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Research Paper

New Variational Principles for Two Kinds of Nonlinear Partial Differential Equation in Shallow Water

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Abstract. Variational principles are very important for a lot of nonlinear problems to be analyzed theoretically or solved numerically. By the popular semi-inverse method and designing trial-Lagrange functionals skillfully, new variational principles are constructed successfully for the Kuramoto-Sivashinsky equation and the Coupled KdV equations, respectively, which can model a lot of nonlinear waves in shallow water. The established variational principles are also proved correct. The procedure reveals that the used technologies are very powerful and applicable, and can be extended to other nonlinear physical and mathematical models.

Keywords: Variational principle, Calculus of variations, Kuramoto-Sivashinsky equation, Coupled KdV equations.

1. Introduction

Nonlinear partial differential equations (PDE) are usually used to model different phenomena, ranging from mechanics to biology, physics, chemistry, ocean and meteorology so on [1-6]. And many mathematical methods have been proposed and developed to get solutions [5-29] of nonlinear PDEs. When contrasted with other approximate analytical methods, variational methods such as Ritz technique [12] and variational iteration method [13-17] show a lot of advantages. Because variational principles are the theoretical basis for many kinds of variational methods, it is a very important task to seek explicit variational formulations for nonlinear and complex PDEs.

In the past decades, various numerical methods have been introduced to tackle the challenges of solving the nonlinear shallow water equations. Durgun et al. [13] addressed the time-fractional non-linear partial differential equations with proportional delays using the Fractional Variational Iteration Method, incorporating the modified Riemann-Liouville fractional derivative. The numerical solutions derived through this method demonstrate superior accuracy compared to those obtained via the Homotopy Perturbation Method and Differential Transform Method, when applied to the same data set and approximation order. The Variational Iteration Method (VIM), which was introduced by He in 1999 [14], has been extensively utilized by numerous researchers for developing approximate solutions to a broad range of scientific and engineering models [15, 16]. The foundational techniques employed by a majority of researchers are grid-based approaches, like the finite difference method (FDM) [18]. Following certain adaptations, the Fractional Variational Iteration Method (FVIM) has been employed in the study of fractional differential equations by He and other researchers [19]. He [19] contrasted the traditional Variational Iteration Method with the Fractional Variational Iteration Method. They introduced the fractional complex transform as a technique to transform a fractional differential equation into its corresponding differential form, thereby facilitating the straightforward construction of its variational iteration algorithm. In recent years, many scholars have shifted their focus towards employing the recently introduced meshless methods to solve the nonlinear shallow water equations, such as natural element method [20], smoothed particle hydrodynamics [21] and etc. From the results of these numerical simulations, it is evident that the method boasts broad applicability, commendable accuracy, and stability. However, its application to the study of nonlinear shock waves with discontinuous issues remains unexplored. Recently, many scientists have made great success for constructing different kinds of variational principles in various fields such as fluid dynamics, ocean, meteorology, mathematical biology, solid state physics, and plasma physics [24-37]. In this paper, we will use the semi-inverse method [24-30] to establish new variational principles for the Kuramoto-Sivashinsky equation and the Coupled KdV equations, respectively, which contains various solitary waves [28, 29].

2. Variational Principle for the Kuramoto-Sivashinsky Equation

The Kuramoto–Sivashinsky equation (also called the KS equation) is a fourth-order nonlinear partial differential equation. It was derived by Yoshiki Kuramoto [38-39] and Gregory Sivashinsky [40] as a model for phase turbulence in reaction–diffusion

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systems and to model the diffusive–thermal instabilities in a laminar flame front. The Kuramoto–Sivashinsky equation is known for its chaotic behavior [41-42]. The KS equation is given as follows:

$$u_t + uu_x + \mu(t)u_{xx} + \lambda(t)u_{xxxx} = 0 \tag{1}$$

where $\mu(t)$ and $\lambda(t)$ are time-varying coefficients. Equation (1) can be rewritten in the following equivalent form:

$$u_t + (\frac{u^2}{2} + \mu u_x + \lambda u_{xxx})_x = 0 \tag{2}$$

A potential function Φ can be introduced, defined as follows:

$$\begin{cases} \Phi_x = u \\ \Phi_t = -(\frac{u^2}{2} + \mu u_x + \lambda u_{xxx}) \end{cases} \tag{3}$$

If Eqs. (3) are adopted, the nonlinear partial differential equation (2) will be automatically satisfied. The objective of this section is to establish a new variational formula, whose Euler-Lagrange equations satisfy Eqs. (2) and (3). To achieve this goal, the semi-inverse method proposed by He [12-18] will be employed here to construct the generalized variational formula for the Kuramoto-Sivashinsky Equation (1) as:

$$J(u,\Phi) = \iint L(u, u_x, u_{xxx}, \Phi_t, \Phi_x) dxdt \tag{4}$$

where L in Eq. (4) is the trial-Lagrange functional, and its specific form is tentatively defined as follows:

$$L = u\Phi_t + (\frac{u^2}{2} + \mu u_x + \lambda u_{xxx})\Phi_x + F \tag{5}$$

Function F in Eq. (5) is an undetermined function that depends solely on u and its derivatives, independent of Φ. There are various choices available to construct the trial-Lagrange functional, and relevant examples can be found in [12-20]. The trial-Lagrange functional represented by Eq. (5) possesses an advantage. By considering its critical condition with respect to Φ,

$$\frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial x}(\frac{\partial L}{\partial \Phi_x}) - \frac{\partial}{\partial t}(\frac{\partial L}{\partial \Phi_t}) = 0 \tag{6}$$

Equation (1) or Eq. (2) can be automatically derived. Now, considering the critical condition of the L with respect to u, we can obtain:

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial L}{\partial u_x}) - \frac{\partial}{\partial x^3}(\frac{\partial L}{\partial u_{xxx}}) + \frac{\delta F}{\delta u} = 0 \tag{7}$$

In Eq. (7), where $\delta F / \delta u$ is called He's variational derivative [12-18] of F with respect to u.

Substituting the trial-Lagrange function Eq. (5) into Eq. (7), we obtain:

$$\Phi_t + u\Phi_x - \mu\Phi_{xx} - \lambda\Phi_{xxxx} + \frac{\delta F}{\delta u} = 0 \tag{8}$$

The purpose of constructing such an undetermined function F is to make Eq. (8) equivalent to the field equations (3). Accordingly, substituting the field equations (3) into Eq. (8), we get: $\delta F / \delta u = -u^2 / 2 + 2\mu u_x + 2\lambda u_{xxx}$. Since we cannot identify F successfully, it indicates that the current trial-Lagrange functional L is not appropriate, so we have to modify the trial-Lagrange function. The modified L is represented as follows:

$$L = Au\Phi_t + B\Phi_x\Phi_t + (\frac{u^2}{2} + \mu u_x + \lambda u_{xxx})\Phi_x + F \tag{9}$$

Then, we calculate the variational operations with respect to Φ and u on Eq. (9), respectively:

$$\frac{\delta L}{\delta \Phi}: \quad -(A+2B)u_t - uu_x - \mu u_{xx} - \lambda u_{xxxx} + \frac{\delta F}{\delta \Phi} = 0 \tag{10}$$

$$\frac{\delta L}{\delta u}: \quad A\Phi_t + u\Phi_x - \mu\Phi_{xx} - \lambda\Phi_{xxxx} + \frac{\delta F}{\delta u} = 0 \tag{11}$$

In view of Eq. (10) and since $\delta F / \delta \Phi = 0$, Eq. (10) can be transformed into:

$$-(A+2B)u_t - uu_x - \mu u_{xx} - \lambda u_{xxxx} = 0 \tag{12}$$

Because Eq. (12) should be identical to Eq. (1). That is:

$$A + 2B = 1 \tag{13}$$

After substituting Eqs. (3) into Eq. (11) and rearranging them, we obtain:

$$(A+1)(\mu u_x + \lambda u_{xxx}) + (\frac{A}{2} - 1)u^2 - \frac{\delta F}{\delta u} = 0 \tag{14}$$

In order to identify the unknown function F successfully, it is necessary to eliminate the terms with u_x and u_{xxx} in Eq. (14), setting the coefficients of u_x and u_{xxx} to zero, thereby determining the undetermined coefficient $A=-1$. Furthermore, from Eq. (13), it can be further determined that $B=1$. Substituting the value of A into Eq. (14), we obtain:

$$\frac{\delta F}{\delta u}=-\frac{3}{2}u^2 \tag{15}$$

As a result, F can be identified successfully as:

$$F=-\frac{u^3}{2} \tag{16}$$

Substituting the expression of F from Eq. (16) into Eq. (9) and then using Eq. (9) into Eq. (4), the new variational principle for the one-dimensional Kuramoto-Sivashinsky equation (1) is obtained, which read:

$$J(u,\Phi)=\iint\{\Phi_x\Phi_t-u\Phi_t+[u^2/2+\mu(t)u_x+\lambda(t)u_{xxx}]\Phi_x-u^3/2\}dxdt \tag{17}$$

The variational principle in Eq. (17) is subject to the constraint $\Phi_x=u$. We will verify the correctness of the obtained generalized variational principle for the Kuramoto-Sivashinsky equation. By taking the first-order variation of the functional, Eq. (17), with respect to Φ and u, and applying the stationary condition, we can obtain the two equations (18) and (19):

$$\delta\Phi:\ \ u_t-(u^2/2+\mu u_x+\lambda u_{xxx})_x-2\Phi_{xt}=0 \tag{18}$$

$$\delta u:\ \ -\Phi_t+u\Phi_x-\mu\Phi_{xx}-\lambda\Phi_{xxxx}-\frac{3}{2}u^2=0 \tag{19}$$

Equations (18) and (19), also called as the Euler-Lagrange equations, are derived from the functional Eq. (17). Substituting $\Phi_x=u$ into Eq. (18) leads to the original Kuramoto-Sivashinsky Eq. (1), obviously. Additionally, by substituting $\Phi_x=u$ into Eq. (19), we can get that: $-\Phi_t-\mu u_x-\lambda u_{xxx}-u^2/2=0$, which is identical to the second relation of equations (3). This confirms that the Euler-Lagrange equations (18) and (19) obtained from the new variational principle formula (17) are equivalent to the second equation in the field equations (3) and also equivalent to Eq. (1). Hence, we proved the obtained generalized variational principle for the Kuramoto-Sivashinsky equation (17) is correct.

3. Variational Principles for the Coupled KdV Equations

The coupled KdV equation is a system of two nonlinear partial differential equations that describes the evolution of two interacting waves. It was first introduced by Hirota and Satsuma [44]. Since its inception, the equation has been studied extensively by mathematicians and physicists, researchers continue to explore the coupled KdV equation's dynamics and implications, making it an essential tool for understanding the behavior of nonlinear waves in various physical contexts [44, 45]. The Coupled KdV equations are given as following:

$$\begin{cases}u_t+(u_{xx}+u^2+\frac{k}{2}v^2+kuv)_x=0\\ v_t+(v_{xx}+v^2+\frac{k}{2}u^2+kuv)_x=0\end{cases} \tag{20}$$

For convenience, we temporarily consider the case when k = 2. Then, the above coupled equations (20) becomes:

$$\begin{cases}u_t+(u_{xx}+u^2+v^2+2uv)_x=0\\ v_t+(v_{xx}+v^2+u^2+2uv)_x=0\end{cases} \tag{21}$$

Equations (20) and (21) can be derived through symmetry constraints from the KP equation and have been applied in various technological fields such as statistical physics, plasma physics, and nonlinear fiber optics communication. Its Painlevé properties and infinite number of symmetries with respect to either the time variable t or the spatial variable y have also been studied. Two potential functions, Φ and Π, can be introduced as:

$$\begin{cases}\Phi_x=u\\ \Phi_t=-(u_{xx}+u^2+v^2+2uv)\end{cases} \tag{22}$$

$$\begin{cases}\Pi_x=v\\ \Pi_t=-(v_{xx}+v^2+u^2+2uv)\end{cases} \tag{23}$$

If we firstly adopt equations (22), then the second relation in equations (20) and (21) is automatically satisfied. The objective of this section is to establish a generalized variational formula, whose Euler-Lagrange equations satisfy the first relation in Eqs. (21) and (22). To achieve this goal, the semi-inverse method, proposed by He [24-27], will be employed here to construct the generalized variational formula for the coupled KdV Eq. (21) as:

$$J(u,v,\Phi)=\iiint Ldxdydt \tag{24}$$

where L is the trial-Lagrange functional, defined by the following formula:

$$L=v\Phi_t+(v_{xx}+v^2+u^2+2uv)\Phi_x+F(u,v) \tag{25}$$

and F in Eq. (25) is the unknown function of u, v, and their derivatives. There are various choices available to construct the trial-Lagrange functional, and relevant examples can be found in references [12-20]. The advantage of the trial-Lagrange functional represented by Eq. (25) is its critical condition with respect to Φ:

$$\frac{\partial L}{\partial \Phi}-\frac{\partial}{\partial t}(\frac{\partial L}{\partial \Phi_t})-\frac{\partial}{\partial x}(\frac{\partial L}{\partial \Phi_x})+\frac{\partial^2}{\partial x^2}(\frac{\partial L}{\partial \Phi_{xx}})=0 \tag{26}$$

The first one in equations (21) can be derived. Now, considering the critical conditions of L with respect to both u and v, and taking into account the presence of derivatives of u and v in L, we obtain:

$$\frac{\partial L}{\partial u}+\frac{\delta F}{\delta u}=0 \tag{27}$$

$$\frac{\partial L}{\partial v}+\frac{\partial^2}{\partial x^2}(\frac{\partial L}{\partial v_{xx}})+\frac{\delta F}{\delta v}=0 \tag{28}$$

where $\delta F/\delta u$ and $\delta F/\delta v$ are referred to as the He's variational derivatives [24-27, 32, 33] of F with respect to u and v, respectively, and they are defined as follows:

$$\frac{\delta F}{\delta u}=\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\frac{\partial F}{\partial u_x}-\frac{\partial}{\partial t}\frac{\partial F}{\partial u_t}+\frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial u_{xx}}+\cdots,$$

$$\frac{\delta F}{\delta v}=\frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\frac{\partial F}{\partial v_x}-\frac{\partial}{\partial t}\frac{\partial F}{\partial v_t}+\frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial v_{xx}}+\cdots.$$

After substituting Eq. (25) into Eqs. (27) and (28), and going through the derivation, we obtain:

$$\frac{\delta L}{\delta v}: \quad \Phi_t+\Phi_{xxx}+2v\Phi_x+2u\Phi_x+\frac{\delta F}{\delta v}=0 \tag{29}$$

$$\frac{\delta L}{\delta u}: \quad 2u\Phi_x+2v\Phi_x+\frac{\delta F}{\delta u}=0 \tag{30}$$

The purpose of constructing the unknown function F is to make Eqs. (29) and (30) equivalent to the field equations in Eqs. (23). Therefore, we substitute the first and second in Eqs. (22) into Eqs. (29) and (30), respectively, resulting in:

$$\frac{\delta F}{\delta u}=-2u^2-2uv \tag{31}$$

$$\frac{\delta F}{\delta v}=-u^2+v^2 \tag{32}$$

As a result, the expression for the unknown function F is obtained:

$$F=-u^2v+\frac{v^3}{3}-\frac{2u^3}{3} \tag{33}$$

By substituting Eq. (33) into Eq. (25), we obtain the new trial-Lagrange functional as:

$$L=v\Phi_t+(v_{xx}+v^2+u^2+2uv)\Phi_x-(2u^3+3u^2v-v^3)/3 \tag{34}$$

By substituting Eq. (34) into Eq. (26) and then using Eq. (26) in Eq. (25), we obtain the variational principle for the coupled KdV equations:

$$J(u,v,\Phi)=\iint[v\Phi_t+(v_{xx}+v^2+u^2+2uv)\Phi_x-(2u^3+3u^2v-v^3)/3]dxdt \tag{35}$$

Below, we will verify the correctness of the obtained variational principle (35) for the coupled KdV nonlinear equations by taking the stationary conditions of the functional Eq. (35) with respect to Φ, u, and v, respectively. This process will lead to the following Euler-Lagrange equations:

$$\begin{cases}\delta\Phi: & -v_t-(v_{xx}+v^2+u^2+2uv)_x=0\\ \delta u: & (2u+2v)\Phi_x-(2u^2+2uv)=0\\ \delta v: & \Phi_t+\Phi_{xxx}+(2u+2v)\Phi_x-u^2+v^2=0\end{cases} \tag{36}$$

In Eqs. (36), $\delta\Phi$, δu, and δv represent the first-order variation of Φ, u, and v, respectively. The first one in Eq. (36) is identical to the first one in Eq. (20), Eq. (21) obviously. From the first one in Eq. (22), we get $\Phi_x=u$, and by substituting $\Phi_t+u_{xx}+u^2+2uv+v^2=0$, we obtain $\Phi_t=-(u_{xx}+u^2+v^2+2uv)$. This shows that the third one in Eq. (36) is identical to the second one in Eq. (22). Hence, it is demonstrated that the obtained generalized variational principle for the coupled KdV equations is correct.

If we introduce the special function Π from Eqs. (23), then the new trial-Lagrange functional L_1 is defined as follows:

$$L_1=u\Pi_t+(u_{xx}+u^2+v^2+2uv)\Pi_x+G(u,v) \tag{37}$$

The function G is an unknown function of u, v, and their derivatives. Various choices exist for constructing the trial-Lagrange functional, and relevant examples can be found in references [12-20]. The advantage of the trial-Lagrange functional represented by Eq. (37) is its critical condition with respect to Π:

$$\frac{\partial L_1}{\partial \Pi}-\frac{\partial}{\partial x}\left(\frac{\partial L_1}{\partial \Pi_x}\right)-\frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial \Pi_t}\right)=0 \tag{38}$$

The second one in Eqs. (21) can be derived. Now, considering the critical conditions of L_1 with respect to both u and v, we get:

$$\frac{\delta L_1}{\delta u}:\quad \frac{\partial L_1}{\partial u}+\frac{\partial^2}{\partial x^2}\left(\frac{\partial L_1}{\partial u_{xx}}\right)+\frac{\delta G}{\delta u}=0 \tag{39}$$

$$\frac{\delta L_1}{\delta u}:\quad \frac{\partial L_1}{\partial v}+\frac{\delta G}{\delta v}=0 \tag{40}$$

where $\delta G/\delta u$ and $\delta G/\delta v$ are referred to as the He's variational derivatives [24-27, 32, 33] of G with respect to u and v, respectively, and they are defined as follows:

$$\frac{\delta G}{\delta u}=\frac{\partial G}{\partial u}-\frac{\partial}{\partial x}\frac{\partial G}{\partial u_x}-\frac{\partial}{\partial t}\frac{\partial G}{\partial u_t}+\frac{\partial^2}{\partial x^2}\frac{\partial G}{\partial u_{xx}}+\cdots,$$

$$\frac{\delta G}{\delta v}=\frac{\partial G}{\partial v}-\frac{\partial}{\partial x}\frac{\partial G}{\partial v_x}-\frac{\partial}{\partial t}\frac{\partial G}{\partial v_t}+\frac{\partial^2}{\partial x^2}\frac{\partial G}{\partial v_{xx}}+\cdots.$$

From Eqs. (37), (39) and (40), we obtain:

$$\Pi_t+2(u+v)\Pi_x+\Pi_{xxx}+\frac{\delta G}{\delta u}=0 \tag{41}$$

$$2v(u+v)+\frac{\delta G}{\delta v}=0 \tag{42}$$

The purpose of constructing the unknow function G is to make Eqs. (41) and (42) equivalent to the field Eqs. (23). Therefore, we substitute Eq. (23) into Eqs. (41) and (42), resulting in:

$$\frac{\delta G}{\delta u}=u^2-v^2 \tag{43}$$

$$\frac{\delta G}{\delta v}=-2v(u+v) \tag{44}$$

Hence, we obtain:

$$G=(u^3-3uv^2-2v^3)/3 \tag{45}$$

By substituting Eq. (45) into Eq. (38) and then using Eq. (38) in Eq. (37), we obtain the variational principle for the coupled KdV nonlinear equations:

$$J(u,v,\Pi)=\iint[u\Pi_t+(u_{xx}+u^2+v^2+2uv)\Pi_x+(u^3-3uv^2-2v^3)/3]dxdt \tag{46}$$

If we take the first variations of the functional Eq. (46) with respect to Π, u, and v, respectively, and apply the stationary condition, we obtain the following form of the Euler-Lagrange equations:

$$\begin{cases}\delta\Pi:\quad -u_t-(u_{xx}+u^2+v^2+2uv)_x=0\\ \delta u:\quad \Pi_t+2(u+v)\Pi_x+\Pi_{xxx}+u^2-v^2=0\\ \delta v:\quad 2\Pi_x(u+v)-2v(u+v)=0\end{cases} \tag{47}$$

From the third one in the above equations, we get: $\Pi_x=v$, and by substituting it into the second one in Eq. (47), we obtain: $\Pi_t=-(v_{xx}+v^2+u^2+2uv)$. Thus, it is verified that the obtained Euler-Lagrange equations are equivalent to the field equations represented by Eqs. (22) and the second one in Eqs. (23). In conclusion, in this section, we have used He's semi-inverse method [24-27, 32, 33] to derive the generalized variational principle for the coupled KdV nonlinear equation (46). Furthermore, through the consistency between the Euler-Lagrange equations derived from the functional equations and the original equations, the correctness of the obtained generalized variational principle has been demonstrated.

4. Conclusion

In the second and third parts of this paper, new variational principles have been successfully constructed for the Kuramoto-Sivashinsky equation and the coupled KdV equations, respectively, by the semi-inverse method [24-30] and designing trial-Lagrange functionals skillfully. Moreover, the obtained variational principles have proved to be correct by minimizing the corresponding functionals. From the above analysis, it is concluded that the variational principle for the coupled KdV equations studied in this paper have different integral formulations, from which the same nonlinear PDEs can be derived. The procedure also reveals that the semi-inverse method [24-30] is effective and powerful. According to the obtained variational principles, we can study possible solution structures for solitary waves. Furthermore, new variational principles can also provide hints for numerical algorithms, equations (1) and (21) can be solved numerically by variational-based methods. Our future work will focus on the dynamics of soliton in the Kuramoto-Sivashinsky equation and the coupled KdV equations, by variational approximation method and choosing an appropriate variational principle established in this paper.

Author Contributions

All authors have significant contributions in this paper. The details are as following. Conceptualization, Xiao-Qun Cao; Methodology, Xiao-Qun Cao, Meng-Ge Zhou and Ke-Cheng Peng; Writing – original draft, Xiao-Qun Cao and Meng-Ge Zhou; Writing – review & editing, Meng-Ge Zhou, Ya-Nan Guo and Si-Hang Xie. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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