J. Appl. Comput. Mech., 10(1) (2024) 64-79
DOI: 10.22055/jacm.2023.44326.4196

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran
University of Ahvaz

Journal of
Applied and Computational Mechanics

Research Paper

A New Class of Linear Canonical Wavelet Transform

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Received July 17 2023; Revised September 11 2023; Accepted for publication September 19 2023.
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Abstract. We define a new class of linear canonical wavelet transform (LCWT) and study its properties like inner product relation, reconstruction formula and also characterize its range. We obtain Donoho-Stark's uncertainty principle for the LCWT and give a lower bound for the measure of its essential support. We also give the Shapiro's mean dispersion theorem for the proposed LCWT.

Keywords: Linear canonical transform; linear canonical wavelet transform; uncertainty principle; Shapiro's theorem.

1. Introduction

We first mention below some important abbreviations that will be used throughout this paper.

List of Abbreviations

FT	Fourier transform
FrFT	Fractional Fourier transform
LCT	Linear canonical transform
WT	Wavelet transform
FrWT	Fractional wavelet transform
WLCT	Windowed linear canonical transform
LCWT	Linear canonical wavelet transform
ONS	Orthonormal sequence
RKHS	Reproducing kernel Hilbert Space
UP	Uncertainty principle

As a generalization of FT [1] and FrFT [2, 3, 4], LCT is a four-parameter family of linear integral transform proposed by Mohinsky and Quesne [5] and is considered as the important tool for non-stationary signal processing. Because of the extra degrees of freedom, as compared to the FT and FrFT, its application can be found in a number of fields, including signal separation [6], signal reconstruction [7], filter designing [8] and many more. Recently, in [9], the authors studied octonion linear canonical transform. For more detail on LCT and its application, we refer the reader to work done by Healy et al. [10].

Even though the wavelet transform (WT) [11] is a potential tool for the analysis of non-stationary signals, it is incompetent for analyzing the signals with not well concentrated energy in the time-frequency plane, for example, the chirp-like signal, which is ubiquitous in nature [12]. On the other hand, for the signal whose energy in the frequency domain is not well concentrated, LCT is an appropriate tool. However, because of its global kernel, it is not capable of indicating the time localization of the LCT spectral components, and thus, LCT is not suitable for processing the non-stationary signal whose LCT spectral characteristics change with time. The WLCT [13], non-separable LCWT [14] is thus proposed to overcome this drawback. In this case, the original signal is first segmented with a time localization window, followed by performing the LCT spectral analysis for these segments. WLCT is capable of offering a joint signal representation in both the time and LCT domains, but its fixed window width limits the practical application; it is impossible to provide good time resolution and spectral resolution simultaneously.

Thus, to circumvent these limitations of LCT, WT, and WLCT, we propose a novel LCWT. Wei et al. [15] and Guo et al. [16] generalized the FrWT, studied in [17], to the LCWT. Wei et al. [15] studied its resolution in time and linear canonical domains, and Guo et al. [16] studied its properties on Sobolev spaces. Dai et al. [12] gave a new definition of the FrWT (also see [18]), which we generalize in the context of the LCT and study the associated UP.

Published online: September 28 2023

In Harmonic analysis, the UP is a relation between a function and its FT, which says that a function (non-zero) and its FT cannot be very well localized simultaneously. This general fact is interpreted in several different ways; for this, we refer the reader to a survey paper by Folland and Sitaram [19]. Shapiro, in [20], studied the localization for an ONS and proved that if an ONS $\{\phi_k\}$ in $L^2(\mathbb{R})$ and the sequence of their FT $\{\hat{\phi}_k\}$ are such that their means and dispersions are uniformly bounded, then $\{\phi_k\}$ is finite. Jaming and Powell [21] proved a quantitative version of Shapiro's theorem, which says that for an ONS $\{\phi_k\}$ in $L^2(\mathbb{R})$ and $N \in \mathbb{N}$,

$$\sum_{k=1}^{N}\left(\|t\phi_k\|^2_{L^2(\mathbb{R})} + \|\xi\hat{\phi}_k\|^2_{L^2(\mathbb{R})}\right) \geq \frac{(N+1)^2}{2\pi}.$$

A multivariable quantitative version of Shapiro's theorem for generalized dispersion was proved by Malinnikova [22]. It states that if $\{\phi_k\}$ be an ONS in $L^2(\mathbb{R}^d)$, $N \in \mathbb{N}$ and $p > 0$ then $\exists\, C_{p,d}$ for which:

$$\sum_{k=1}^{N}\left(\left\||t|^{\frac{p}{2}}\phi_k\right\|^2_{L^2(\mathbb{R}^d)} + \left\||\xi|^{\frac{p}{2}}\hat{\phi}_k\right\|^2_{L^2(\mathbb{R}^d)}\right) \geq C_{p,d}N^{1+\frac{p}{2d}}.$$

Recently, in this direction Shapiro's mean dispersion theorem has been proved for many integral transforms like short-time FT [23], WT [24], Hankel WT [25], Hankel Stockwell transform [26], Shearlet transform [27], Windowed LCT [28], etc. The main objectives of this paper are as follows:

(i) To define a novel time-frequency analyzing tool, namely LCWT, which generalizes the FrWT studied in [12] in the context of LCT, and study some of its basic properties along with the inner product relation, reconstruction formula and also characterize its range. To the best of our knowledge, this LCWT has not been analyzed and does not exist in the literature.
(ii) To study the time-LCT frequency analysis and the associated constant Q–factor.
(iii) To establish an UP for the LCWT for a finite energy signal. The UP for the LCWT can be derived from the UP of the LCT following the strategy adopted by Wilczok [29] and Verma et al. [30] while deriving the UP for the WT and the FrWT respectively. Similar UP has been introduced for several integral transforms like fractional WT [31], non-isotropic angular Stockwell transform [32], etc. However, we are interested in proposing an uncertainty principle directly for the LCWT without using the UP associated with LCT. In this regard, we establish the Donoho-Stark's UP for the LCWT, which in turn provides a lower bound for the measure of essential support of the LCWT. See also [13, 33], for similar results in the case of other integral transforms.
(iv) To study the Shapiro's mean dispersion theorem for the LCWT which gives the uncertainty principle for the orthonormal sequences.

The paper is arranged as follows. In section 2, we recall the definition of LCT and some of its properties. In section 3, we define LCWT and study some of its basic properties, including inner product relation, reconstruction formula, and also characterize its range. Donoho-Stark's UP for the proposed LCWT is studied in section 4. Section 5 is devoted to Shapiro's mean dispersion theorem for LCWT. Finally, in section 6, we conclude our paper.

2. Preliminaries

We briefly recall the definition of LCT and its important properties that we will be using in the sequel.

Definition 2. 1. The LCT of $\boldsymbol{f} \in \boldsymbol{L^2}(\mathbb{R})$ with respect to a matrix parameter $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$ such that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}$ and $\boldsymbol{AD} - \boldsymbol{BC} = \mathbf{1}$, is defined as:

$$(\mathcal{L}^M f)(\xi) = \begin{cases} \displaystyle\int_{\mathbb{R}} f(t)K_M(t,\xi)\,dt, B \neq 0 \\ \sqrt{D}\; e^{\frac{i}{2}CD\xi^2} f(D\xi), B = 0, \end{cases}$$

where $K_M(t,\xi)$ is a kernel given by:

$$K_M(t,\xi) = \frac{1}{\sqrt{2\pi iB}} e^{\frac{i}{2}\left(\frac{A}{B}t^2 - \frac{2}{B}\xi t + \frac{D}{B}\xi^2\right)}, \xi \in \mathbb{R}. \tag{1}$$

Among several important properties of the LCT the important among them that will be used in the sequel is the Parseval's formula:

$$\int_{\mathbb{R}} f(t)\overline{g(t)}dt = \int_{\mathbb{R}} (\mathcal{L}^M f)(\xi)\overline{(\mathcal{L}^M g)(\xi)}d\xi, \text{ where } f, g \in L^2(\mathbb{R}). \tag{2}$$

Particularly, if $f = g$, then we have the Plancherel's formula:

$$\|f\|_{L^2(\mathbb{R})} = \|\mathcal{L}^M f\|_{L^2(\mathbb{R})}. \tag{3}$$

The LCTs satisfies the additive property, i.e.,

$$\mathcal{L}^M \mathcal{L}^N f = \mathcal{L}^{MN} f, \text{where } f \in L^2(\mathbb{R}), \tag{4}$$

and the inversion property,

$$\mathcal{L}^{M^{-1}}(\mathcal{L}^M f) = f, \tag{5}$$

where, M^{-1} denotes the inverse of M. For convenience, we now denote the matrix M by $(A, B; C, D)$.

3. LCWT

We propose a new integral transform namely the LCWT. This definition generalizes the definition of FrWT defined by Dai et al. [12]. To the best of our knowledge this definition does not exist in the literature. We shall discuss some of its basic properties along with the inner product relation, reconstruction formula and also prove that its range is a RKHS.

Motivated by the definition of the admissible wavelet pair in [34], we first define it in the setting of LCT domain.

Definition 3. 1. A pair $\{\psi, \phi\}$ of functions in $L^2(\mathbb{R})$ is said to be an admissible linear canonical wavelet pair (ALCWP) if they satisfy the following admissibility condition:

$$C_{\psi,\phi,M} := \int_{\mathbb{R}^+} \overline{(\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right)}\ (\mathcal{L}^M\phi)\left(\frac{\xi}{a}\right)\frac{da}{a} \tag{6}$$

is a non-zero complex constant independent of $\xi = \pm 1$. In case $\psi = \phi$, we denote $C_{\psi,\psi,M}$ by $C_{\psi,M}$ and the required admissibility condition reduces to:

$$C_{\psi,M} := \int_{\mathbb{R}^+} \left|\ (\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right)\right|^2 \frac{da}{a} \tag{7}$$

is a positive constant independent of $\xi = \pm 1$. We call $\psi \in L^2(\mathbb{R})$, satisfying equation (7), the admissible linear canonical wavelet (ALCW).

For Example: Take $M = (A, B; C, D)$ with $B > 0$ and:

$$\psi(t) = (1-t^2)e^{-i\frac{At^2}{2B}-\frac{t^2}{2}}. \tag{8}$$

Then,

$$\begin{aligned}(\mathcal{L}^M\psi)(\xi) &= \frac{1}{\sqrt{2\pi i B}}e^{\frac{i}{2}\left(\frac{D\xi^2}{B}\right)}\int_{\mathbb{R}}(1-t^2)e^{-\frac{t^2}{2}}e^{-\frac{it\xi}{B}}\,dt \\ &= \frac{1}{\sqrt{iB}}\left(\frac{\xi}{B}\right)^2 e^{-\frac{i}{2}\left(\frac{\xi}{B}\right)^2}e^{\frac{i}{2}\left(\frac{D\xi^2}{B}\right)}.\end{aligned}$$

Thus using (7), we have:

$$\begin{aligned}C_{\psi,M} &= \frac{1}{B}\int_{\mathbb{R}^+}\left(\frac{\xi}{Ba}\right)^4 e^{-\left(\frac{\xi}{Ba}\right)^2}\ \frac{da}{a} \\ &= \frac{1}{2B},\end{aligned}$$

which is a positive constant independent of $\xi = \pm 1$. Thus, ψ given by (8) is an ALCW. In particular, for $M = (1, \frac{1}{2}; 0, 1)$, the plot of real part of ψ is given in Fig. 1.

Remark 3. 1. The function ψ given by (8) is not the only example of ALCW. Many examples can be constructed from a given ALCW. If ψ is an ALCW and $\psi_0 \in L^1(\mathbb{R})$ is any function satisfying $(\mathcal{L}^M\psi_0)(\xi) = (\mathcal{L}^M\psi_0)(-\xi)$, then the function $\psi \star_M \psi_0$, where $\star_M$ denotes the linear canonical convolution [34] given by:

$$(\psi \star_M \psi_0)(t) = e^{-\frac{iA}{2B}t^2}\left(\left[\psi(\cdot)e^{\frac{iA}{2B}(\cdot)^2}\right] \star \left[\psi_0(\cdot)e^{\frac{iA}{2B}(\cdot)^2}\right]\right)(t)$$

is also an ALCW. This can be concluded using the fact that $(\mathcal{L}^M(\psi \star_M \psi_0))(\xi) = \sqrt{2\pi i B}\ (\mathcal{L}^M\psi)(\xi)\ (\mathcal{L}^M\psi_0)(\xi)e^{-\frac{iD}{2B}\xi^2}$ and the function $\mathcal{L}^M\psi_0$ is bounded.

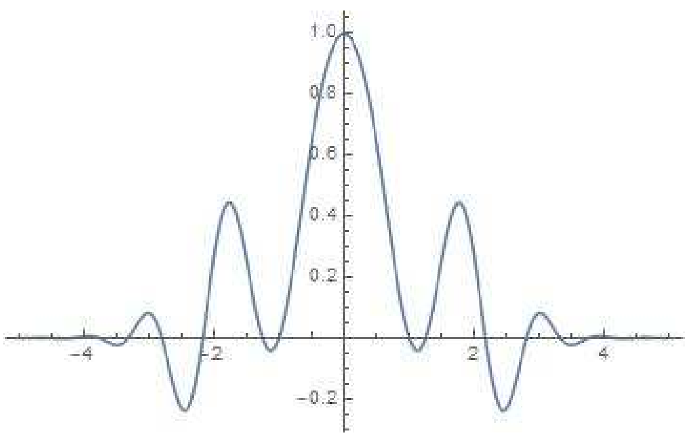

Fig. 1. Real part of ALCW ψ for $M = (1, \frac{1}{2}; 0, 1)$.

We now give the definition of the novel LCWT.

Definition 3. 2. Let $f \in L^2(\mathbb{R})$, $M = (A, B; C, D)$ be a matrix with $AD - BC = 1$ and $B \neq 0$ then we define the LCWT of f with respect to M and an ALCW ψ by:

$$(W_\psi^M f)(a,b) = e^{-\frac{iA}{2B}b^2} \left\{ f(t)e^{\frac{iA}{2B}t^2} \star \overline{\sqrt{a}\psi(-at)e^{\frac{iA}{2B}(at)^2}} \right\}(b), a \in \mathbb{R}^+, b \in \mathbb{R},$$

where $\star$ denote the convolution given by:

$$(f \star g)(\nu) = \int_{\mathbb{R}} f(x)g(\nu - x)dx, \nu \in \mathbb{R}.$$

Equivalently,

$$\begin{aligned}(W_\psi^M f)(a,b) &= e^{-\frac{iA}{2B}b^2} \int_{\mathbb{R}} f(t)e^{\frac{iA}{2B}t^2} \overline{\sqrt{a}\psi(-a(b-t))e^{\frac{iA}{2B}(a(t-b))^2}}\, dt \\ &= \int_{\mathbb{R}} f(t)\overline{e^{-\frac{iA}{2B}\left\{(t^2-b^2)-(a(t-b))^2\right\}} \sqrt{a}\psi(a(t-b))}\, dt \\ &= \int_{\mathbb{R}} f(t)\overline{\psi_{a,b}^M(t)}\, dt,\end{aligned}$$

where, with $\psi_{a,b}(t) = \sqrt{a}\psi(a(t-b))$:

$$\psi_{a,b}^M(t) = e^{-\frac{iA}{2B}\left\{(t^2-b^2)-(a(t-b))^2\right\}}\psi_{a,b}(t). \tag{9}$$

Thus, we have an equivalent definition of the LCWT as:

$$(W_\psi^M f)(a,b) = \langle f, \psi_{a,b}^M \rangle_{L^2(\mathbb{R})}. \tag{10}$$

It should to be noted that depending on the different choice of the matrix M, we have different integral transform:
1. For $M = (\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$, $\alpha \neq n\pi$, we obtain the FrWT as discussed in [12].
2. For $M = (0,1;-1,0)$ we obtain the traditional WT [35].

We now establish a fundamental relation between LCWT and the LCT. This relation will be useful in obtaining the resolution of time and linear canonical spectrum in the time-LCT-frequency plane and inner product relation associated with the LCWT.

Proposition 3. 1. If $W_\psi^M f$ and $\mathcal{L}^M f$ are respectively the LCWT and the LCT of $f \in L^2(\mathbb{R})$. Then,

$$\mathcal{L}^M\left((W_\psi^M f)(a,\cdot)\right)(\xi) = \frac{\sqrt{-2\pi i\ B}}{\sqrt{a}} e^{\frac{iD}{2B}\left(\frac{\xi}{a}\right)^2} (\mathcal{L}^M f)(\xi)\overline{(\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right)}. \tag{11}$$

Proof: Form the definition of the LCT and $\psi_{a,b}^M$, it follows that:

$$\begin{aligned}(\mathcal{L}^M\psi_{a,b}^M)(\xi) &= \int_{\mathbb{R}} \sqrt{a}\,\psi\big(a(t-b)\big) \sqrt{\frac{1}{2\pi iB}} e^{\frac{i}{2}\left\{\frac{Ab^2}{B}+\frac{A}{B}(a(t-b))^2-\frac{2}{B}\xi t+\frac{D}{B}\xi^2\right\}} dt \\ &= \int_{\mathbb{R}} \sqrt{a}\,\psi(at) \sqrt{\frac{1}{2\pi iB}} e^{\frac{i}{2}\left\{\frac{Ab^2}{B}+\frac{A}{B}(at)^2-\frac{2}{Ba}(at+ab)\xi+\frac{D}{B}\xi^2\right\}} dt \\ &= \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \psi(t) \sqrt{\frac{1}{2\pi iB}} e^{\frac{i}{2}\left(\frac{Ab^2}{B}-\frac{2}{B}\xi b+\frac{D}{B}\xi^2\right)} e^{\frac{i}{2}\left\{\frac{Ab^2}{B}-\frac{2}{B}t\left(\frac{\xi}{a}\right)+\frac{D}{B}\left(\frac{\xi}{a}\right)^2\right\}} e^{-\frac{iD}{2B}\left(\frac{\xi}{a}\right)^2} dt \\ &= \frac{1}{\sqrt{a}} e^{-\frac{iD}{2B}\left(\frac{\xi}{a}\right)^2} \sqrt{2\pi iB} \int_{\mathbb{R}} \psi(t)K_M(b,\xi)K_M\left(t,\frac{\xi}{a}\right) dt.\end{aligned}$$

Therefore, we have:

$$(\mathcal{L}^M\psi_{a,b}^M)(\xi) = \frac{\sqrt{2\pi iB}}{\sqrt{a}} e^{-\frac{iD}{2B}\left(\frac{\xi}{a}\right)^2} K_M(b,\xi)(\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right). \tag{12}$$

Using (2) in (10), we get:

$$(W_\psi^M f)(a,b) = \langle \mathcal{L}^M f, \mathcal{L}^M\psi_{a,b}^M \rangle_{L^2(\mathbb{R})}.$$

Using equation (12), we have:

$$(W_\psi^M f)(a,b) = \frac{\sqrt{-2\pi i\ B}}{\sqrt{a}} \int_{\mathbb{R}} e^{\frac{iD}{2B}\left(\frac{\xi}{a}\right)^2} (\mathcal{L}^M f)(\xi)\overline{(\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right)} K_{M^{-1}}(\xi,b)\, d\xi. \tag{13}$$

Therefore, it follows that:

$$\mathcal{L}^M\left((W_\psi^M f)(a,\cdot)\right)(\xi) = \frac{\sqrt{-2\pi i\ B}}{\sqrt{a}} e^{\frac{iD}{2B}\left(\frac{\xi}{a}\right)^2} (\mathcal{L}^M f)(\xi)\overline{(\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right)}.$$

This completes the proof.

3.1. Time-LCT frequency analysis

From equation (10) it follows that if $\psi_{a,b}^M$ is localized in the time domain, then the transform $(W_\psi^M f)(a,b)$ gives the local information of the f in the time domain. Also, from equation (13), it follows that the LCWT can provide the local property of f in the linear canonical domain. Thus, the LCWT is capable of producing simultaneously the time-LCT frequency information and represent the signal in the time-LCT frequency domain. More precisely, if ψ and $\mathcal{L}^M\psi$ are window functions in time and linear canonical domain respectively with E_ψ and $E_{\mathcal{L}^M\psi}$ as centers and Δ_ψ and $\Delta_{\mathcal{L}^M\psi}$ are radii, respectively. Then the center and radius of $\psi_{a,b}^M$ are given respectively by:

$$E[\psi_{a,b}^M] = \frac{1}{a}E_\psi + b,$$

and

$$\Delta[\psi_{a,b}^M\ ] = \frac{1}{a}\Delta_\psi.$$

Similarly, the center and radius of window function $(\mathcal{L}^M\psi)(\frac{\xi}{a})$ are given by:

$$E\left[(\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right)\right] = aE_{\mathcal{L}^M\psi}$$

and

$$\Delta\left[(\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right)\right] = a\Delta_{\mathcal{L}^M\psi}.$$

Thus, the Q-factor of the window function of the linear canonical transform domain is:

$$Q = \frac{\Delta_{\mathcal{L}^M\psi}}{E_{\mathcal{L}^M\psi}},$$

which is independent of the scaling parameter a for a given parameter M. This is called the constant Q–property of the LCWT.

3.2. Time-LCT frequency resolution

The LCWT $(W_\psi^M f)(a,b)$ localizes the signal f in the time window:

$$\left[\frac{1}{a}E_\psi + b - \frac{1}{a}\Delta_\psi, \frac{1}{a}E_\psi + b + \frac{1}{a}\Delta_\psi\right].$$

Similarly, we get that the LCWT gives linear canonical spectrum content of f in the window:

$$[aE_{\mathcal{L}^M\psi} - a\Delta_{\mathcal{L}^M\psi}, aE_{\mathcal{L}^M\psi} + a\Delta_{\mathcal{L}^M\psi}].$$

Thus, the joint resolution of the LCWT in the time and linear canonical domain is given by the window:

$$\left[\frac{1}{a}E_\psi + b - \frac{1}{a}\Delta_\psi, \frac{1}{a}E_\psi + b + \frac{1}{a}\Delta_\psi\right] \times [aE_{\mathcal{L}^M\psi} - a\Delta_{\mathcal{L}^M\psi}, aE_{\mathcal{L}^M\psi} + a\Delta_{\mathcal{L}^M\psi}],$$

with constant area $4\Delta_\psi\Delta_{\mathcal{L}^M\psi}$ in the time-LCT-frequency plane. Thus, it follows that for a given parameter M, the window area depends on the linear canonical admissible wavelets and is independent of the parameters a and b. But it is to be noted that the window gets narrower for large value of a and wider for small value of a. Thus, the window given by the transform is flexible and hence, it is capable of simultaneously providing the time linear canonical domain information. This flexibility of the window makes the proposed LCWT more advantageous then the WLCT as in this case the window is rigid.

Some basic properties of LCWT is given below.

Theorem 3. 1. Let $\boldsymbol{g}, \boldsymbol{h} \in \boldsymbol{L^2}(\mathbb{R})$, $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ are ALCWs, $\boldsymbol{\alpha}, \boldsymbol{\beta}\ \in \mathbb{C}$, $\boldsymbol{\lambda}\ >\ \mathbf{0}$ and $\boldsymbol{y}\ \in \mathbb{R}$. Then:

1. $W_\psi^M(\alpha g + \beta h) = \alpha(W_\psi^M g) + \beta(W_\psi^M h)$
2. $W_{\alpha\psi+\beta\phi}^M(g) = \bar{\alpha}(W_\psi^M g) + \bar{\beta}(W_\phi^M g)$
3. $(W_\psi^M \delta_\lambda g)(a,b) = \left(W_\psi^{\widetilde{M}} g\right)(\frac{a}{\lambda}, b\lambda)$, where $(\delta_\lambda g)(t) = \sqrt{\lambda}g(\lambda t)$ and $\widetilde{M} = (A, \lambda^2 B; \frac{C}{\lambda^2}, D)$
4. $(W_\psi^M \tau_y g)(a,b) = e^{\frac{iA}{B}y(y-b)}\left(W_\psi^M e^{\frac{iA}{B}yt} g\right)(a, b-y)$, where $(\tau_y g)(y) = g(t-y)$.

Proof: The proofs are immediate and can be omitted.

If $\{\psi, \phi\}$ is admissible linear canonical wavelet pair such that each ϕ and ψ are ALCWs and $f, g\ \in L^2(\mathbb{R})$ are such that they are orthogonal then $W_\psi^M f$ and $W_\psi^M g$ are orthogonal in $L^2(\mathbb{R}^+\ \times \mathbb{R}, dadb)$. This result is justified by the following theorem, which further gives the resolution of identity for the LCWT.

Theorem 3. 2. (Inner product relation for LCWT). Let $\{\psi, \phi\}$ be an ALCWP such that ψ and ϕ are ALCWs and $f, g \in L^2(\mathbb{R})$, then:

$$\langle W_\psi^M f, W_\phi^M g\rangle_{L^2(\mathbb{R}^+\times\mathbb{R})} = 2\pi|B|C_{\psi,\phi,M}\langle f,g\rangle_{L^2(\mathbb{R})}, \tag{14}$$

where $C_{\psi,\phi,M}$ is provided in (6).

Proof: Using equation (11), we get:

$$\begin{aligned}
\langle W_\psi^M f, W_\phi^M g\rangle_{L^2(\mathbb{R}^+\times\mathbb{R})} &= \int_{\mathbb{R}^+\times\mathbb{R}} (W_\psi^M f)(a,b)\overline{(W_\phi^M g)(a,b)}dadb \\
&= \int_{\mathbb{R}^+\times\mathbb{R}} \left(\mathcal{L}^M\left((W_\psi^M f)(a,\cdot)\right)\right)(\xi)\overline{\left(\mathcal{L}^M\left((W_\phi^M g)(a,\cdot)\right)\right)(\xi)d\xi da} \\
&= \int_{\mathbb{R}^+\times\mathbb{R}} \frac{2\pi|B|}{a}(\mathcal{L}^M f)(\xi)\ \overline{(\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right)}\ \ \overline{(\mathcal{L}^M g)(\xi)}(\mathcal{L}^M\phi)\left(\frac{\xi}{a}\right)d\xi da \\
&= 2\pi|B|\int_{\mathbb{R}}(\mathcal{L}^M f)(\xi)\ \overline{(\mathcal{L}^M g)(\xi)}\left\{\int_{\mathbb{R}^+}\overline{(\mathcal{L}^M\psi)\left(\frac{\xi}{a}\right)}\ (\mathcal{L}^M\phi)\left(\frac{\xi}{a}\right)\ \frac{da}{a}\right\}d\xi \\
&= 2\pi|B|C_{\psi,\phi,M}\langle\mathcal{L}^M f,\mathcal{L}^M g\rangle_{L^2(\mathbb{R})} \\
&= 2\pi|B|C_{\psi,\phi,M}\langle f,g\rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

Remark 3. 2. Replacing $\psi = \phi$ in equation (14), we have:

$$\langle W_\psi^M f, W_\psi^M g\rangle_{L^2(\mathbb{R}^+\times\mathbb{R})} = 2\pi|B|C_{\psi,M}\langle f,g\rangle_{L^2(\mathbb{R})}, \tag{15}$$

where $C_{\psi,M}$ is provided in (7).

Remark 3. 3. (Plancherel's theorem for $\boldsymbol{W_\psi^M f}$) Replacing $\boldsymbol{f} = \boldsymbol{g}$ and $\boldsymbol{\phi} = \boldsymbol{\psi}$ in equation (14) we have the Plancherel's theorem for $\boldsymbol{W_\psi^M}$ given by:

$$\|W_\psi^M f\|_{L^2(\mathbb{R}^+\times\mathbb{R})} = (2\pi|B|C_{\psi,M})^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R})}. \tag{16}$$

Thus, from equation (16), it follows that LCWT from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^+\times\mathbb{R})$ is a continuous linear operator. If further ALCW ψ is such that $C_{\psi,M} = \frac{1}{2\pi|B|}$, then the operator is an isometry.

Theorem 3. 3. (Reconstruction formula). Let $\{\psi, \phi\}$ be an ALCWP such that ψ and ϕ are ALCWs and $f \in L^2(\mathbb{R})$, then f can be given by the formula:

$$f(t) = \frac{1}{2\pi|B|C_{\psi,\phi,M}}\int_{\mathbb{R}^+\times\mathbb{R}}(W_\psi^M f)(a,b)\phi_{a,b}^M(t)\,dadb\ a.e.\,t\in\mathbb{R}. \tag{17}$$

Proof: From equation (14), we get:

$$\begin{aligned}
2\pi|B|C_{\psi,\phi,M}\langle f,g\rangle_{L^2(\mathbb{R})} &= \langle W_\psi^M f, W_\phi^M g\rangle_{L^2(\mathbb{R}^+\times\mathbb{R})} \\
&= \int_{\mathbb{R}^+\times\mathbb{R}}(W_\psi^M f)(a,b)\overline{\left(\int_{\mathbb{R}} g(t)\overline{\phi_{a,b}^M(t)}\right)dadb} \\
&= \left\langle\int_{\mathbb{R}^+\times\mathbb{R}}(W_\psi^M f)(a,b)\,\phi_{a,b}^M(t)dadb, g(t)\right\rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

Since $g \in L^2(\mathbb{R})$ is arbitrary, we have:

$$f(t) = \frac{1}{2\pi|B|C_{\psi,\phi,M}}\int_{\mathbb{R}^+\times\mathbb{R}}(W_\psi^M f)(a,b)\phi_{a,b}^M(t)\,dadb\ a.e.$$

The proof is complete.

In particular, if $\psi = \phi$ then we have the following reconstruction formula:

$$f(t) = \frac{1}{2\pi|B|C_{\psi,\phi,M}}\int_{\mathbb{R}^+\times\mathbb{R}}(W_\psi^M f)(a,b)\psi_{a,b}^M(t)\,dadb\ a.e.\,t\in\mathbb{R}.$$

The following theorem characterizes the range of the LCWT and proves that the range is a RKHS. It also gives the explicit expression for the reproducing kernel.

Theorem 3. 4. For ψ being ALCW, $\boldsymbol{W_\psi^M}(\boldsymbol{L^2}(\mathbb{R}))$ is a RKHS with the kernel:

$$K_\psi^M(x,y;a,b) = \frac{1}{2\pi|B|C_{\psi,M}}\langle\psi_{a,b}^M,\psi_{x,y}^M\rangle_{L^2(\mathbb{R})},\ (x,y),(a,b)\in\mathbb{R}^+\times\mathbb{R}.$$

Moreover, the kernel is such that $\left|K_\psi^M(x,y;a,b)\right| \le \frac{1}{2\pi|B|C_{\psi,M}}\|\psi\|^2_{L^2(\mathbb{R})}$.

Proof: For $(a,b)\in\mathbb{R}^+\times\mathbb{R}$, we see that:

$$K_\psi^M(x,y;a,b)=\frac{1}{2\pi|B|C_{\psi,M}}(W_\psi^M\psi_{a,b}^M)(x,y) \text{ for all } (x,y)\in\mathbb{R}^+\times\mathbb{R}.$$

Now,

$$\begin{aligned}\|K_\psi^M(\cdot,\cdot;a,b)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}&=\frac{1}{(2\pi|B|C_{\psi,M})^2}\|W_\psi^M\psi_{a,b}^M\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}\\&=\frac{1}{2\pi|B|C_{\psi,M}}\|\psi\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, for $(a,b)\in\mathbb{R}^+\times\mathbb{R}$, $K_\psi^M(x,y;a,b)\in L^2(\mathbb{R}^+\times\mathbb{R})$. Now, let $f\in L^2(\mathbb{R})$:

$$\begin{aligned}(W_\psi^M f)(a,b)&=\langle f,\psi_{a,b}^M\rangle_{L^2(\mathbb{R})}\\&=\frac{1}{2\pi|B|C_{\psi,M}}\langle W_\psi^M f, 2\pi|B|C_{\psi,M}K_\psi^M(\cdot,\cdot;a,b)\rangle_{L^2(\mathbb{R}^+\times\mathbb{R})}\\&=\langle W_\psi^M f, K_\psi^M(\cdot,\cdot;a,b)\rangle_{L^2(\mathbb{R}^+\times\mathbb{R})}.\end{aligned}$$

Thus, it follows that:

$$K_\psi^M(x,y;a,b)=\frac{1}{2\pi|B|C_{\psi,M}}\langle\psi_{a,b}^M,\psi_{x,y}^M\rangle_{L^2(\mathbb{R})},$$

is the reproducing kernel of $W_\psi^M(L^2(\mathbb{R}))$.

Again,

$$\begin{aligned}\left|K_\psi^M(x,y;a,b)\right|&=\frac{1}{2\pi|B|C_{\psi,M}}\left|\langle\psi_{a,b}^M,\psi_{x,y}^M\rangle_{L^2(\mathbb{R})}\right|\\&\le\frac{1}{2\pi|B|C_{\psi,M}}\|\psi_{a,b}^M\|_{L^2(\mathbb{R})}\|\psi_{x,y}^M\|_{L^2(\mathbb{R})}\\&=\frac{\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}.\end{aligned}$$

This completes the proof.

4. Uncertainty Principle

We prove some UPs that limits the concentration of the LCWT in some subset in $\mathbb{R}^+\times\mathbb{R}$ of small measure. For related results in case of Fourier transform and windowed Fourier transform we refer the reader to [36, 37]. Kou et al. [38] studied the same for the WLCT.

Definition 4. 1. Let $0\le\epsilon<1$, $f\in L^2(\mathbb{R})$ and $E\subset\mathbb{R}$ be measurable, then f is ϵ –concentrated on E if:

$$\left(\int_{E^c}|f(x)|^2\,dx\right)^{\frac{1}{2}}\le\epsilon\,\|f\|_{L^2(\mathbb{R})}.$$

If $0\le\epsilon\le\frac{1}{2}$, then we say that most of the energy of f is concentrated on E and E is called an essential support of f. If $\epsilon=0$, then support of f is contained in E.

Lemma 4. 1. If ψ is an ALCW and $f\in L^2(\mathbb{R})$. Then $W_\psi^M f\in L^p(\mathbb{R}^+\times\mathbb{R})$, for all $p\in[2,\infty]$. Moreover,

$$\|W_\psi^M f\|_{L^p(\mathbb{R}^+\times\mathbb{R})}\le(2\pi|B|)^{\frac{1}{p}}C_{\psi,M}^{\frac{1}{p}}\,\|f\|_{L^2(\mathbb{R})}\|\psi\|^{1-\frac{2}{p}}_{L^2(\mathbb{R})},\qquad p\in[2,\infty) \tag{18}$$

$$\|W_\psi^M f\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\le\|\psi\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}. \tag{19}$$

Proof: Since ψ is an ALCW, it follows that $W_\psi^M f\in L^2(\mathbb{R}^+\times\mathbb{R})$. Again:

$$\left|(W_\psi^M f)(a,b)\right|\le\ \|\psi\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}.$$

Thus, $W_\psi^M f\in L^\infty(\mathbb{R}^+\times\mathbb{R})$. Also, since $W_\psi^M f\in L^2(\mathbb{R}^+\times\mathbb{R})$, we have $W_\psi^M f\in L^p(\mathbb{R}^+\times\mathbb{R})$, $p\in[2,\infty)$. Moreover,

$$\|W_\psi^M f\|_{L^p(\mathbb{R}^+\times\mathbb{R})} \le \|W_\psi^M f\|_{L^2(\mathbb{R}^+\times\mathbb{R})}^{\frac{2}{p}} \|W_\psi^M f\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}^{1-\frac{2}{p}}$$
$$\le (2\pi|B|C_{\psi,M})^{\frac{1}{p}} \, \|f\|_{L^2(\mathbb{R})}^{\frac{2}{p}} \, \|f\|_{L^2(\mathbb{R})}^{1-\frac{2}{p}} \, \|\psi\|_{L^2(\mathbb{R})}^{1-\frac{2}{p}}.$$

This proves the lemma.

Definition 4. 2. Let $\mathbf{0} \le \boldsymbol{\epsilon} < \mathbf{1}$, $\boldsymbol{F} \in \boldsymbol{L^2}(\mathbb{R}^+ \times \mathbb{R})$ and $\boldsymbol{\Omega} \subset \mathbb{R}^+ \times \mathbb{R}$ be measurable, then $\boldsymbol{F}$ is $\boldsymbol{\epsilon}-$ concentrated on $\boldsymbol{\Omega}$ if:

$$\left(\int_{\Omega^c} |F(x,y)|^2 \, dxdy\right)^{\frac{1}{2}} \le \epsilon \|F\|_{L^2(\mathbb{R}^+\times\mathbb{R})}.$$

If $0 \le \epsilon \le \frac{1}{2}$, then we say that most of the energy of F is concentrated on Ω and Ω is called an essential support of F. If $\epsilon = 0$, then support of F is contained in Ω.

We now prove the Donoho-Stark's UP for the propose LCWT.

Theorem 4. 1. Let $\mathbf{0} \le \boldsymbol{\epsilon} < \mathbf{1}$, $\boldsymbol{\psi}$ is an ALCW. Also let there exists a non-zero $\boldsymbol{f} \in \boldsymbol{L^2}(\mathbb{R})$ such that $\boldsymbol{W_\psi^M f}$ is $\boldsymbol{\epsilon}-$ concentrated on $\boldsymbol{\Omega} \subset \mathbb{R}^+ \times \mathbb{R}$ then:

$$|\Omega|\|\psi\|_{L^2(\mathbb{R})}^2 \ge 2\pi|B|C_{\psi,M}(1-\epsilon^2), \tag{20}$$

where $|\Omega|$ denotes the measure of Ω.

Proof: In equation (16), we have:

$$\|W_\psi^M f\|_{L^2(\mathbb{R}^+\times\mathbb{R})}^2 = 2\pi|B|C_{\psi,M} \, \|f\|_{L^2(\mathbb{R})}^2.$$

Now,

$$\int_{\mathbb{R}^+\times\mathbb{R}} \left|(W_\psi^M f)(a,b)\right|^2 dadb \le \int_{\mathbb{R}^+\times\mathbb{R}} \chi_\Omega(a,b)\left|(W_\psi^M f)(a,b)\right|^2 dadb + \epsilon^2 \, \|W_\psi^M f\|_{L^2(\mathbb{R}^+\times\mathbb{R})}^2.$$

This gives:

$$(1-\epsilon^2)\|W_\psi^M f\|_{L^2(\mathbb{R}^+\times\mathbb{R})}^2 \le |\Omega|\|W_\psi^M f\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}^2.$$

Thus, using (19), we get:

$$2\pi|B|C_{\psi,M}\,(1-\epsilon^2)\,\|f\|_{L^2(\mathbb{R})}^2 \le |\Omega|\|f\|_{L^2(\mathbb{R})}^2\,\|\psi\|_{L^2(\mathbb{R})}^2.$$

The result follows, since $f \ne 0$.

Corollary 4. 1. If $\boldsymbol{f} \in \boldsymbol{L^2}(\mathbb{R}) \cap \boldsymbol{L^4}(\mathbb{R})$, in $\boldsymbol{L^2}(\mathbb{R})$ - norm, is $\boldsymbol{\epsilon_E}-$ concentrated on $\boldsymbol{E} \subset \mathbb{R}$ and $\boldsymbol{W_\psi^M f}$ is $\boldsymbol{\epsilon_\Omega}-$concentrated on $\boldsymbol{\Omega} \subset \mathbb{R}^+ \times \mathbb{R}$, then:

$$|\Omega| m(E)\|\psi\|_{L^2(\mathbb{R})}^2\|f\|_{L^4(\mathbb{R})}^4 \ge 2\pi|B|C_{\psi,M}\,(1-\epsilon_\Omega^2)(1-\epsilon_E^2)^2\,\|f\|_{L^2(\mathbb{R})}^4,$$

where $m(E)$ denotes the measure of E.

Proof: Since $W_\psi^M f$ is ϵ_Ω $-$concentrated on $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$ in $L^2(\mathbb{R}^+ \times \mathbb{R})$ $-$norm, so we have $|\Omega|\|\psi\|_{L^2(\mathbb{R})}^2 \ge 2\pi|B|C_{\psi,M}(1-\epsilon_\Omega^2)$. Again, since f is $\epsilon_E -$ concentrated, we have:

$$\left(\int_{E^c} |f(x)|^2 \, dx\right)^{\frac{1}{2}} \le \epsilon_E \, \|f\|_{L^2(\mathbb{R})},$$

which further implies that:

$$\|f\|_{L^2(\mathbb{R})}^2(1-\epsilon_E^2) \le \int_{\mathbb{R}} \chi_E(x)|f(x)|^2 dx.$$

We have by Hölder's inequality:

$$\int_{\mathbb{R}} \chi_E(x)|f(x)|^2 dx \le \left(\int_{\mathbb{R}} |\chi_E(x)|^2 \, dx\right)^{\frac{1}{2}} \|f\|_{L^4(\mathbb{R})}^2.$$

Thus,

$$(1-\epsilon_E^2)\|f\|_{L^2(\mathbb{R})}^2 \le \big(m(E)\big)^{\frac{1}{2}}\|f\|_{L^4(\mathbb{R})}^2. \tag{21}$$

Therefore,

$$|\Omega|m(E)\|\psi\|_{L^2(\mathbb{R})}^2\ \|f\|_{L^4(\mathbb{R})}^4\ \geq 2\pi|B|C_{\psi,M}(1-\epsilon_\Omega^2)(1-\epsilon_E^2)^2\ \|f\|_{L^2(\mathbb{R})}^4.$$

This proof is complete.

Corollary 4. 2. If $\boldsymbol{f}\in \boldsymbol{L^2}(\mathbb{R})\cap \boldsymbol{L^\infty}(\mathbb{R})$, in $\boldsymbol{L^2}(\mathbb{R})$-norm, is $\boldsymbol{\epsilon_E}$ –concentrated on $\boldsymbol{E}\subset\mathbb{R}$ and $\boldsymbol{W_\psi^M f}$ is $\boldsymbol{\epsilon_\Omega}$ –concentrated on $\boldsymbol{\Omega}\subset \mathbb{R}^+\times\mathbb{R}$, then:

$$|\Omega|m(E)\|\psi\|_{L^2(\mathbb{R})}^2\|f\|_{L^\infty(\mathbb{R})}^2\geq 2\pi|B|C_{\psi,M}\ (1-\epsilon_\Omega^2)(1-\epsilon_E^2)\ \|f\|_{L^2(\mathbb{R})}^2.$$

Proof: Since $W_\psi^M f$ is ϵ_Ω –concentrated on $\Omega\subset\mathbb{R}^+\times\mathbb{R}$ in $L^2(\mathbb{R}^+\times\mathbb{R})-$ norm, so we have $|\Omega|\|\psi\|_{L^2(\mathbb{R})}^2\geq 2\pi|B|C_{\psi,M}(1-\epsilon_\Omega^2)$. Again, since f is ϵ_E- concentrated, we have:

$$\left(\int_{E^c}|f(x)|^2\ dx\right)^{\frac{1}{2}}\leq\epsilon_E\ \|f\|_{L^2(\mathbb{R})},$$

which further implies that:

$$\|f\|_{L^2(\mathbb{R})}^2(1-\epsilon_E^2)\leq\int_{\mathbb{R}}|f(x)|^2\chi_E(x)dx.$$

Since $f\in L^\infty(\mathbb{R})$, so:

$$\int_{\mathbb{R}}\chi_E(x)|f(x)|^2\ \ dx\leq m(E)\|f\|_{L^\infty(\mathbb{R})}^2.$$

Thus,

$$\|f\|_{L^\infty(\mathbb{R})}^2 m(E)\geq(1-\epsilon_E^2)\|f\|_{L^2(\mathbb{R})}^2. \tag{22}$$

Therefore,

$$|\Omega|m(E)\|\psi\|_{L^2(\mathbb{R})}^2\|f\|_{L^\infty(\mathbb{R})}^2\geq 2\pi|B|C_{\psi,M}\ (1-\epsilon_\Omega^2)(1-\epsilon_E^2)\ \|f\|_{L^2(\mathbb{R})}^2.$$

The proof is complete.

5. Orthonormal Sequences and Uncertainty Principle

We now express the UP in term of the generalized dispersion of W_ψ^M , which is defined by:

$$\rho_p(W_\psi^M f)=\left(\int_{\mathbb{R}^+\times\mathbb{R}}|(a,b)|^p\ \left|(W_\psi^M f)(a,b)\right|^2\ dadb\right)^{\frac{1}{p}}, \tag{23}$$

where $|(a,b)|=\sqrt{a^2+b^2},\ p>0$.

Definition 5. 1. Let $\boldsymbol{T}$ be a bounded linear operator on a Hilbert space $\mathbb{H}$ over the field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$) and $\{\boldsymbol{u_n}\}_{\boldsymbol{n}\in\mathbb{N}}$ be an orthonormal basis of $\mathbb{H}$, then $\boldsymbol{T}$ is called a Hilbert-Schmidt operator if:

$$\|T\|_{HS}=\left(\sum_{n=1}^{\infty}\|Tu_n\|^2\right)^{\frac{1}{2}}<\infty.$$

It is to be noted that the Hilbert-Schmidt norm does not depend on the choice of orthonormal basis.

Before discussing the main result of this section, we estimate the Hilbert-Schmidt norm of the product of some orthogonal projection operators and use it to estimate the concentration of $W_\psi^M f$ on subset of $\mathbb{R}^+\times\mathbb{R}$. Similar results were first studied by Wilczok [29] in the case of windowed FT and WT.

Theorem 5. 1. Let $\boldsymbol{f}\ \in\ \boldsymbol{L^2}(\mathbb{R})$, $\boldsymbol{\psi}$ is an ALCW and $\boldsymbol{\Omega}\subset\mathbb{R}^+\times\mathbb{R}$ such that $|\boldsymbol{\Omega}|<\frac{\boldsymbol{2\pi|B|C_{\psi,M}}}{\|\boldsymbol{\psi}\|_{\boldsymbol{L^2}(\mathbb{R})}^{\boldsymbol{2}}}$. Then:

$$\left\|\chi_{\Omega^c}W_\psi^M f\right\|_{L^2(\mathbb{R}^+\times\mathbb{R})}\geq\sqrt{2\pi|B|C_{\psi,M}-|\Omega|\|\psi\|_{L^2(\mathbb{R})}^2}\ \|f\|_{L^2(\mathbb{R})}.$$

Proof: We consider the orthogonal projections P_ψ from $L^2(\mathbb{R}^+\times\mathbb{R},dadb)$ on the RKHS $W_\psi^M(L^2(\mathbb{R}))$ and P_Ω on $L^2(\mathbb{R}^+\times\mathbb{R},dadb)$ defined by $P_\Omega F\ =\ \chi_\Omega F$, for all $F\in L^2(\mathbb{R}^+\times\mathbb{R},dadb)$. According to Saitoh [39], for every $(a,b)\in\mathbb{R}^+\times\mathbb{R}$ and $F\in L^2(\mathbb{R}^+\times\mathbb{R},dadb)$, we get:

$$\begin{aligned}(P_\Omega P_\psi F)(a,b)&=\chi_\Omega(a,b)\langle F,K_\psi^M(\cdot,\cdot;a,b)\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}\\&=\int_{\mathbb{R}^+\times\mathbb{R}}\chi_\Omega(a,b)F(x,y)\overline{K_\psi^M(x,y;a,b)}\,dxdy.\end{aligned}$$

Thus, the integral operator $P_\Omega P_\psi$ has the kernel $\mathcal{N}^M_{\psi,\Omega}$ defined on $(\mathbb{R}^+\times\mathbb{R})^2$ by:

$$\mathcal{N}^M_{\psi,\Omega}(x,y;a,b)=\chi_\Omega(a,b)\overline{K^M_\psi(x,y;a,b)}$$

such that,

$$\begin{aligned}\int_{\mathbb{R}^+\times\mathbb{R}}\int_{\mathbb{R}^+\times\mathbb{R}}&|\mathcal{N}^M_{\psi,\Omega}(x,y;a,b)|^2dxdydadb\\ &=\int_{\mathbb{R}^+\times\mathbb{R}}\left(\int_{\mathbb{R}^+\times\mathbb{R}}|K^M_\psi(x,y;a,b)|^2\,dxdy\right)|\chi_\Omega(a,b)|^2dadb\\ &=\int_{\mathbb{R}^+\times\mathbb{R}}\chi_\Omega(a,b)\|K^M_\psi(\cdot,\cdot;a,b)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}dadb\\ &=\frac{|\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}.\end{aligned}$$

Now,

$$\chi_\Omega W^M_\psi f=P_\Omega P_\psi(W^M_\psi f)$$

Implies,

$$\|\chi_\Omega W^M_\psi f\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}\le\|P_\Omega P_\psi\|^2\|W^M_\psi f\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}.$$

Therefore,

$$\|W^M_\psi f\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}\le\|P_\Omega P_\psi\|^2\|W^M_\psi f\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}+\|\chi_{\Omega^c}W^M_\psi f\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}$$

$$\text{i. e.},\|\chi_{\Omega^c}W^M_\psi f\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}\ge(1-\|P_\Omega P_\psi\|^2)2\pi|B|C_{\psi,M}\|f\|^2_{L^2(\mathbb{R})}.$$

Now using the fact that $\|P_\Omega P_\psi\|\le\|P_\Omega P_\psi\|_{HS}$, where $\|\cdot\|$ denotes the operator norm, we obtain:

$$\|\chi_{\Omega^c}W^M_\psi f\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})}\ge(1-\|P_\Omega P_\psi\|^2_{HS})2\pi|B|C_{\psi,M}\|f\|^2_{L^2(\mathbb{R})}.$$

Hence, we obtain:

$$\|\chi_{\Omega^c}W^M_\psi f\|_{L^2(\mathbb{R}^+\times\mathbb{R})}\ge\sqrt{2\pi|B|C_{\psi,M}-|\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}\|f\|_{L^2(\mathbb{R})}.$$

This proves the theorem.

Theorem 5. 2. If ψ is an ALCW, $\{\boldsymbol{\phi_n}\}_{\boldsymbol{n}\in\mathbb{N}}\subset\ \boldsymbol{L^2}(\mathbb{R})$ be an ONS and $\boldsymbol{\Omega}\subset\mathbb{R}^+\times\mathbb{R}$ be such that its measure $|\boldsymbol{\Omega}|<\infty$, then for any non-empty $\wedge\subset\mathbb{N}$,

$$\sum_{n\in\wedge}\left(1-\left\|\chi_{\Omega^c}W^M_\psi\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\|_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}\right)\le\frac{|\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}.\tag{24}$$

Proof: Consider the orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$ of $L^2(\mathbb{R}^+\times\mathbb{R},dadb)$. It has been proved in the above theorem that $P_\Omega P_\psi$ is a Hilbert-Schmidt operator such that $\|P_\Omega P_\psi\|^2_{HS}=\frac{|\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}$.

Since $P_\Omega^2=P_\Omega$ and both $P_\psi,\ P_\Omega$ are self-adjoint, the operator $T=(P_\Omega P_\psi)^\star(\ P_\Omega P_\psi)=P_\psi P_\Omega P_\psi$ is positive and is such that:

$$\begin{aligned}\sum_{n\in\mathbb{N}}\langle Th_n,h_n\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}&=\sum_{n\in\mathbb{N}}\langle P_\Omega P_\psi h_n,P_\Omega P_\psi h_n\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}\\&=\sum_{n\in\mathbb{N}}\|P_\Omega P_\psi h_n\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}\\&=\|P_\Omega P_\psi\|^2_{HS}\\&=\frac{|\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}<\infty.\end{aligned}$$

Therefore, T is a trace class operator with $Tr(T)=\frac{|\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}$.

Now as $\{\phi_n\}_{n\in\mathbb{N}}$ is an ONS, from equation (15), it follows that $\left\{W^M_\psi\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\}_{n\in\mathbb{N}}$ is an ONS in $L^2(\mathbb{R}^+\times\mathbb{R},dadb)$.

Hence, we have:

$$\sum_{n\in\wedge}\left\langle P_{\Omega}W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right),W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}$$
$$=\sum_{n\in\wedge}\left\langle P_{\psi}P_{\Omega}P_{\psi}\left(W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right),W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}$$
$$=\sum_{n\in\wedge}\left\langle T\left(W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right),W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}$$
$$\leq\sum_{n\in\mathbb{N}}\left\langle T\left(W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right),W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}$$
$$=Tr(T)=\frac{|\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}.$$

For each n ∈ ∧, we have:

$$\left\langle P_{\Omega}W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right),W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}$$
$$=\left\langle \chi_{\Omega}W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right),W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}$$
$$=1-\left\langle \chi_{\Omega^c}W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right),W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}$$
$$\geq 1-\left\|\chi_{\Omega^c}W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\|_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}.$$

Thus, we have:

$$\sum_{n\in\wedge}\left(1-\left\|\chi_{\Omega^c}W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\|_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}\right)\leq\frac{|\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}.$$

This proves the theorem.

The theorem below shows that, if the LCWT of each member of an ONS are $\epsilon-$ concentrated in a set of finite measure then the sequence is necessarily finite. The theorem also gives an upper bound of the cardinality of the so proved finite sequence.

Theorem 5. 3. Let $s,\epsilon>0$ such that $\epsilon<1$. Let $G_s=\{(a,b)\in\mathbb{R}^+\times\mathbb{R}:\ a^2+b^2\leq s^2\}$ and ψ is an ALCW. Also let $\wedge\subset\mathbb{N}$ be non-empty and $\{\phi_n\}_{n\in\wedge}\subset L^2(\mathbb{R})$ be an ONS. If $W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)$ is ϵ –concentrated in G_s for all $n\in\wedge$, then $\wedge$ is finite and:

$$Card(\wedge)\leq\frac{s^2\|\psi\|^2_{L^2(\mathbb{R})}}{4|B|C_{\psi,M}(1-\epsilon)},\tag{25}$$

where $Card(\wedge)$ denotes the cardinality of $\wedge$.

Proof: Applying **Theorem 5. 2**, we have:

$$\sum_{n\in\wedge}\left(1-\left\|\chi_{G_s^c}W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\|_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)}\right)\leq\frac{|G_s|\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}.$$

Again, since for each $W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)$ is $\epsilon-$ concentrated in G_s, we have:

$$\left\|\chi_{G_s^c} W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right\|_{L^2(\mathbb{R}^+\times\mathbb{R},dadb)} \leq \epsilon.$$

Therefore, it follows that:

$$\sum_{n\in\wedge}(1-\epsilon) \leq \frac{|G_s|\|\psi\|_{L^2(\mathbb{R})}^2}{2\pi|B|C_{\psi,M}},$$

$$\text{i. e.}, Card(\wedge)(1-\epsilon) \leq \frac{|G_s|\|\psi\|_{L^2(\mathbb{R})}^2}{2\pi|B|C_{\psi,M}}.$$

Thus, $Card(\wedge)$ is finite and using $|G_s| = \frac{\pi s^2}{2}$, we obtain:

$$Card(\wedge) \leq \frac{s^2\|\psi\|_{L^2(\mathbb{R})}^2}{4|B|(1-\epsilon)C_{\psi,M}}.$$

The proof is complete.

Corollary 5. 1. Let $\boldsymbol{p>0},\ \boldsymbol{R>0}$ and $\boldsymbol{\psi}$ is an ALCW. Also let $\wedge \subset \mathbb{N}$, be non-empty and $\{\boldsymbol{\phi_n}\}_{\boldsymbol{n}\in\wedge} \subset \boldsymbol{L^2}(\mathbb{R})$ be an ONS. Then $\wedge$ is finite if $\left\{\boldsymbol{\rho_p}\left(\boldsymbol{W_{\psi}^{M}}\left(\frac{\boldsymbol{\phi_n}}{\sqrt{\boldsymbol{2\pi|B|C_{\psi,M}}}}\right)\right)\right\}_{\boldsymbol{n}\in\wedge}$ is uniformly bounded. Moreover, if it is uniformly bounded by $\boldsymbol{R}$, then:

$$Card(\wedge) \leq \frac{2^{\frac{4}{p}+1}R^2\|\psi\|_{L^2(\mathbb{R})}^2}{4|B|C_{\psi,M}}.$$

Proof: Since for each $n \in \wedge$, $\rho_p\left(W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right) \leq R$, and thus:

$$\begin{aligned}
&\int_{|(a,b)|\geq R2^{\frac{2}{p}}}\left|\left(W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)(a,b)\right|^2 dadb \\
&= \int_{|(a,b)|\geq R2^{\frac{2}{p}}} |(a,b)|^{-p}|(a,b)|^p\left|\left(W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)(a,b)\right|^2 dadb \\
&\leq \frac{1}{\left(R2^{\frac{2}{p}}\right)^p}\int_{\mathbb{R}^+\times\mathbb{R}} |(a,b)|^p\left|\left(W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)(a,b)\right|^2 dadb \\
&\leq \frac{1}{4}.
\end{aligned}$$

Thus, it follows that, for each $n \in\wedge, W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)$ is $\frac{1}{2}-$concentrated in:

$$G_{R2^{\frac{2}{p}}} = \left\{(a,b) \in \mathbb{R}^+\times\mathbb{R}\colon |(a,b)| < R2^{\frac{2}{p}}\right\}.$$

Thus, from **Theorem 5. 3**, it follows that $\wedge$ is finite and:

$$Card(\wedge) \leq \frac{\left(R2^{\frac{2}{p}}\right)^2\|\psi\|_{L^2(\mathbb{R})}^2}{4|B|\left(1-\frac{1}{2}\right)C_{\psi,M}},$$

$$\text{i. e.}, Card(\wedge) \leq \frac{2^{\frac{4}{p}+1}R^2\|\psi\|_{L^2(\mathbb{R})}^2}{4|B|C_{\psi,M}}.$$

Thus, the proof is complete.

Lemma 5. 1. Let $\boldsymbol{p>0},\ \boldsymbol{\psi}$ is an ALCW and $\{\boldsymbol{\phi_n}\}_{\boldsymbol{n}\in\mathbb{N}} \subset \boldsymbol{L^2}(\mathbb{R})$ be an ONS, then $\exists\ \boldsymbol{m_0} \in \mathbb{Z}$ for which:

$$\rho_p\left(W_{\psi}^{M}\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right) \geq 2^{m_0}, \forall\ n \in \mathbb{N}.$$

Proof: Define $P_m = \left\{ n \in \mathbb{N}\colon \rho_p\left(W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right) \in [2^{m-1}, 2^m)\right\}$, for each $m \in \mathbb{Z}$.

Then for each $n \in P_m$, we get:

$$\int_{\mathbb{R}^+\times\mathbb{R}} |(a,b)|^p \left| \left(W_\psi^M \left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right)(a,b) \right|^2 dadb < 2^{mp}.$$

Now,

$$\begin{aligned}\int_{|(a,b)|\geq 2^{m+\frac{2}{p}}} &\left| \left(W_\psi^M \left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right)(a,b) \right|^2 dadb \\ &\leq \frac{1}{2^{mp+2}} \int_{\mathbb{R}^+\times\mathbb{R}} |(a,b)|^p \left| \left(W_\psi^M \left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right)(a,b) \right|^2 dadb \\ &\leq \frac{1}{2^{mp+2}} \left\{ \rho_p \left(W_\psi^M \left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \right\}^p .\end{aligned}$$

This gives,

$$\int_{|(a,b)|\geq 2^{\frac{2}{p}+m}} \left| \left(W_\psi^M \left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right)(a,b) \right|^2 dadb \leq \frac{1}{4}.$$

Thus, it follows that, for each $n \in P_m$, $W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)$ is $\frac{1}{2}$ −concentrted on the set:

$$G_{2^{m+\frac{2}{p}}} = \left\{ (a,b) \in \mathbb{R}^+ \times \mathbb{R}\colon |(a,b)| < 2^{m+\frac{2}{p}} \right\}.$$

Therefore, P_m is finite and:

$$Card(P_m) \leq \frac{2^{2m+\frac{4}{p}+1} \|\psi\|_{L^2(\mathbb{R})}^2}{4|B|C_{\psi,M}}, \text{for all} \quad m \in \mathbb{Z}.$$

Letting $m \to -\infty$, we get:

$$\lim_{m\to-\infty} Card(P_m) = 0.$$

Hence, $\exists\ m_0 \in \mathbb{Z}$ such that for all $m < m_0$, P_m are empty sets. Therefore, $\rho_p\left(W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right) \geq 2^{m_0}, \forall\ n \in \mathbb{N}$.

Theorem 5. 4. (Shapiro's Dispersion theorem). Let $\boldsymbol{\psi}$ be an ALCW and $\{\boldsymbol{\phi_n}\}_{\boldsymbol{n}\in\mathbb{N}} \subset \boldsymbol{L^2}(\mathbb{R})$ be an ONS, then for every $\boldsymbol{p} > \boldsymbol{0}$ and non-empty finite $\wedge \subset \mathbb{N}$,

$$\sum_{n\in\wedge} \left\{ \rho_p \left(W_\psi^M \left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \right\}^p \geq \frac{(Card(\wedge))^{\frac{p}{2}+1}}{2^{p+1}} \left(\frac{3|B|C_{\psi,M}}{2^{\frac{4}{p}+2} \|\psi\|_{L^2(\mathbb{R})}^2} \right)^{\frac{p}{2}}.$$

Proof. Let m_0 be an integer defined in the **Lemma 5. 1**. Let $k \in \mathbb{Z}$ such that $k \geq m_0$. Define $Q_k = \bigcup_{m=m_0}^k P_m$. Then we have:

$$\begin{aligned} Card(Q_k) = \sum_{m=m_0}^{k} Card(P_m) &\leq \sum_{m=m_0}^{k} \frac{2^{2m+\frac{4}{p}+1} \|\psi\|_{L^2(\mathbb{R})}^2}{4|B|C_{\psi,M}} \\ &= \frac{2^{\frac{4}{p}+1} \|\psi\|_{L^2(\mathbb{R})}^2}{4|B|C_{\psi,M}} \sum_{m=m_0}^{k} 2^{2m} \\ &\leq \frac{2^{\frac{4}{p}+1} \|\psi\|_{L^2(\mathbb{R})}^2}{4|B|C_{\psi,M}} \frac{2^{2k+2}}{3}, \end{aligned}$$

$$\text{i. e.}, Card(Q_k) \leq \frac{2^{\frac{4}{p}+1} \|\psi\|_{L^2(\mathbb{R})}^2}{3|B|C_{\psi,M}} 2^{2k}.$$

Let $C = \frac{2^{\frac{4}{p}+2} \ \|\psi\|^2_{L^2(\mathbb{R})}}{3|B|C_{\psi,M}}$. Then $Card(Q_k) \leq \frac{C}{2}\ 2^{2k}$. If $Card(\wedge) > 2^{2(m_0+1)}C$, then $\frac{1}{2\log 2}\log\left(\frac{Card(\wedge)}{C}\right) > m_0 + 1$. Let us choose an integer $k > m_0 + 1$ such that:

$$k - 1 \leq \frac{1}{2\log 2}\log\left(\frac{Card(\wedge)}{C}\right) < k.$$

Then, it results in:

$$C2^{2(k-1)} \leq Card(\wedge) < C2^{2k}.$$

Thus, we have:

$$Card(Q_{k-1}) \leq \frac{C}{2}2^{2(k-1)} \leq \frac{Card(\wedge)}{2}.$$

This shows that at least half of the elements of $\wedge$ are not in Q_{k-1}. Thus, we have:

$$\begin{aligned}\sum_{n\in\wedge}\left\{\rho_p\left(W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)\right\}^p &\geq \sum_{n\in\wedge\setminus Q_{k-1}}\left\{\rho_p\left(W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)\right\}^p \\ &\geq \frac{Card(\wedge)}{2}2^{(k-1)p} \\ &= \frac{Card(\wedge)}{2^{p+1}}2^{kp}.\end{aligned}$$

Since, $Card(\wedge) \leq C2^{2k}$, we have $\left(\frac{Card(\wedge)}{C}\right)^{\frac{p}{2}} \leq 2^{kp}$.
Therefore,

$$\sum_{n\in\wedge}\left\{\rho_p\left(W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)\right\}^p \geq \frac{(Card(\wedge)\ )^{\frac{p}{2}+1}}{2^{p+1}}\left(\frac{1}{C}\right)^{\frac{p}{2}}.$$

Again, if $Card(\wedge) \leq C2^{2(m_0+1)}$, then:

$$\sum_{n\in\wedge}\left\{\rho_p\left(W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)\right\}^p \geq\ Card(\wedge)2^{m_0p}\ ,(\text{using Lemma } 5.1).$$

Now, $Card(\wedge) \leq C2^{2(m_0+1)}$ implies $\frac{1}{2^p}\ \left(\frac{Card(\wedge)}{C}\right)^{\frac{p}{2}} \leq 2^{m_0p}$. Thus, we have:

$$\sum_{n\in\wedge}\left\{\rho_p\left(W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)\right\}^p \geq \frac{(Card(\wedge)\ )^{\frac{p}{2}+1}}{2^{p}}\left(\frac{1}{C}\right)^{\frac{p}{2}}.$$

Hence, for any non-empty finite $\wedge \subset \mathbb{N}$, we have:

$$\sum_{n\in\wedge}\left\{\rho_p\left(W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)\right\}^p \geq \frac{(Card(\wedge)\ )^{\frac{p}{2}+1}}{2^{p+1}}\left(\frac{1}{C}\right)^{\frac{p}{2}}.$$

Therefore, putting the value of C we get:

$$\sum_{n\in\wedge}\left\{\rho_p\left(W_\psi^M\left(\frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}}\right)\right)\right\}^p \geq \frac{(Card(\wedge))^{\frac{p}{2}+1}}{2^{p+1}}\left(\frac{3|B|C_{\psi,M}}{2^{\frac{4}{p}+2}\ \|\psi\|^2_{L^2(\mathbb{R})}}\right)^{\frac{p}{2}}.$$

This completes the proof.

6. Conclusions

We have proposed a novel time-frequency analyzing tool, namely LCWT, which combines the advantages of the LCT and the WT and offers time and linear canonical domain spectral information simultaneously in the time LCT-frequency plane. We have studied its properties like inner product relation, reconstruction formula and also characterized its range. We also gave a lower bound of the measure of essential support of the LCWT via UP of Donoho-Stark. Finally, we have studied the Shapiro's mean dispersion theorem associated with the LCWT.

Author Contributions

B. Gupta and A.K. Verma proposed the problem and C. Cattani verified that proposed problem is well defined. B. Gupta formally wrote the proofs in consultation with A.K. Verma and C. Cattani. Examples were constructed by B. Gupta and C. Cattani. Entire manuscript is checked, reviewed and revised by all authors. C. Cattani supervised the entire work.

Acknowledgments

We are thankful to the anonymous reviewers for their valuable comments.

Conflict of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

Funding

This work is partially supported by DST SERB FILE NO. MTR/2021/000907, and UGC File No. 16-9(June 2017)/2018(NET/CSIR), New Delhi, India.

Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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How to cite this article: Gupta B., Verma A.K., Cattani C. A New Class of Linear Canonical Wavelet Transform, *J. Appl. Comput. Mech.*, 10(1), 2024, 64–79. https://doi.org/10.22055/jacm.2023.44326.4196