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A Dual Lagrange Multiplier Approach for the Dynamics of the Mechanical Systems

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Abstract. The variational iteration method, a powerful tool for solving nonlinear oscillators with initial conditions, has been expanded to accommodate two-point boundary conditions. In this modification, two Lagrange multipliers are introduced, and their identification process mirrors that of the conventional approach. A generalized equation is provided for a category of highly nonlinear mechanical systems, followed by three illustrative examples derived from this generalized equation. These examples serve to demonstrate the effectiveness of the method. The solutions produced by this modified approach not only show remarkable agreement with numerical results but also exhibit superior accuracy when compared to outcomes obtained through other established methods.

Keywords: Amplitude-frequency relationship, Lagrange multiplier, variational principle, nonlinear conservative oscillator.

1. Introduction

Nonlinear oscillations, characterized by complex and often chaotic behaviors, have advanced applications across diverse scientific and engineering realms. From the intricate dynamics of biological systems and structural vibrations in engineering to the design of secure communication systems and the exploration of nonlinear effects in optics and lasers, the study of nonlinear oscillations contributes crucial insights for understanding, controlling, and innovating in complex dynamic systems across multiple disciplines [1-3]. Understanding the dynamics of nonlinear oscillations is essential because it provides insights into the complex behaviors exhibited by a wide range of dynamic systems. Unlike linear systems, where responses are directly proportional to inputs, nonlinear systems can display intricate phenomena such as chaos, bifurcations, and strange attractors. This understanding is crucial in various fields, including physics, engineering, biology, and control theory, enabling researchers and engineers to predict, control, and harness the behavior of complex systems. It facilitates advancements in fields like structural design, biological modeling, communication systems, and materials science, contributing to innovation and the development of more robust and efficient technologies.

Many effective methods have been proposed to solve nonlinear oscillatory problems, for example, the proximal policy optimization [4, 5], the multiple scale technique [6, 7], the variational approach [8-10], the homotopy perturbation method [11-14], Taylor series method [15], Adomian decomposition method [16] and so on. The energy balance method [17] and the frequency-formulation [18] are two other methods based on the variational principle and Nofel et al. [19] combined these two and found that this coupling gave superior results compared to those obtained by using the energy balance method alone, furthermore, Ma [20] suggested a modified frequency formulation based on the Hamilton principle.

The variational iteration method (VIM) was mainly introduced in 1998 [21] and has since been widely used to solve various nonlinear problems [22-27], as well as for fractal-fractional differential equations [28]. The method introduces a Lagrange multiplier to generate an appropriate correction iteration, called a correction functional. The optimal determination of this Lagrange multiplier is achieved through the application of the variational theory, which is an effective tool in engineering [29, 30]. It is worth noting that the correction functional for a nonlinear vibration system is particularly well suited to dealing with initial conditions, but no effort has yet been made for nonlinear oscillators without initial conditions. Although the method was originally developed for the problems involving initial conditions, it also works in the case of boundary conditions [31, 32].

This article extends the variational iteration method [21] to a kind of special nonlinear oscillators with boundary conditions. We modify the correction functional to include two points instead of just one point, which was considered previously [21]. We construct a general nonlinear model of a vibrating system which, under certain assumptions, reduces to various problems such as the oscillatory motion of a bead in an arranged parabola, the vibration of a conical beam, the Mathews and Lakshmanan oscillator, etc. This modification in the variational iteration method is useful to obtain frequency-amplitude association as well as fairly accurate analytical solution of general nonlinear oscillatory system of these problems. We compare the results obtained

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with this modification with the results obtained with the frequency formulation based on the energy balance method (FF-EBM) and numerical calculations to show the effectiveness and reliability of this modification. The rest of the paper is organized as follows:

In Section 2 of the paper, the fundamental concept of dual Lagrange multipliers for nonlinear oscillators is elucidated, laying the theoretical groundwork for the subsequent discussions. Moving to Section 3, the paper introduces and illustrates a novel technique applicable to a broad spectrum of oscillatory systems encountered in science and engineering. This methodology is then employed in three diverse examples showcased in the paper: firstly, the analysis of particle motion on a rotating parabola; secondly, the examination of a nonlinear oscillator pertinent to the construction industry; and finally, the application of the proposed technique to the Mathews-Lakshmanan nonlinear oscillator. Through these practical instances, the paper aims to demonstrate the versatility and efficacy of the introduced approach in tackling real-world problems across different domains.

2. Basic Idea of Dual Lagrange Multiplier for Nonlinear Oscillators

Generally, a nonlinear oscillator with displacement χ, time τ and amplitude a can be expressed as an initial problem:

$$\chi'' + f(\tau, \chi, \chi', \chi'') = 0, \quad \chi(0) = a, \qquad \chi'(0) = 0 \tag{1}$$

where $(\bullet)'$ and $(\bullet)''$, respectively, represent the first and second order time derivatives. It is well known that for a conservative oscillator with unchanged amplitude a and period T, the following property holds:

$$\chi(0) = a, \qquad \chi\left(\frac{T}{4}\right) = 0, \qquad \chi\left(\frac{T}{2}\right) = -a, \qquad \chi\left(\frac{3T}{4}\right) = 0, \qquad \chi(T) = a \tag{2}$$

This is the main reason why we can change an initial value problem into a two-point problem. We can express Eq. (1) in the form:

$$\chi'' + \Omega^2\chi + g(\tau, \chi, \chi', \chi'') = 0 \tag{3}$$

where $g(\tau, \chi, \chi', \chi'') = f(\tau, \chi, \chi', \chi'') - \Omega^2\chi$ and Ω is the nonlinear frequency of oscillator to be determined.

For Eq. (3), the variational iteration algorithm can be considered as [18]:

$$\chi_{n+1}(\tau) = \chi_n(\tau) + \int_0^{\tau} \lambda(\xi,\tau)\left[\chi_n''(\xi) + \Omega^2\chi_n(\xi) + \tilde{g}\left(\xi, \chi_n(\xi), \chi_n'(\xi), \chi_n''(\xi)\right)\right]d\xi \tag{4}$$

where λ and $\tilde{g}$ are, respectively, the Lagrange multiplier and the restricted variation. Also, n and $n+1$, respectively, refer to the current and next iteration of the technique.

The starting approximate solution is:

$$\chi_0 = a\cos\Omega\tau \tag{5}$$

Now we consider the two-point boundary conditions:

$$\chi(\alpha) = A, \qquad \chi(\beta) = B \tag{6}$$

The correction functional can be modified as:

$$\chi_{n+1}(\tau) = \chi_n(\tau) + \int_{\alpha}^{\tau} \lambda_1(\xi,\tau)\left[\chi_n''(\xi) + \Omega^2\chi_n(\xi) + \tilde{g}\left(\xi, \chi_n(\xi), \chi_n'(\xi), \chi_n''(\xi)\right)\right]d\xi + \int_{\tau}^{\beta} \lambda_2(\xi,\tau)\left[\chi_n''(\xi) + \Omega^2\chi_n(\xi) + \tilde{g}\left(\xi, \chi_n(\xi), \chi_n'(\xi), \chi_n''(\xi)\right)\right]d\xi \tag{7}$$

where $\lambda_1(\xi,\tau)$ and $\lambda_2(\xi,\tau)$ are Lagrange's multipliers described respectively on the intervals $[\alpha,\tau]$ and $[\tau,\beta]$. The initial term χ_0 is selected to comply with the criteria on both points $\tau = \alpha$ and $\tau = \beta$.

In order to elucidate the identification of the multipliers, we express Eq. (7) as:

$$\begin{aligned}\chi_{n+1}(\tau) = \chi_n(\tau) &+ \int_{\alpha}^{\tau} \lambda_1(\xi,\tau)\chi_n''(\xi)d\xi + \Omega^2\int_{\alpha}^{\tau} \lambda_1(\xi,\tau)\chi_n(\xi)d\xi + \int_{\alpha}^{\tau} \lambda_1(\xi,\tau)\tilde{g}\left(\xi, \chi_n(\xi), \chi_n'(\xi), \chi_n''(\xi)\right)d\xi \\ &+ \int_{\tau}^{\beta} \lambda_2(\xi,\tau)\chi_n''(\xi)d\xi + \Omega^2\int_{\tau}^{\beta} \lambda_2(\xi,\tau)\chi_n(\xi)d\xi + \int_{\tau}^{\beta} \lambda_2(\xi,\tau)\tilde{g}\left(\xi, \chi_n(\xi), \chi_n'(\xi), \chi_n''(\xi)\right)d\xi\end{aligned} \tag{8}$$

Integration by parts twice of the following terms yields:

$$\int_{\alpha}^{\tau} \lambda_1(\xi,\tau)\chi_n''(\xi)d\xi = \lambda_1(\tau,\tau)\chi_n'(\tau) - \lambda_1(\alpha,\tau)\chi_n'(\alpha) - \lambda_1'(\tau,\tau)\chi_n(\tau) + \lambda_1'(\alpha,\tau)\chi_n(\alpha) + \int_{\alpha}^{\tau} \lambda_1''(\xi,\tau)\chi_n(\xi)d\xi \tag{9}$$

$$\int_{\tau}^{\beta} \lambda_2(\xi,\tau)\chi_n''(\xi)d\xi = \lambda_2(\beta,\tau)\chi_n'(\beta) - \lambda_2(\tau,\tau)\chi_n'(\tau) - \lambda_2'(\beta,\tau)\chi_n(\beta) + \lambda_{21}'(\tau,\tau)\chi_n(\tau) + \int_{\tau}^{\beta} \lambda_2''(\xi,\tau)\chi_n(\xi)d\xi \tag{10}$$

After applying above relations in Eq. (8), we have:

$$\begin{aligned}\chi_{n+1}(\tau) = &\left[1 - \lambda_1'(\tau,\tau) + \lambda_2'(\tau,\tau)\right]\chi_n(\tau) + \left[\lambda_1(\tau,\tau) - \lambda_2(\tau,\tau)\right]\chi_n'(\tau) - \lambda_1(\alpha,\tau)\chi_n'(\alpha) + \lambda_1'(\alpha,\tau)\chi_n(\alpha) + \lambda_2(\beta,\tau)\chi_n'(\beta) - \lambda_2'(\beta,\tau)\chi_n(\beta) \\ &+ \int_{\alpha}^{\tau}\left[\lambda_1''(\xi,\tau) + \Omega^2\lambda_1(\xi,\tau)\right]\chi_n(\xi)d\xi + \int_{\alpha}^{t} \lambda_1(\xi,\tau)\tilde{g}\left(\xi, \chi_n(\xi), \chi_n'(\xi), \chi_n''(\xi)\right)d\xi \\ &+ \int_{\tau}^{\beta}\left[\lambda_2''(\xi,\tau) + \Omega^2\lambda_2(\xi,\tau)\right]\chi_n(\xi)d\xi + \int_{\tau}^{\beta} \lambda_2(\xi,\tau)\tilde{g}\left(\xi, \chi_n(\xi), \chi_n'(\xi), \chi_n''(\xi)\right)d\xi\end{aligned} \tag{11}$$

Considering the variation on both sides of Eq. (11) yields:

$$\delta\chi_{n+1}(\tau)=\left[1-\lambda_1'(\tau,\tau)+\lambda_2'(\tau,\tau)\right]\delta\chi_n(\tau)+\left[\lambda_1(\tau,\tau)-\lambda_2(\tau,\tau)\right]\delta\chi_n'(\tau)+\delta\left(\int_\alpha^\tau\left[\lambda_1''(\xi,\tau)+\Omega^2\lambda_1(\xi,\tau)\right]\chi_n(\xi)d\xi\right)+\delta\left(\int_\tau^\beta\left[\lambda_2''(\xi,\tau)+\Omega^2\lambda_2(\xi,\tau)\right]\chi_n(\xi)d\xi\right)=0 \tag{12}$$

Therefore, we have the following stationary conditions:

$$\begin{aligned}&\lambda_1''(\xi,\tau)+\Omega^2\lambda_1(\xi,\tau)=0, && \alpha<\xi<\tau\\&\lambda_2''(\xi,\tau)+\Omega^2\lambda_2(\xi,\tau)=0, && \tau<\xi<\beta\\&1-\lambda_1'(\xi,\tau)+\lambda_2'(\xi,\tau)\big|_{\xi=\tau}=0, &&\\&\lambda_1(\xi,\tau)-\lambda_2(\xi,\tau)\big|_{\xi=\tau}=0. &&\end{aligned} \tag{13}$$

Upon solving Eq. (13), we have:

$$\begin{aligned}\lambda_1(\xi,\tau)&=\frac{\sin\left[\Omega(\beta-\tau)\right]\sin\left[\Omega(\alpha-\xi)\right]}{\Omega\sin\left[\Omega(\alpha-\beta)\right]}, && \alpha<\xi<\tau\\\lambda_2(\xi,\tau)&=\frac{\sin\left[\Omega(\alpha-\tau)\right]\sin\left[\Omega(\beta-\xi)\right]}{\Omega\sin\left[\Omega(\alpha-\beta)\right]}, && \tau<\xi<\beta\end{aligned} \tag{14}$$

Hence, the VIM algorithm for the nonlinear oscillator with two-point boundary conditions is:

$$\begin{aligned}\chi_{n+1}(\tau)=\chi_n(\tau)&+\int_\alpha^\tau\frac{\sin\left[\Omega(\beta-\tau)\right]\sin\left[\Omega(\alpha-\xi)\right]}{\Omega\sin\left[\Omega(\alpha-\beta)\right]}\left[\chi_n''(\xi)+\Omega^2\chi_n(\xi)+g\left(\xi,\chi_n(\xi),\chi_n'(\xi),\chi_n''(\xi)\right)\right]d\xi\\&+\int_\tau^\beta\frac{\sin\left[\Omega(\alpha-\tau)\right]\sin\left[\Omega(\beta-\xi)\right]}{\Omega\sin\left[\Omega(\alpha-\beta)\right]}\left[\chi_n''(\xi)+\Omega^2\chi_n(\xi)+g\left(\xi,\chi_n(\xi),\chi_n'(\xi),\chi_n''(\xi)\right)\right]d\xi\end{aligned} \tag{15}$$

3. Applications

This section demonstrates the proposed technique for a general oscillatory system that describes various problems, such as the cubic duffing equation, the cubic-quintic duffing equation, the oscillatory motion of a bead in an arranged parabola, the vibration of a conical beam, the Mathews and Lakshmanan oscillator, etc. arising in science and engineering [30, 33-35] to show its reliability and validity. Consider this general equation as:

$$\chi''+\frac{\kappa\chi+\gamma_1\chi\chi'^2+\gamma_2\chi^3\chi'^2+\gamma_3\chi^3+\gamma_4\chi^5}{1+\eta_1\chi^2+\eta_2\chi^4}=0 \tag{16}$$

where $\kappa,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\eta_1$ and η_2 are arbitrary parameters. Equation (16) can be expressed as:

$$\chi''+\Omega^2\chi+\left(\kappa-\Omega^2\right)\chi+\gamma_1\chi\chi'^2+\gamma_2\chi^3\chi'^2+\gamma_3\chi^3+\gamma_4\chi^5+\eta_1\chi^2\chi''+\eta_2\chi^4\chi''=0 \tag{17}$$

Using Eq. (2) for two-points as 0 and $T/4$, the iteration formula for Eq. (17) by employing Eq. (15) can be represented as:

$$\begin{aligned}\chi_{n+1}(\tau)=\chi_n(\tau)&+\frac{\cos\Omega\tau}{\Omega}\int_0^\tau\sin\Omega\xi\left[\chi_n''(\xi)+\kappa\chi_n(\xi)+\gamma_1\chi_n(\xi)\chi_n'^2(\xi)+\gamma_2\chi_n^3(\xi)\chi_n'^2(\xi)\right]d\xi\\&+\frac{\cos\Omega\tau}{\Omega}\int_0^\tau\sin\Omega\xi\left[\gamma_3\chi_n^3(\xi)+\gamma_4\chi_n^5(\xi)+\eta_1\chi_n^2(\xi)\chi_n''(\xi)+\eta_2\chi_n^4(\xi)\chi_n''(\xi)\right]d\xi\\&+\frac{\sin\Omega\tau}{\Omega}\int_\tau^{T/4}\cos\Omega\xi\left[\chi_n''(\xi)+\kappa\chi_n(\xi)+\gamma_1\chi_n(\xi)\chi_n'^2(\xi)+\gamma_2\chi_n^3(\xi)\chi_n'^2(\xi)\right]d\xi\\&+\frac{\sin\Omega\tau}{\Omega}\int_\tau^{T/4}\cos\Omega\xi\left[\gamma_3\chi_n^3(\xi)+\gamma_4\chi_n^5(\xi)+\eta_1\chi_n^2(\xi)\chi_n''(\xi)+\eta_2\chi_n^4(\xi)\chi_n''(\xi)\right]d\xi\end{aligned} \tag{18}$$

The initial solution is chosen as $\chi_0=a\cos\Omega\tau$, by Eq. (18) we obtain:

$$\begin{aligned}\chi_1(\tau)=a\cos\Omega\tau&+\frac{\cos\Omega\tau}{\Omega}\int_0^\tau\sin\Omega\xi\left[\left(-a\Omega^2+\kappa a+\gamma_1a^3\Omega^2\right)\cos\Omega\xi\right]d\xi\\&+\frac{\cos\Omega\tau}{\Omega}\int_0^\tau\sin\Omega\xi\left[\left(-\gamma_1a^3\Omega^2+\gamma_2a^5\Omega^2+\gamma_3a^3-\eta_1a^3\Omega^2\right)\cos^3\Omega\xi\right]d\xi\\&+\frac{\cos\Omega\tau}{\Omega}\int_0^\tau\sin\Omega\xi\left[\left(-\gamma_2a^5\Omega^2+\gamma_4a^5-\eta_2a^5\Omega^2\right)\cos^5\Omega\xi\right]d\xi\\&+\frac{\sin\Omega\tau}{\Omega}\int_\tau^{T/4}\cos\Omega\xi\left[\left(-a\Omega^2+\kappa a+\gamma_1a^3\Omega^2\right)\cos\Omega\xi\right]d\xi\\&+\frac{\sin\Omega\tau}{\Omega}\int_\tau^{T/4}\cos\Omega\xi\left[\left(-\gamma_1a^3\Omega^2+\gamma_2a^5\Omega^2+\gamma_3a^3-\eta_1a^3\Omega^2\right)\cos^3\Omega\xi\right]d\xi\\&+\frac{\sin\Omega\tau}{\Omega}\int_\tau^{T/4}\cos\Omega\xi\left[\left(-\gamma_2a^5\Omega^2+\gamma_4a^5-\eta_2a^5\Omega^2\right)\cos^5\Omega\xi\right]d\xi\end{aligned}$$

After simplification, we have:

$$\chi_1(t) = a\cos\Omega\tau + \frac{\cos\Omega\tau}{\Omega}\int_0^\tau \sin\Omega\xi[\Lambda_1\cos\Omega\xi + \Lambda_2\cos3\Omega\xi + \Lambda_3\cos5\Omega\xi]d\xi + \frac{\sin\Omega\tau}{\Omega}\int_\tau^{T/4}\cos\Omega\xi[\Lambda_1\cos\Omega\xi + \Lambda_2\cos3\Omega\xi + \Lambda_3\cos5\Omega\xi]d\xi \tag{19}$$

where,

$$\Lambda_1 = -a\Omega^2 + \kappa a + \frac{\gamma_1 a^3\Omega^2}{4} + \frac{\gamma_2 a^5\Omega^2}{8} + \frac{3\gamma_3 a^3}{4} + \frac{5\gamma_4 a^5}{8} - \frac{3\eta_1 a^3\Omega^2}{4} - \frac{5\eta_2 a^5\Omega^2}{8} \tag{20}$$

$$\Lambda_2 = -\frac{\gamma_1 a^3\Omega^2}{4} - \frac{\gamma_2 a^5\Omega^2}{16} + \frac{\gamma_3 a^3}{4} + \frac{5\gamma_4 a^5}{16} - \frac{\eta_1 a^3\Omega^2}{4} - \frac{5\eta_2 a^5\Omega^2}{16} \tag{21}$$

and,

$$\Lambda_3 = \frac{1}{16}\left(-\gamma_2 a^5\Omega^2 + \gamma_4 a^5 - \eta_2 a^5\Omega^2\right) \tag{22}$$

Upon integration of Eq. (19), we have:

$$\begin{aligned}\chi_1(t) = a\cos\Omega\tau - \frac{\cos\Omega\tau}{\Omega^2}&\left[\Lambda_1\left(-\frac{1}{4} - \frac{\cos2\Omega\xi}{4}\right) + \Lambda_2\left(\frac{\cos2\Omega\xi}{4} - \frac{\cos4\Omega\xi}{8}\right) + \Lambda_3\left(\frac{\cos4\Omega\xi}{8} - \frac{\cos6\Omega\xi}{12}\right)\right]\Bigg|_0^\tau \\ + \frac{\sin\Omega\tau}{\Omega^2}&\left[\Lambda_1\left(\frac{\Omega\xi}{2} + \frac{\sin2\Omega\xi}{4}\right) + \Lambda_2\left(\frac{\sin2\Omega\xi}{4} + \frac{\sin4\Omega\xi}{8}\right) + \Lambda_3\left(\frac{\sin4\Omega\xi}{8} + \frac{\sin6\Omega\xi}{12}\right)\right]\Bigg|_\tau^{T/4}\end{aligned} \tag{23}$$

After simple calculation, we obtain:

$$\begin{aligned}\chi_1(\tau) = a\cos\Omega\tau + \frac{\cos\Omega\tau}{\Omega^2}&\left[\frac{\Lambda_1}{4} + \left(-\frac{\Lambda_1}{4} + \frac{\Lambda_2}{4}\right)\cos2\Omega\tau + \left(-\frac{\Lambda_2}{8} + \frac{\Lambda_3}{8}\right)\cos4\Omega\tau - \frac{\Lambda_3}{12}\cos6\Omega\tau - \left(\frac{\Lambda_2}{8} + \frac{\Lambda_3}{24}\right)\right] \\ + \frac{\sin\Omega\tau}{\Omega^2}&\left[\frac{\Lambda_1\pi}{4} - \frac{\Lambda_1\Omega\tau}{2} - \left(\frac{\Lambda_1}{4} + \frac{\Lambda_2}{4}\right)\sin2\Omega\tau - \left(\frac{\Lambda_2}{8} + \frac{\Lambda_3}{8}\right)\sin4\Omega\tau - \frac{\Lambda_3}{12}\sin6\Omega\tau\right]\end{aligned} \tag{24}$$

Using simple trigonometric identities, we have:

$$\chi_1(\tau) = a\cos\Omega\tau + \frac{1}{\Omega^2}\left[\left(-\frac{\Lambda_2}{8} - \frac{\Lambda_3}{24}\right)\cos\Omega\tau + \frac{\Lambda_2}{8}\cos3\Omega\tau + \frac{\Lambda_3}{24}\cos5\Omega\tau + \frac{\Lambda_1\pi}{4}\sin\Omega\tau - \frac{\Lambda_1\Omega}{2}\tau\sin\Omega\tau\right] \tag{25}$$

In order to maintain the oscillatory behavior, no secular term [21] appears. Therefore:

$$\Lambda_1 = 0 \tag{26}$$

Using Eq. (20), we have:

$$-a\Omega^2 + \kappa a + \frac{\gamma_1 a^3\Omega^2}{4} + \frac{\gamma_2 a^5\Omega^2}{8} + \frac{3\gamma_3 a^3}{4} + \frac{5\gamma_4 a^5}{8} - \frac{3\eta_1 a^3\Omega^2}{4} - \frac{5\eta_2 a^5\Omega^2}{8} = 0$$

which provides the following result:

$$\Omega = \sqrt{\frac{8\kappa + 6\gamma_3 a^2 + 5\gamma_4 a^4}{8 - 2\gamma_1 a^2 - \gamma_2 a^4 + 6\eta_1 a^2 + 5\eta_2 a^4}} \tag{27}$$

Thus, approximate solution of Eq. (16) from Eq. (25) as:

$$\chi_{GVIM} = a\cos\Omega\tau - \frac{\Lambda_2}{8\Omega^2}(\cos\Omega\tau - \cos3\Omega\tau) - \frac{\Lambda_3}{24\Omega^2}(\cos\Omega\tau - \cos5\Omega\tau)$$

or,

$$\begin{aligned}\chi_{GVIM} = a\cos\Omega\tau + \frac{1}{8\Omega^2}&\left[\frac{1}{4}(\gamma_1 + \eta_1)a^3\Omega^2 + \frac{1}{16}(\gamma_2 + 5\eta_2)a^5\Omega^2 - \frac{1}{4}\gamma_3 a^3 - \frac{5}{16}\gamma_4 a^5\right](\cos\Omega\tau - \cos3\Omega\tau) \\ + \frac{1}{24\Omega^2}&\left[\frac{1}{16}(\gamma_2 + \eta_2)a^5\Omega^2 - \frac{1}{16}\gamma_4 a^5\right](\cos\Omega\tau - \cos5\Omega\tau)\end{aligned} \tag{28}$$

The methodology described above is now going to be applied to the following three examples.

3.1. Example I: Movement of a particle on a rotating parabola

By setting the parameters $\kappa = \Omega_0^2, \gamma_1 = 4\varepsilon^2, \gamma_2 = \gamma_3 = \gamma_4 = 0,$ $\eta_1 = 4\varepsilon^2$ and $\eta_2 = 0$ of general oscillatory Eq. (16), we have the equation of motion of the vibration of a particle moving freely down a parabola [36, 37]:

$$\chi'' + \frac{\Omega_0^2 + 4\varepsilon^2\chi\chi'^2}{1 + 4\varepsilon^2\chi^2} = 0 \tag{29}$$

Equation (29) also characterizes the motion of the double slider mechanism [34]. The solution of Eq. (29) can be seen from the frequency formulation [38] and the FF-EBM [19]. By substituting the above parameters in Eq. (27), the expression for the angular frequency of the oscillator can be obtained as:

$$\Omega = \frac{\Omega_0}{\sqrt{1+2\varepsilon^2 a^2}} \tag{30}$$

which is exactly identical to that given in [19, 37]. Equation (28) can be obtained analytically the approximate solution of Eq. (29) as:

$$\chi_{\mathrm{GVIM}} = a\cos\Omega\tau + \frac{\varepsilon^2 a^3}{4}(\cos\Omega\tau - \cos 3\Omega\tau) \tag{31}$$

The approximate solution due to [19] is:

$$\chi_{\mathrm{FF-EBM}} = a\cos\left(\frac{\Omega_0}{\sqrt{1+2\varepsilon^2 a^2}}\tau\right) \tag{32}$$

Figure 1 shows the results gained from the modification in VIM, Runge-Kutta (RK) method of order 4 using MATLAB and frequency formulation followed by energy balance method [19]. Three set of parameters $A(\Omega_0 = 0.5,\ \varepsilon = 0.5,\ a = 0.8)$ in the top panel, $B(\Omega_0 = 0.8,\ \varepsilon = 0.5,\ a = 0.5)$ in the middle panel and $C(\Omega_0 = 0.8,\ \varepsilon = 0.5,\ a = 0.5)$ in the bottom panel are considered for this oscillator.

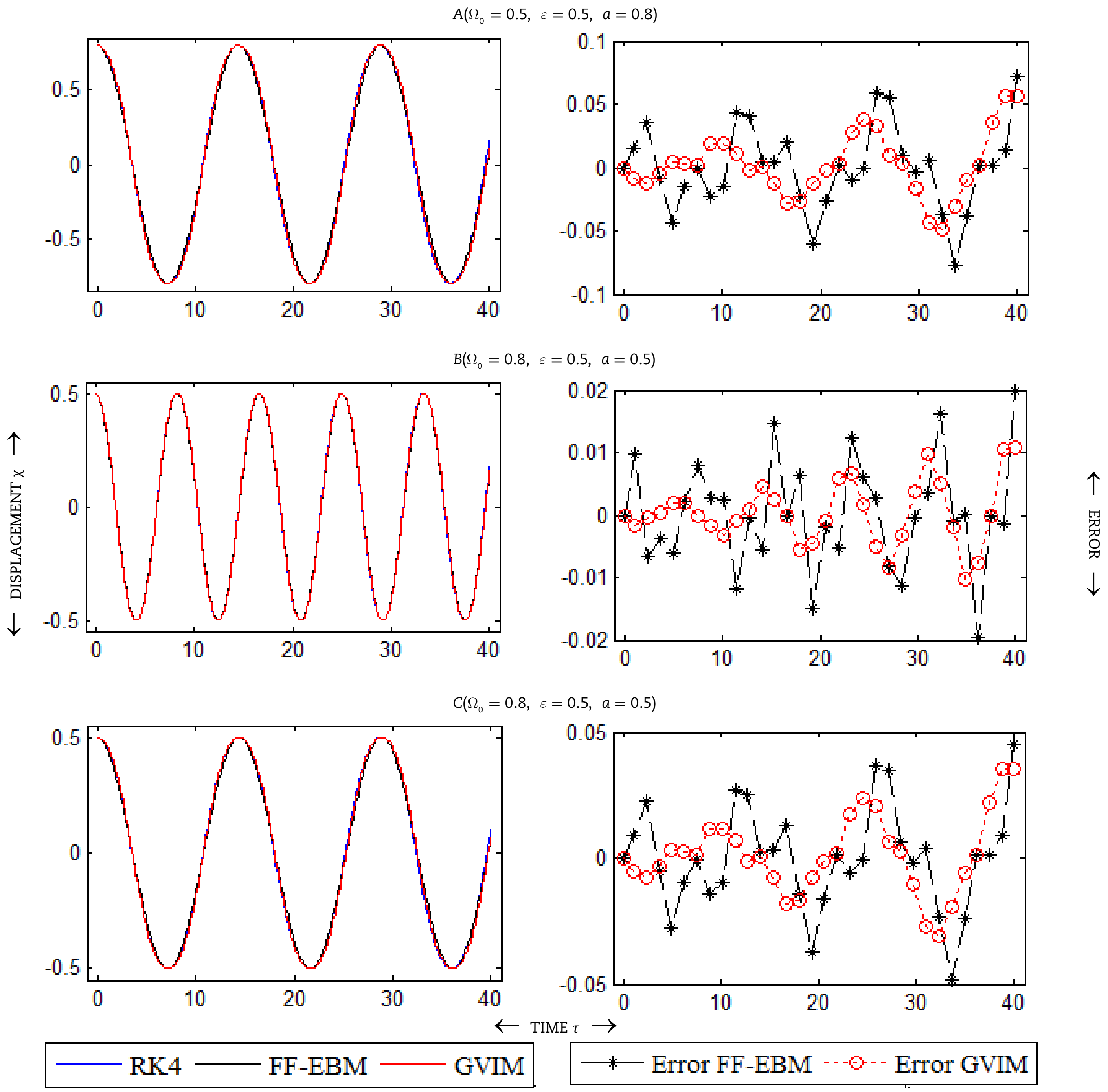

Fig. 1. Comparison of displacement and errors for Eq. (29).

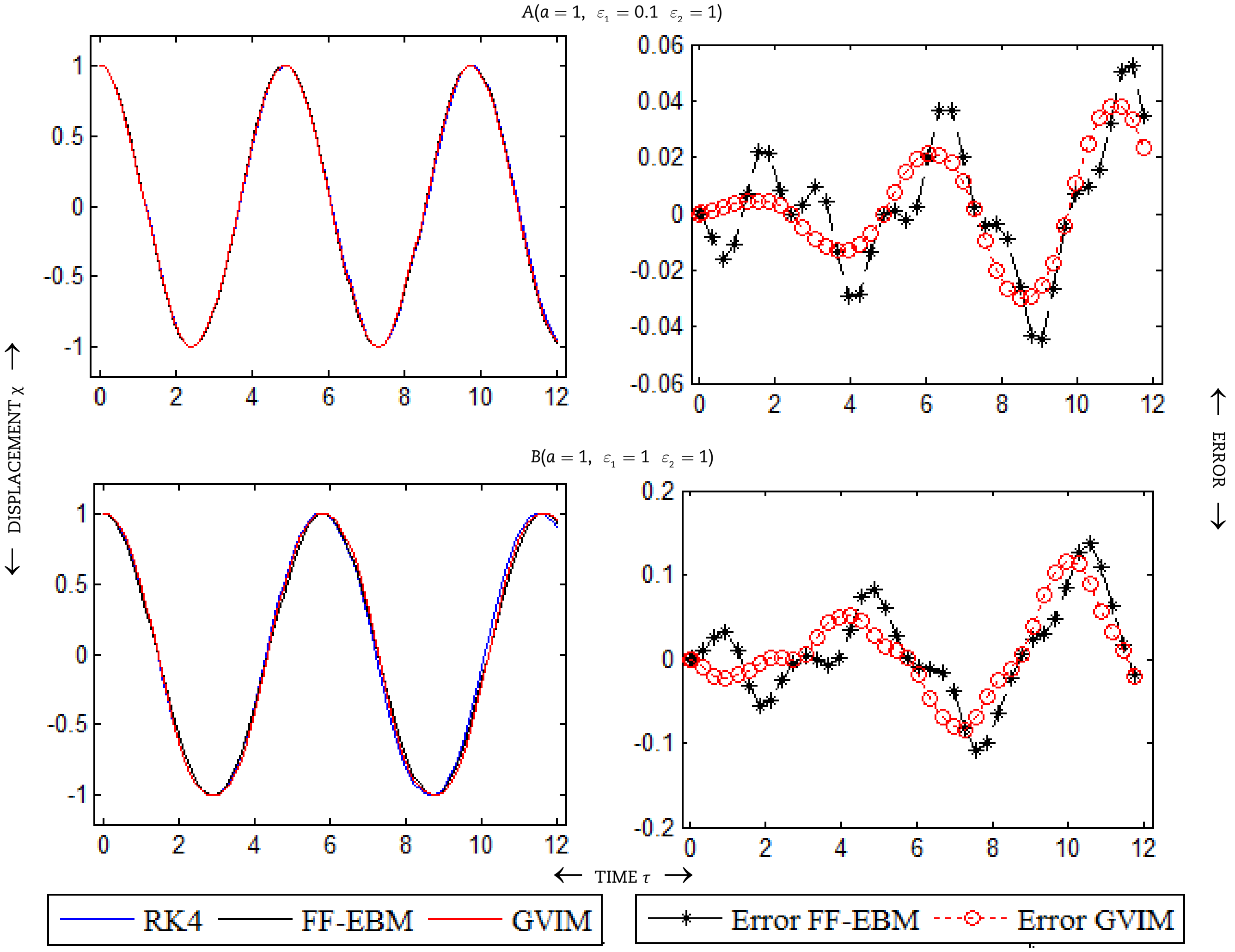

Fig. 2. Comparison of displacement and errors for Eq. (33).

The left column shows displacement obtained from RK method (blue line), Eq. (32) (black line) and Eq. (31) (red line) for above mentioned parameters and this comparison authenticate the modification in the correction functional because the approximate analytical results from Eq. (31) using proposed technique match remarkably well with the numerical results of RK method. For the same parameter values, we also display errors, ϵ_{GVIM} $(=\chi_{\mathrm{RK4}}-\chi_{\mathrm{GVIM}},$ red circles with dotted line) and $\epsilon_{\mathrm{FF-EBM}}$ $(=\chi_{\mathrm{RK4}}-\chi_{\mathrm{FF-EBM}},$ black asterisks with solid line) against time in the right column of Figure 1. Both approximate solutions Eq. (31) and Eq. (32) have small errors but all right-side panels ensure the accuracy of the proposed modification when compare with energy balance method [19].

3.2. Example II: Nonlinear oscillator from construction industry

Considering the forced parameters $\kappa=1, \gamma_1=\varepsilon_1, \gamma_2=0, \gamma_3=\varepsilon_2, \gamma_4=0, \eta_1=\varepsilon_1$ and $\eta_2=0$ of general Eq. (16), we have the equation [39-40]:

$$\chi''+\frac{\chi+\varepsilon_1\chi\chi'^2+\varepsilon_2\chi^3}{1+\varepsilon_1\chi^2}=0 \tag{33}$$

Equation (33) represents model of tapered beam used in construction industry [39-40]. The angular frequency for this oscillator using Eq. (27) is:

$$\Omega=\sqrt{\frac{4+3\varepsilon_2 a^2}{4+2\varepsilon_1 a^2}} \tag{34}$$

which is perfect match with given by [19]. The approximate analytical solution using Eq. (28) is:

$$\chi_{\mathrm{GVIM}}=a\cos\Omega\tau+\frac{1}{8\Omega^2}\left[\frac{1}{2}\varepsilon_1 a^3\Omega^2-\frac{1}{4}\varepsilon_2 a^3\right](\cos\Omega\tau-\cos3\Omega\tau) \tag{35}$$

whereas the solution given by FF-EBM [19] is:

$$\chi_{\mathrm{FF-EBM}}=a\cos\left(\sqrt{\frac{4+3\varepsilon_2 a^2}{4+2\varepsilon_1 a^2}}\,\tau\right) \tag{36}$$

The variation in displacement and their corresponding errors are plotted in Fig. 2 for tapered beam. Displacement yielded from RK method using MATLAB χ_{RK4} (blue line), Eq. (35) χ_{GVIM} (red line) and Eq. (36) χ_{FF-EBM} (black line) against time τ for two parameters $A(a=1,\ \varepsilon_1=0.1\ \varepsilon_2=1)$ in the top left panel, and $B(a=1,\ \varepsilon_1=1\ \varepsilon_2=1)$ in the left bottom panel ensures the efficiency of the proposed modification. $\in_{GVIM}(=\chi_{RK4}-\chi_{GVIM}$, red circles with dotted line) and $\in_{FF-EBM}(=\chi_{RK4}-\chi_{FF-EBM}$, black asterisks with solid line) with time τ for the similar parameters in the right column concludes that the proposed strategy is better than energy balance method.

3.3. Example III: Mathews and Lakshmanan nonlinear oscillator

Mathews-Lakshmanan nonlinear oscillator [41] can be gained from general oscillatory Eq. (16) by setting the values of the parameters $\kappa=\alpha^2$, $\gamma_1=\pm k$, $\gamma_2=\gamma_3=\gamma_4=0$, $\eta_1=\pm k$ and $\eta_2=0$,

$$\chi''+\frac{\alpha^2\chi\pm k\chi\chi'^2}{1\pm k\chi^2}=0 \tag{37}$$

Equation (37) plays significant role in elementary particle theory and the Lagrangian minimum density for relativity scalar field [41] can be used to model it. The angular frequency of Eq. (37) can be yielded using Eq. (27) by applying aforementioned parameters as:

$$\Omega=\frac{\alpha}{\sqrt{1\pm ka^2}} \tag{38}$$

Using Eq. (28) the analytic solution of Eq. (37) is:

$$\chi_{GVIM}=a\cos\left[\frac{\alpha\tau}{\sqrt{1\pm ka^2}}\right] \tag{39}$$

Equations (38) and (39) are, respectively, the exact nonlinear frequency and analytic solution of the Mathews-Lakshmanan oscillator.

4. Conclusion

This paper introduced a novel extension of the variational iteration method (VIM) to address nonlinear oscillators with boundary conditions. It tackled the dynamics of oscillatory systems marked by generalization, simplifying them into equations with significant nonlinearity. Examples included the motion of a bead in a rotating parabola, the large oscillation of a conical beam, and the Mathews-Lakshmanan oscillator. Through a modified correction functional, considering two points instead of the previously explored single point, the study successfully obtained the approximate analytic solutions and established the frequency-amplitude relationship for the general oscillatory problem and its associated systems. The proposed modification yields rapidly converging solutions, where even a first-order approximation proves sufficient to achieve highly accurate results. The examples showed that the modification is reliable. Consequently, this technique can be used to solve other nonlinear oscillatory problems. It is also extensible to nonlinear oscillators with fractional and/or fractal derivatives.

Author Contributions

All authors contribute equally in preparation of this manuscript. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

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Not applicable.

Nomenclature

A	Amplitude of the oscillator	γ_2	Coefficient of nonlinear term
T	Time period	γ_3	Cubic power coefficient
χ	Approximate solution	γ_4	Quintic power coefficient
τ	Time	η_1	Quadratic nonlinearity coefficient
κ	Coefficient of linear term	η_2	Quatral nonlinearity coefficient
γ_1	Coefficient of nonlinear term	Ω	Nonlinear frequency of the oscillator

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