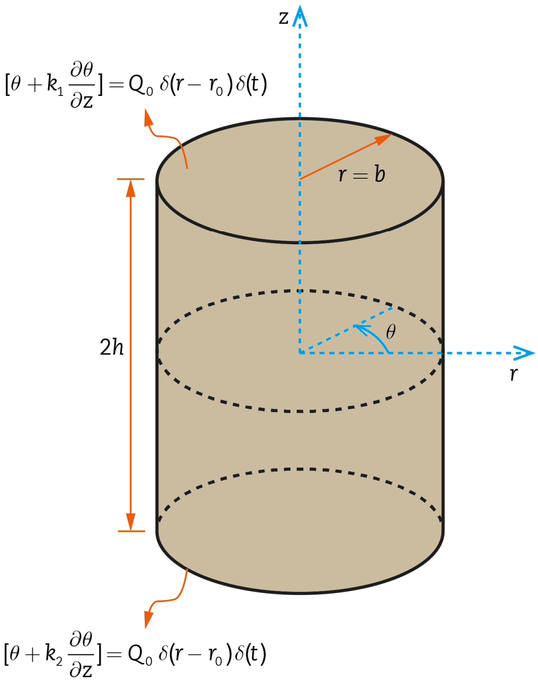


**Fig. 1.** Geometry of the solid cylinder with boundary conditions.

The present article deals with the transient thermoelastic problem of a solid circular cylinder of dimensions $0 \le r \le b$ and $-h \le z \le h$ (the geometry is shown in Fig. 1). The equation of heat transfer in the considered problem is subjected to a memory-dependent derivative. In addition, the convective heat boundaries at the top and bottom surfaces are used with additional heat sources with radiation constants. The finite Marchi-Fasulo, Laplace, and Hankel transforms are used to determine the exact expression of the temperature, displacement, and stress functions. Finally, numerical calculations are performed by considering the material properties of aluminium solid cylinder, and the graphical results are shifted with the effects of time delay parameters for better understanding of the presented findings.

To the author's knowledge, there is no research that investigates the thermal stresses due to heat generation in a solid circular cylinder with memory-dependent derivatives and this is the main contribution of the present work. The results presented here will be useful for engineering applications, especially for the development of advanced structural materials.

## 2. Mathematical Modelling of Thermoelastic Problem

### 2.1 Description of the Problem

Let's consider a solid circular cylinder whose origins radiate according to the linear relationship between temperature and the heat source. The radius and thickness of the assumed cylindrical region lie in regions $0 \le r \le b$ and $-h \le z \le h$. Each of the features is assumed to be constant, and the material of the cylinder is isotropic and homogeneous. The aim of this work is to show the influence of memory under the action of heat sources and radiation boundaries in solid objects, especially circular cylinders.

### 2.2 Generation of Heat Transfer Equation with MDD

Karamany and Ezzat [18] presented a novel energy equation with a memory-dependent derivative and time delay as:

$$q(1 + \tau D_\Omega) = -k\nabla T. \tag{1}$$

The heat conduction equation with a heat source is given by [23]:

$$\frac{\partial}{\partial t}(\rho_m c_E T) = -\nabla q + \hbar. \tag{2}$$

where $\hbar$ is the source of heat, $\rho_m$ denotes the mass density and $c_E$ is the specific heat capacity. The equation that governs the thermal heat transfer obtained by converting equation (1) into form (2) as:

$$(1 + \tau D_\Omega)\frac{\partial}{\partial t}(\rho_m c_E T) = k\nabla^2 T + (1 + \tau D_\Omega)\hbar. \tag{3}$$

Considering the use of the memory-dependent derivative concept, one has:

$$D_\Omega T(t) = \frac{1}{\Omega}\int_{t-\Omega}^{t} K(t - \psi) T'_\psi(\psi) d\psi. \tag{4}$$

Similarly, the order $n$ of $T(t)$ memory-dependent derivative takes the form:

$$D_\Omega^n T(t) = \frac{\partial^{n-1}}{\partial t^{n-1}} D_\Omega T(t) = \frac{1}{\Omega}\int_{t-\Omega}^{t} K(t - \psi)\frac{\partial^n T(\psi)}{\partial \psi^n} d\psi. \tag{5}$$

where the Kernel function $K(t-\psi)$ and time delay $\Omega$ are chosen arbitrarily in order to capture materials real behaviours. In particular, the Kernel functions can be chosen as:

$$K(t-\psi)=1-\frac{2l_2}{\Omega}(t-\psi)+\frac{l_1^2}{\Omega^2}(t-\psi)^2. \tag{6}$$

where $l_1$ and $l_2$ are constants.

Kernel function $K(t-\psi)$ in general lies in the range 0 to 1 for $\psi\in[t-\Omega,t]$, so that:

$$|D_\Omega T(r,z,t)|\le\left|\frac{\partial T(r,z,t)}{\partial t}\right|.$$

For the sake of simplicity, the number of variables is listed in dimensionless form as:

$$r'=\frac{r(t',\tau',\Omega')}{r_0}=\frac{1}{\rho_m c_E r_0^2}(t,\tau,\Omega),\ z'=\frac{z(t',\tau',\Omega')}{z_0}=\frac{1}{\rho_m c_E r_0^2}(t,\tau,\Omega)\ ,\ \theta'=\frac{\theta}{\theta_0}.$$

Equation (3) has the following form (by dropping the prime symbols) using the non-dimensional variables mentioned above:

$$k\nabla^2\theta+(1+\tau D_\Omega)\hbar=(1+\tau D_\Omega)\frac{\partial\theta}{\partial t}. \tag{7}$$

For convenience, the heat source is assumed to be described as below:

$$\hbar=\frac{1}{2\pi r_0}\,\delta(r-r_0)\,\delta(z-z_0)\,e^{-\omega t}.$$

where $0\le r_0\le b$, $-h\le z_0\le h$, $\omega>0$, $k=\lambda/\rho C$, $k$ and $\lambda$ are the thermal diffusivity and conductivity of the cylinder's material, respectively. Also, constants $\rho$ and $C$ are density and calorific capacity, respectively.

### 2.3 Boundaries with Radiations

The various boundary constraints of the problem for solid cylinders under memory-related derivatives are outlined below:

$$\theta(t=0)=0. \tag{8}$$

$$\theta(r=b)=0. \tag{9}$$

$$\left[\theta(z=h)+k_1\frac{\partial\theta(z=h)}{\partial z}\right]=Q_0\delta(r-r_0)\delta(t). \tag{10}$$

$$\left[\theta(z=-h)+k_2\frac{\partial\theta(z=-h)}{\partial z}\right]=Q_0\delta(r-r_0)\delta(t). \tag{11}$$

where $\delta$ stands for the Dirac Delta function, $Q_0\delta(t)\delta(r-r_0)$ stands for the additional sectional heat that is present on the solid cylinder's top and lower surfaces, and $k_1, k_2$ denote the radiation constants.

### 2.4 Basic Two-dimension Thermoelastic Equation

For the axisymmetric two-dimensional thermoelastic problem, without the body forces, the Navier's equations are stated as [24]:

$$\nabla^2 S_r-\frac{S_r}{r^2}+\frac{1}{1-2\upsilon}\frac{\partial e}{\partial r}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial\theta}{\partial r}=0, \tag{12}$$

$$\nabla^2 S_z-\frac{1}{1-2\upsilon}\frac{\partial e}{\partial z}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial\theta}{\partial z}=0. \tag{13}$$

where $S_r$ and $S_z$ are the elements of displacement in the axial and radial directions, respectively, and the dilation $e$ is given by:

$$e=\frac{\partial S_r}{\partial r}+\frac{S_r}{r}+\frac{\partial S_r}{\partial z}.$$

The expressions of displacements in terms of Goodier's displacement potential $U(r,z,t)$ and Michell's function $M$ are written as:

$$S_r=\frac{\partial U}{\partial r}-\frac{\partial^2 M}{\partial r\partial z}, \tag{14}$$

$$S_z=\frac{\partial U}{\partial z}+2(1-\upsilon)\nabla^2 M-\frac{\partial^2 M}{\partial z^2}. \tag{15}$$

where the equation for Goodier's thermoelastic potential must hold:

$$\nabla^2 U = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \theta. \tag{16}$$

Hence, the equation must be satisfied by Michell's function $M$ as:

$$\nabla^2(\nabla^2 M) = 0. \tag{17}$$

The potential $U(r,z,t)$ and Michell's function $M$ are used to depict the stress component as:

$$\Theta_{rr} = 2G\left\{\left(\frac{\partial^2 U}{\partial r^2} - \nabla^2 U\right) + \frac{\partial}{\partial z}\left(\nu\,\nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right\}, \tag{18}$$

$$\Theta_{\theta\theta} = 2G\left\{\left(\frac{1}{r}\frac{\partial U}{\partial r} - \nabla^2 U\right) + \frac{\partial}{\partial z}\left(\nu\,\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right\}, \tag{19}$$

$$\Theta_{zz} = 2G\left\{\left(\frac{\partial^2 U}{\partial z^2} - \nabla^2 U\right) + \frac{\partial}{\partial z}\left((2-\nu)\,\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right\}, \tag{20}$$

and

$$\Theta_{rz} = 2G\left\{\frac{\partial^2 U}{\partial r\,\partial z} + \frac{\partial}{\partial r}\left((1-\nu)\,\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right\}. \tag{21}$$

where $G$ and $\nu$ represent the shear modulus and the Poisson's ratio, respectively. The traction-free surfaces of a solid cylinder's boundary conditions are:

$$\Theta_{rr} = \Theta_{rz} = 0 \ \text{ at } \ r = b. \tag{22}$$

The mathematical formulation of the problem under investigation is outlined within section 2.

## 3. Solution in Laplace Domain

### 3.1 Solution of Heat Transfer Equation with MDD

To obtain the expression of the temperature distribution, the first technique named the finite Hankel integral transformation is applied to equations (7), (8), (10) and (11), and using equation (9), we obtain:

$$k\left[-\xi_n^2\theta^*(\xi_n,z,t) + \frac{\partial^2\theta^*(\xi_n,z,t)}{\partial z^2}\right] + (1+\tau D_\Omega)\hbar^* = (1+\tau D_\Omega)\frac{\partial\theta^*}{\partial t} \tag{23a}$$

$$\hbar^* = \frac{1}{2\pi}\delta(z-z_0)\,e^{-\omega t} f_0(\xi_n, r_0) \tag{23b}$$

where the finite Hankel transform's nucleus is defined by $f_0(\xi_n, r)$ and the symbol (*) designates a function in the transformed domain. The expression for the function $f_0(\xi_n, r)$ is given by:

$$f_0(\xi_n, r) = \frac{-\sqrt{2}}{b}\left(\frac{J_0(\xi_n r)}{\xi_n\, J_0(\xi_n b)}\right)$$

here, the eigenvalues $\xi_n$ are the positive roots of characteristic equation $J_0(\xi_n b) = 0$, and $J_n(x)$ is $n$th-order of Bessel's function of the first kind. The transformed initial and boundary conditions are:

$$\theta^*(\xi_n, z, t=0) = 0. \tag{24}$$

$$\left[\theta^*(\xi_n, z=h, t) + k_1\frac{\partial\theta^*(\xi_n, z=h, t)}{\partial z}\right] = Q_0\delta(t) r_0\, f_0(\xi_n, r_0). \tag{25}$$

$$\left[\theta^*(\xi_n, z=-h, t) + k_2\frac{\partial\theta^*(\xi_n, z=-h, t)}{\partial z}\right] = Q_0\delta(t) r_0\, f_0(\xi_n, r_0). \tag{26}$$

Equations (23) and (24) are further subjected to the finite Marchi-Fasulo transform [22], and by utilizing equations (25) and (26), one gets:

$$\left[-k(\xi_n^2+\mu_m^2)\bar{\theta}^*(\xi_n,m,t) + k\left[\frac{p_m(h)}{k_1} - \frac{p_m(-h)}{k_2}\right]Q_0\,\delta(t)\,r_0 f_0(\xi_n,r_0)\right] + \frac{1}{2\pi}(1+\tau D_\Omega)\,p_m(z_0)\,e^{-\omega t} f_0(\xi_n, r_0) = (1+\tau D_\Omega)\frac{\partial\bar{\theta}^*}{\partial t} \tag{27}$$

where $m$ denotes the Marchi-Fasulo transform parameter and $\bar{\theta}^*$ is the converted function of $\bar{\theta}$. The orthogonal functions within the interval $-h \le z \le h$ provide the nucleus as:

$$p_m(z) = X_m \cos(\mu_m z) - Y_m \sin(\mu_m z)$$

where

$$X_m = \mu_m(k_1 + k_2)\cos(\mu_m h)$$

$$Y_m = 2\cos(\mu_m h) + (k_2 - k_1)\,\mu_m \sin(\mu_m h)$$

$$\lambda_m^2 = \int_{-h}^{h} p_m^2(z)\,dz = h[X_m^2 + Y_m^2] + \frac{\sin(2\mu_m h)}{2\mu_m}[X_m{}^2 - Y_m^2]$$

The eigenvalues $\mu_m$ are the positive roots of the characteristic equation as:

$$[k_1\mu\cos(\mu h) + \sin(\mu h)][\cos(\mu h) + k_2\mu\sin(\mu h)] = [k_2\mu\cos(\mu h) - \sin(\mu h)][\cos(\mu h) - k_1\mu\sin(\mu h)]$$

The transformed initial condition becomes:

$$\bar{\theta}^{*}(\xi_n, m, t = 0) = 0. \tag{28}$$

Further applying Laplace transform rule on equation (27) and utilizing the transformed initial condition (28), yields:

$$\hat{\bar{\theta}}^{*} = \frac{H(\xi_n, \mu_m)}{L(s)} + \frac{p_m(z_0)}{2\pi}\frac{(L(s) - k\,\beta_{m,n})}{s(s+\omega)L(s)} f_0(\xi_n, r_0), \tag{29a}$$

$$H(\xi_n, \mu_m) = k\left[\frac{p_m(h)}{k_1} - \frac{p_m(-h)}{k_2}\right] Q_0\, r_0\, f_0(\xi_n, r_0), \tag{29b}$$

$$\left((1+G)s + k\,\beta_{m,n}\right) = L(s), \tag{29c}$$

$$\beta_{m,n} = \mu_m^2 + \xi_n^2, \tag{29d}$$

$$G = \frac{\tau}{\Omega}\left\{\left(1 - e^{-s\Omega}\right)\left(1 - \frac{2l_2}{\Omega s} + \frac{2l_1^2}{\Omega^2 s^2}\right) - \left(l_1^2 - 2l_2 + \frac{2l_1^2}{\Omega s}\right)e^{-s\Omega}\right\}. \tag{29e}$$

The resulting temperature distribution is derived in the Laplace transform domain by inverting the finite Marchi-Fasulo transform and the finite Hankel transform as:

$$\hat{\theta} = \frac{2}{b^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{P_m(z)}{\lambda_m}\left\{\frac{H(\xi_n, \mu_m)}{L(s)} + \frac{p_m(z_0)}{2\pi}\frac{(L(s) - k\,\beta_{m,n})}{s(s+\omega)L(s)} f_0(\xi_n, r_0)\right\}\frac{J_0(r\,\xi_n)}{\left[J_0'(b\,\xi_n)\right]^2}. \tag{30}$$

### 3.2 Evaluation of Thermal Displacement Components

The Goodier's thermoelastic potential in the domain of the Laplace transform is obtained by utilizing equation (30) in (16) as:

$$\hat{U} = -\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{2}{b^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{P_m(z)}{\beta_{m,n}\,\lambda_m}\left\{\frac{H(\xi_n, \mu_m)}{L(s)} + \frac{p_m(z_0)}{2\pi}\frac{(L(s) - k\,\beta_{m,n})}{s(s+\omega)L(s)} f_0(\xi_n, r_0)\right\}\frac{J_0(r\,\xi_n)}{\left[J_0'(b\,\xi_n)\right]^2}. \tag{31}$$

Similarly, it is believed that the Michell's function solutions in the Laplace transform domain meet the governed condition of equation (17):

$$\hat{M} = -\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{2}{b^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{P_m(z)}{\beta_{m,n}\,\lambda_m}\left\{\frac{H(\xi_n, \mu_m)}{L(s)} + \frac{p_m(z_0)}{2\pi}\frac{(L(s) - k\,\beta_{m,n})}{s(s+\omega)L(s)} f_0(\xi_n, r_0)\right\}\times[X_n J_0(\xi_n r) + Y_n(\xi_n r)\,J_1(\xi_n r)][\cos h(\xi_n z)]\frac{J_0(r\,\xi_n)}{\left[J_0'(b\,\xi_n)\right]^2}. \tag{32}$$

Using equations (31) and (32) in (14) and (15), one obtains the displacement components as below:

$$\hat{S}_r = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{2}{b^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{P_m(z)}{\beta_{m,n}\,\lambda_m}\left\{\frac{H(\xi_n, \mu_m)}{L(s)} + \frac{p_m(z_0)}{2\pi}\frac{(L(s) - k\,\beta_{m,n})}{s(s+\omega)L(s)} f_0(\xi_n, r_0)\right\}\times\left[[\xi_n \sin h(\xi_n z)][X_n(-\xi_n)J_1(\xi_n r) + Y_n(\xi_n r)J_0(\xi_n r)] - \frac{\sqrt{2}}{b}\frac{J_1(\xi_n r)}{J_1(\xi_n b)}p_m(z)\right]\frac{\xi_n\,J_0'(r\,\xi_n)}{\left[J_0'(b\,\xi_n)\right]^2}. \tag{33}$$

$$\begin{aligned}\hat{S}_z = {} & \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{2}{b^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{P_m(z)}{\beta_{m,n}\,\lambda_m}\left\{\frac{H(\xi_n, \mu_m)}{L(s)} + \frac{p_m(z_0)}{2\pi}\frac{(L(s) - k\,\beta_{m,n})}{s(s+\omega)L(s)} f_0(\xi_n, r_0)\right\}\times[\mu_m[X_m \sin\mu_m z + Y_m\cos\mu_m z]\,f_0(\xi_n, r) + [X_n\xi_n^2 J_0(\xi_n r)][\cos h\,\xi_n z] \\ & - Y_n\xi_n^2[4(1-\upsilon)J_0(\xi_n r) - (\xi_n r)\,J_1(\xi_n r)][\cos h\xi_n z]\frac{J_0(r\,\xi_n)}{\left[J_0'(b\,\xi_n)\right]^2}.\end{aligned} \tag{34}$$

### 3.3 Evaluation of Thermal Stress Components

The values of the Michell's function from equation (32) and the thermoelastic displacement potential from equation (31) are inserted into equations (18) to (21) in order to evaluate the stress components as follows:

$$\hat{\Theta}_{rr} = -2G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{2}{b^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{P_m(z)}{\beta_{m,n}\lambda_m}\left\{\frac{H(\xi_n,\mu_m)}{L(s)}+\frac{p_m(z_0)}{2\pi}\frac{(L(s)-k\beta_{m,n})}{s(s+\omega)L(s)}f_0(\xi_n,r_0)\right\}\times[(\mu_m^2+2\xi_n^2)J_0(\xi_n r)-\xi_n^2 J_1(\xi_n r)]\left[\frac{\sqrt{2}}{b\,\xi_n J_1(\xi_n b)}\right]p_m(z)$$
$$-X_n\xi_n^2\left[\frac{J_1(\xi_n r)}{(\xi_n r)}-J_0(\xi_n r)\right][\xi_n \sin h(\xi_n z)]+Y_n\xi_n^2[(2\upsilon-1)J_0(\xi_n r)+(\xi_n r)J_1(\xi_n r)][\xi_n \sin h(\xi_n z)]\frac{\xi_n^2 J_0''(r\xi_n)}{[J_0'(b\xi_n)]^2}. \quad (35)$$

$$\hat{\Theta}_{\theta\theta} = -2G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{2}{b^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{P_m(z)}{\beta_{m,n}\lambda_m}\left\{\frac{H(\xi_n,\mu_m)}{L(s)}+\frac{p_m(z_0)}{2\pi}\frac{(L(s)-k\beta_{m,n})}{s(s+\omega)L(s)}f_0(\xi_n,r_0)\right\}\times\left[\beta_{m,n}J_0(\xi_n r)+\xi_n\frac{J_1(\xi_n r)}{r}\right]\left[\frac{\sqrt{2}}{b\,\xi_n J_1(\xi_n b)}\right]p_m(z)$$
$$+\left[X_n\xi_n^2\frac{J_1(\xi_n r)}{r}[\sin h(\xi_n z)]+Y_n\xi_n^3[(2\upsilon-1)J_0(\xi_n r)][\sin h(\xi_n z)]\right]\frac{\xi_n J_0'(r\xi_n)}{[J_0'(b\xi_n)]^2}. \quad (36)$$

$$\hat{\Theta}_{zz} = -2G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{2}{b^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{P_m(z)}{\beta_{m,n}\lambda_m}\left\{\frac{H(\xi_n,\mu_m)}{L(s)}+\frac{p_m(z_0)}{2\pi}\frac{(L(s)-k\beta_{m,n})}{s(s+\omega)L(s)}f_0(\xi_n,r_0)\right\}\times[-2\mu_m^2-\xi_n^2]\,p_m(z)\,f_0(\xi_n,r)-[X_n^2\xi_n^3 J_0(\xi_n r)[\sin h\,\xi_n z]$$
$$-Y_n\,\xi_n^3[2(\upsilon-2)J_0(\xi_n r)-(\xi_n r)J_1(\xi_n r)][\sin h\,\xi_n z]\frac{J_0(r\xi_n)}{[J_0'(b\xi_n)]^2}. \quad (37)$$

$$\hat{\Theta}_{rz} = -2G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{2}{b^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{P_m(z)}{\beta_{m,n}\lambda_m}\left\{\frac{H(\xi_n,\mu_m)}{L(s)}+\frac{p_m(z_0)}{2\pi}\frac{(L(s)-k\beta_{m,n})}{s(s+\omega)L(s)}f_0(\xi_n,r_0)\right\}\times[-\mu_m^2][X_m\sin(\mu_m z)+Y_m\cos(\mu_m z)]$$
$$\times\left[\frac{\sqrt{2}\,J_1(\xi_n r)}{b\,J_1(\xi_n b)}\right]+[X_n\xi_n^3 J_1(\xi_n r)][\cos h(\xi_n z)]-Y_n[\xi_n^3 J_0(\xi_n r)+2(1-\upsilon)\,\xi_n^2 J_1(\xi_n r)][\cos h(\xi_n z)]\frac{\xi_n J_0'(r\xi_n)}{[J_0'(b\xi_n)]^2}. \quad (38)$$

### 3.4 Finding the Arbitrary Functions $X_n$ and $Y_n$

The unknown function $X_n$ and $Y_n$ are evaluated using the boundary conditions in equation (22) of a solid cylinder with traction-free surfaces as follows:

$$X_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Upsilon_4\Upsilon_1\sin h(\xi_n z)-\Upsilon_3\Upsilon_2\,b\cos h(\xi_n z)\,p_m(z)}{b\xi_n^3[\Upsilon_1-\Upsilon_2\Upsilon_5]\cos h(\xi_n z)\sin h(\xi_n z)}, \quad (39)$$

$$Y_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{b\Upsilon_3 J_1(\xi_n b)p_m(z)\cos h(\xi_n z)-\Upsilon_4\Upsilon_5\sin h(\xi_n z)}{b\xi_n^2[\Upsilon_1-\Upsilon_2\Upsilon_5]\cos h(\xi_n z)\sin h(\xi_n z)}. \quad (40)$$

where

$$\Upsilon_1 = B_0 + C_0 = [(2\upsilon-1)J_0(\xi_n b)+(\xi_n b)\,J_1(\xi_n b)]\xi_n J_1(\xi_n b),$$

$$\Upsilon_2 = \hat{B}+\hat{C} = \xi_n J_0(\xi_n b)+2(1-\upsilon)\,J_1(\xi_n b),$$

$$\Upsilon_3 = D_0\times E_0 = (\mu_m^2+2\xi_n^2-\xi_n)\times\left[\frac{J_0(\xi_n b)}{\xi_n J_1(\xi_n b)}-\xi_n\right]\frac{\sqrt{2}}{b},$$

$$\Upsilon_4 = \hat{D} = \frac{2\mu_m^2}{b}[X_m\sin(\mu_m z)+Y_m\cos(\mu_m z)],$$

$$\Upsilon_5 = A_0 = \left[\frac{J_1(\xi_n b)}{\xi_n b}-J_6(\xi_n b)\right].$$

## 4. Numerical Results and Discussion

Due to its low density, aluminium is often used in the aerospace industry and other areas of transportation. We consider the material properties of the aluminium metal to understand the numerical calculations as defined in [25]:

Modulus of Elasticity, $E = 6.9\times10^{11}\,(dynes/cm^2)$
Shear modulus, $G = 2.7\times10^{11}\,(dynes/cm^2)$
Poisson ratio, $\upsilon = 0.281$
Thermal expansion coefficient, $\alpha_t = 25.5\times10^{-6}\,(cm/cm-{}^0C)$
Thermal diffusivity, $k = 0.86\,(cm^2/\sec)$
Thermal conductivity, $\lambda = 0.48\,(cal-cm/{}^0C/\sec/cm^2)$
Outer radius, $b = 1\,cm$
Thickness, $h = 0.5\,cm$

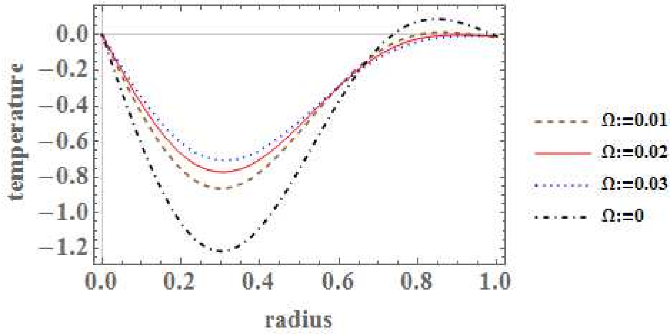


**Fig. 2.** Time delay's influence over dimensionless temperature in a dimensionless radial direction.

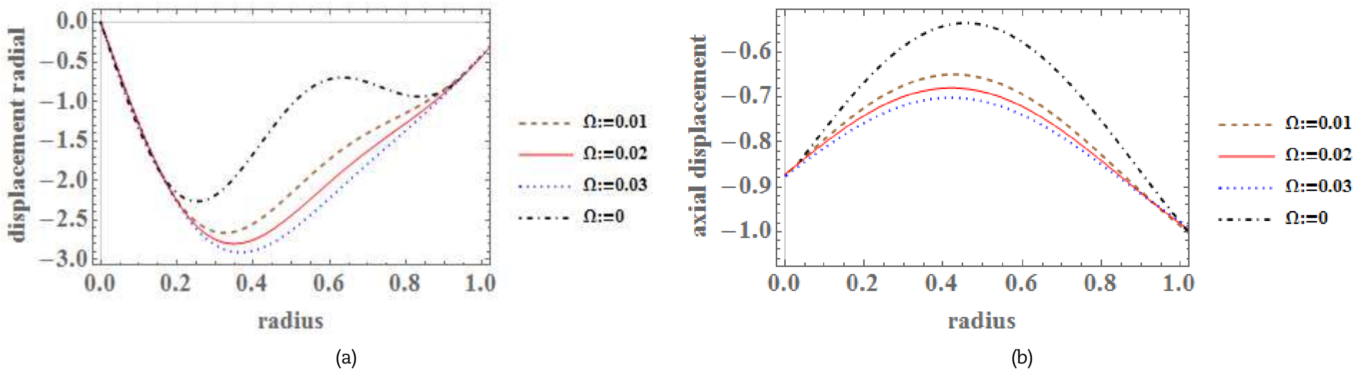


**Fig. 3.** (a) Time delay's influence over dimensionless radial displacement in a dimensionless radial direction and (b) Time delay's influence over dimensionless axial displacement in a dimensionless radial direction.

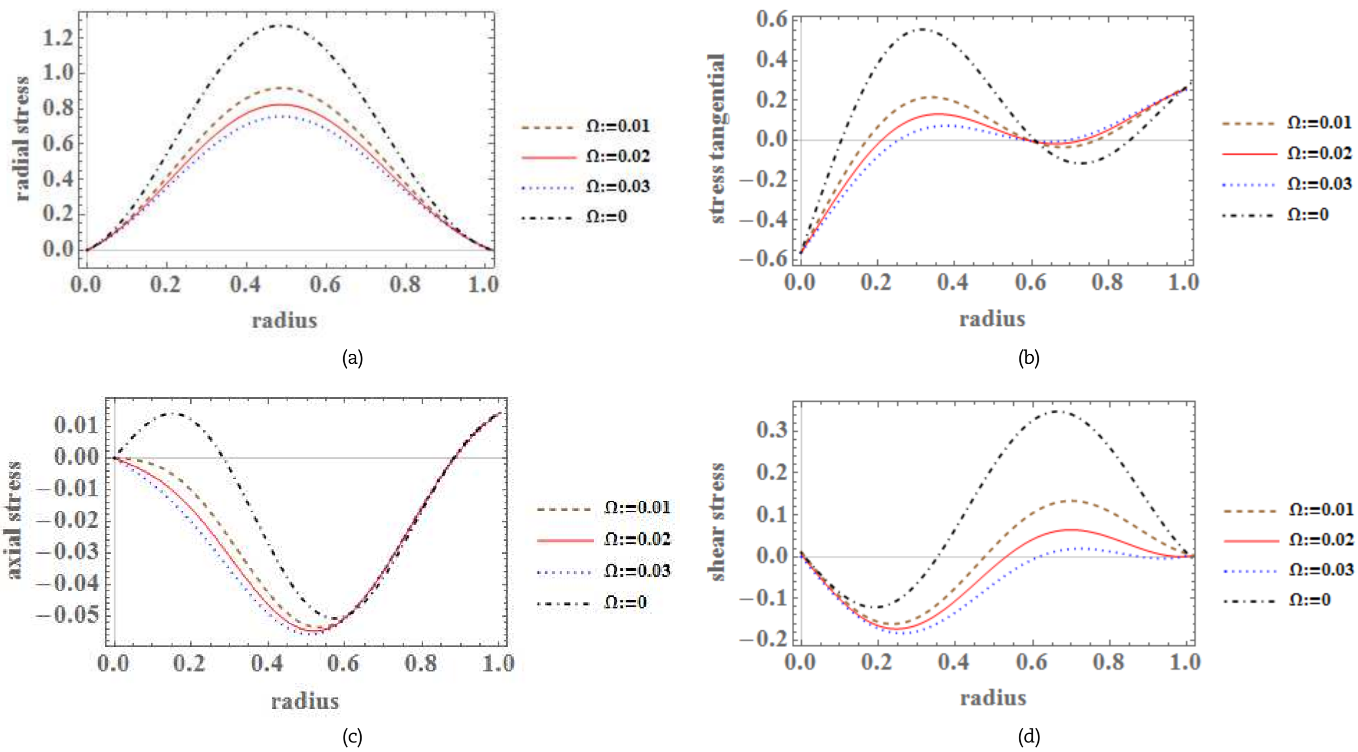


**Fig. 4.** (a) Time delay's influence over dimensionless radial stress in a dimensionless radial direction, (b) Time delay's influence over dimensionless tangential stress in a dimensionless radial direction, (c) Time delay's influence over dimensionless axial stress in a dimensionless radial direction and (d) Time delay's influence over dimensionless shear stress in a dimensionless radial direction.

In this section, we examine how the time delay affects the temperature, displacement, and thermal stress functions. These distributions are shown graphically in Figs. 2 to 4. It should be noted that the present heat transfer model reverts to the CV model when the time delay approaches infinitesimal when the kernel function is considered to be 1.

According to Fig. 2, the dimensionless temperature distribution function is significantly affected by the time delay parameters $\Omega = 0, 0.1, 0.2, 0.3$ in the dimensionless radial direction for $t = 0.5$ and $z = 0.5$. For longer time delays, a more uniform temperature distribution is seen, or one could say that the temperature decreases as the value of parameter $\Omega$ increases. In the radial direction, the temperature first decreases to $r = 0.3$ and then continues to increase outward, which is due to the action of the heat source and the additional cross-sectional heating. Another temperature wave reaches zero value at the outer radii $r = 1$ which corresponds to the condition defined in mathematical equation (9).

Figures 3(a) and 3(b) show the effect of time delay on the dimensionless radial and axial displacements in a dimensionless radial direction for different delay parameters by fixing $t = 0.5$ and $z = 0.5$, respectively. For large values of the delay parameter, a higher distribution of curve variations is found. The finite propagation velocity of the displacement waves is observed when recording a solid cylinder. In the center of the circular cylinder there is a peak in the displacement function which is due to the influence of the heat source and the additional cross-sectional heating.

The effect of time delay on the dimensionless radial stress is shown in Fig. 4(a) for various parameters $\Omega = 0, 0.1, 0.2, 0.3$. The above variation is recorded by fixing $t = 0.5$ and $z = 0.5$. The stress distribution first increases, peaks at the center of the cylinder, and then decreases outward in the radial direction. On the outer surface, where the stress is zero, it satisfies the mathematical condition given in equation (22). The larger stress distribution in the center shows the influence of the applied heat source and the additional cross-sectional heating.

Figures 4(b), 4(c), and 4(d) show the significant effect of time delay on the dimensionless tangential, axial, and shear stresses, respectively, along a dimensionless, radially outward direction by fixing $t = 0.5$ and $z = 0.5$. For a large value of the time delay parameter, a small distribution of thermal stress waves is observed for a solid circular cylinder. All curves show compression near the center of the cylinder and tensile stress near the inner and outer radii.

## 5. Conclusion

Modelling of memory-related heat transfer equations subjected to a heat source in the case of a solid circular cylinder was successfully determined in the present study, and the influence of time delay variables on the temperature distributions, displacements, and stress functions was established. The bottom and top surfaces of the solid cylinder were additionally heated section by section. The temperature, displacement and stress functions were expressed in the Laplace transform domain. Numerical calculations were performed for a special case involving a solid cylinder made of aluminium metal, and the results were presented graphically. The following important points are summarized below:

- The graphical analysis showed that the changes in the temperature, displacement and stress functions in the assumed thermoelastic problem are within a limited range and are not observed outside this range.
- The different values of the time delay parameter had a significant effect on the variations of the temperature, displacement and stress functions.
- Thermal tracking of various curves under memory response resulted in a significant response of the heat source and cross-section heating.
- The phenomenon of total variation showed that the waves move at a finite speed.
- Classification of materials based on time delay features capable of reflecting the effects of memory on temperature and stress history may be useful for various structural designs.

The present study may also be useful to the researchers and mathematicians working on the development of thermoelasticity by considering memory-related derivatives which are important in describing the behaviour of many physical processes. Engineering applications may use the obtained thermal variations for different parameters with memory-related responses to develop the realistic structures or machines.

## Author Contributions

The author discussed the results, reviewed, and approved the final version of the manuscript.

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## Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

## References

[1] Oldham, K.B., Spanier, J., *The Fractional Calculus*, Academic Press, New York, 1974.
[2] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Amsterdam, 1993.
[3] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley–Blackwell, New York, 1993.
[4] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[5] Diethelm, K., *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[6] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[7] Povstenko, Y.Z., Fractional heat conduction equation and associated thermal stresses, *Journal of Thermal Stresses*, 28(1), 2005, 83–102.
[8] Povstenko, Y.Z., Two-dimensional axisymmetric stresses exerted by instantaneous pulses and sources of diffusion in an infinite space in a case of time-fractional diffusion equation, *International Journal of Solids and Structures*, 44(7–8), 2007, 2324–2348.
[9] Povstenko, Y.Z., Thermoelasticity which uses fractional heat conduction equation, *Mathematical Methods and Physicomechanical Fields,* 51(2), 2008, 239–246.
[10] Povstenko, Y.Z., Fractional radial diffusion in a cylinder, *Journal of Molecular Liquids,* 137(1–3), 2008, 46–50.
[11] Povstenko, Y.Z., Thermoelasticity which uses fractional heat conduction equation, *Journal of Mathematical Sciences*, 162, 2009, 296–305.
[12] Povstenko, Y.Z., Theory of thermoelasticity based on the space-time-fractional heat conduction equation, *Physica Scripta*, 136, 2009, 014017-1-6.
[13] Lamba, N.K., Thermosensitive response of a functionally graded cylinder with fractional order derivative, *International Journal of Applied Mechanics and Engineering*, 27(1), 2022, 107-124.
[14] Lamba, N.K., Deshmukh, K.C., Hygrothermoelastic response of a finite solid circular cylinder, *Multidiscipline Modeling in Materials and Structures*, 16(1), 2020, 37-52.
[15] Lamba, N.K., Deshmukh, K.C., Hygrothermoelastic response of a finite hollow circular cylinder, *Waves in Random and Complex Media*, 2022, doi: 10.1080/17455030.2022.2030501.
[16] Thakare, S., Warbhe, M.S., Lamba, N.K., Time fractional heat transfer analysis in nonhomogeneous thick hollow cylinder with internal heat generation and its thermal stresses, *International Journal of Thermodynamics*, 23(4), 2020, 281-302.
[17] Wang J.L., Li, H.F., Surpassing the fractional derivative: Concept of the memory-dependent derivative, *Computers and Mathematics with Applications*, 62, 2011, 1562–1567.
[18] Ahmed, S.K., Ezzat, M.A., Modified Fourier's Law with Time-Delay and Kernel Function: Application in Thermoelasticity, *Journal of Thermal Stresses*, 38(7), 2015, 811-834.
[19] Sun, W.W., Wang, J.L., Reconstruct the Heat Conduction Model with Memory Dependent Derivative, *Applied Mathematics*, 9, 2018, 1072-1080.
[20] Xue, Z.N., Chen, Z.T., Tian, X.G., Transient thermal stress analysis for a circumferentially cracked hollow cylinder based on memory-dependent heat conduction model, *Theoretical and Applied Fracture Mechanics,* 96, 2018, 123–133.
[21] Ma, Y., Gao, Y., Dynamic response of a hollow cylinder subjected to thermal shock considering scale effect and memory dependent effect, *Mechanics of Advanced Materials and Structures*, 29(25), 2022, 4468-4477.
[22] Marchi, E., Fasulo, A., Heat conduction in sector of hollow cylinder with radiation, *Atti della Reale Accademia delle scienze di Torino*, 1, 1967, 373-382.
[23] Ozisik, M.N., *Boundary Value Problems of Heat Conductions*, International text book Company, Scranton, Pennsylvania, 1986.
[24] Noda, N., Hetnarski, R.B., Tanigawa, Y., *Thermal stresses*, Second ed., Taylor and Francis, New York, 2003.
[25] Kumar, R., Lamba N.K., Varghese, V., Analysis of thermoelastic disc with radiation conditions on the curved surfaces, *Materials Physics and Mechanics*, 16, 2013, 175-186.
[26] Marin M., Contributions on uniqueness in thermoelastodynamics on bodies with voids, *Ciencias Matematicas (Havana)*, 16(2), 1998, 101-109.
[27] Marin M., A temporally evolutionary equation in elasticity of micropolar bodies with voids, *UPB Scientific Bulletin, Series A: Applied Mathematics Physics,* 60(3-4), 1998, 3-12.
[28] Kamdi, D., Lamba, N.K., Thermoelastic Analysis of Functionally Graded Hollow Cylinder Subjected to Uniform Temperature Field, *Journal of Applied and Computational Mechanics*, 2(2), 2016, 118-127.
[29] Manthena, V.R., Lamba, N.K., Kedar, G.D., Springbackward Phenomenon of a Transversely Isotropic Functionally Graded Composite Cylindrical Shell, *Journal of Applied and Computational Mechanics*, 2(3), 2016, 134-143.
[30] Varghese, V., An Analysis of Thermal-Bending Stresses in a Simply Supported Thin Elliptical Plate, *Journal of Applied and Computational Mechanics*, 4(4), 2018, 299-309.
[31] Abouelregal, A.E., Moustapha, M.V., Nofal, T.A., Rashid, S., Ahmad, H., Generalized thermoelasticity based on higher-order memory-dependent derivative with time delay, *Results in Physics*, 20, 2021, 103705.
[32] Abouelregal, A.E., Sedighi, H.M., Faghidian, S.A., Shirazi, A.H., Temperature-dependent physical characteristics of the rotating nonlocal nanobeams subject to a varying heat source and a dynamic load, *Facta Universitatis, Series: Mechanical Engineering*, 19(4), 2021, 633-656.
[33] Abouelregal, A.E., Atta, D., Sedighi, H.M., Vibrational behavior of thermoelastic rotating nanobeams with variable thermal properties based on memory-dependent derivative of heat conduction model, *Archive of Applied Mechanics*, 93, 2023, 197–220.
[34] Abouelregal, A.E., Askar, S.S., Marin, M., Badahiould M., The theory of thermoelasticity with a memory-dependent dynamic response for a thermo-piezoelectric functionally graded rotating rod, *Scientific Reports*, 13, 2023, 9052.
[35] Atta, D., Thermal Diffusion Responses in an Infinite Medium with a Spherical Cavity using the Atangana-Baleanu Fractional Operator, *Journal of Applied and Computational Mechanics*, 8(4), 2022, 1358-1369.

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