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Research Paper

Vibration of a Flexible Hanging Chain with a Mass on the End in a Nonstationary Regime of the Motion Mechanism in Hoisting Machines

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Abstract. The article is focused on the vibrations of a heavy chain with a mass on the end when the hoisting machines are in a nonstationary regime of motion, assuming that motor or brake torque is a linear function of the speed of the motion mechanism. An approach based on the application of approximate analytical solution of the system of differential equations of motion has been presented. The solution was obtained by two iterations based on the Fourier and small parameter methods, the representation of the motion of any point of the chain, and the intensity of the inertial perturbation forces by Eigenfunction series of the free vibration task. Our analytical results have been illustrated with a numerical example on the basis of the technical characteristics of the crane trolley and a visual Fortran computer program has been created.

Keywords: Heavy chain, Vibrations, Analytical solution, Small parameter, Fourier method.

1. Introduction

Historically, the first solid deformable body model is the flexible inextensible rope. The mechanics of ropes has developed continuously over the years, with emphasis on different aspects in certain periods. Now, the modern flexible rope theory is a widely branched discipline covering various facets of this science. Interestingly, a number of published scientific works present this theory from different points of view, with different degrees of completeness and rigor. In recent decades, the main position has been occupied by research on the dynamics of ropes. The theory of flexible ropes and their specific elements have been examined in various books and monographs [1-10].

Mechanical systems with different flexible constraints have wide practical applications. Both inextensible and extensible ropes, both massless and heavy ropes, are used as models of these ropes. The thread theory is commonly applied to the design of various engineering structures, machines, and musical instruments. The flexible rope model in its varieties is used to describe the vibrations of power lines, hoisting devices, ropes of suspension bridges, ropes of textile machines, etc. [5-7, 10-12].

Sandilo and coauthors in [11] studied the free lateral responses of vertically translating media of variable length, velocity and tension, subject to general initial conditions. There, the translating media were modeled as taut strings with fixed boundaries. The problem can be used as a simple model to describe the lateral vibrations of an elevator cable for which the length changes linearly in time, or for which the length changes harmonically about a constant mean length. For linear length variations, it was assumed that: 1) the axial velocity of the string is small compared to the nominal wave velocity; 2) the string mass is small compared to the car mass, and small oscillation amplitudes take place for the harmonic length variations; and 3) the string mass is small compared to the total mass of the string and the car. A multiple-timescales perturbation method was used to construct formal asymptotic approximations of the solutions to show the complicated dynamical behavior of the string. The linearly varying length analytical approximations of the exact solution were compared with a numerical solution.

Smolnikov and coauthors in [12] investigated the oscillation of a string in the presence of internal friction. It was demonstrated that equations of a string finite-dimensional model with dissipative forces could be derived. The equation of a string motion in partial derivatives for the continual model in the presence of internal friction forces was obtained using the passage to the limit from the equations of the finite-dimensional model. The influence of dissipative factors on the dynamic behavior of the string was approximately investigated for each model based on the harmonic balance method. As a result of the analysis, expressions for the

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oscillation amplitudes of a string when it moved on one of the oscillation modes separately were found. In conclusion, it was shown that the results obtained for both models were in agreement with each other.

In [13], Malyshev studied the spatial oscillations of an elastic thread in flat harmonic motion of one end, when the other end of the thread was fixed. In numerical modeling, the monotonic multiplicity of the second degree of accuracy was used. Thus, amplitude-frequency characteristics, oscillations in the vertical and horizontal plane, and the shape of the thread were studied. The stationary modes of oscillations of heavy filaments under harmonic and random disturbances at the top end were discussed in [14]. Amplitude-frequency characteristics were studied using universal numerical methods. On the other hand, in [15], using the combination of homogenization and the method of matching of asymptotic extensions, this case was examined when the string with a concentrated mass had an inhomogeneous rapidly oscillating density. The homogenized (limit) solution to a boundary value problem was obtained, up to $O(\mu+\varepsilon)$, where μ is a period of rapid oscillations of the density, and ε is the order of the length of the small part of the string on which the concentrated mass was located.

The motion of a particle suspended on an ideal thread in a uniform gravitational field was investigated in [16]. The orbital stability problem of the periodic motion of a particle along the vertical axis was solved. The nonlinear oscillations in the neighborhood of the periodic motion were considered when the motion was unstable. In this investigation, a normalization of the Hamilton function using symplectic mappings was employed. The dynamic string motion, its displacement being unilaterally constrained by the rigid termination condition of arbitrary geometry, was simulated and analyzed in [17]. |A numerical calculation based on the traveling wave solution (inducing interactions between the vibrating string and the contact condition at the point of string termination) was proposed for the nonlinearity modeling. In laboratory experiments, Perov and coauthors [18] recently described the harmonic content of standing waves in guitar strings. The experimental data were taken by using the magnetic pickup from a guitar and a digital oscilloscope with a Fast Fourier transform capability. The amplitudes of the harmonics in the measured signal were found to depend on the location where the string was plucked, resulting in a different timbre of the sound. The relative amplitudes of transverse standing waves in a string were determined from the experimental data and also predicted from the wave equation with the boundary and initial conditions corresponding to the initial shape of the string.

In recent literature, an increasing number of theoretical approaches have been presented in order to explain vibration and stability in nonlinear dynamical systems with two or three degrees of freedom (DOF), especially the existence of probable steady-state solutions near the resonances. In some cases, the equations of motion have been derived utilizing Lagrangian equations [19-25]. According to theoretical considerations, high consistency must exist between the approximate and the numerical solutions. The dynamical models investigated in [23-25] seem to be related to real-life applications, i.e. charging electronics and medicine.

The mechanical systems with suspended loads are widely used in industry, construction, etc. [7, 26] These systems are basic for hoisting and mining machines, inter alia. The design of such machines requires the creation of rational dynamical models of mechanical systems and the solution of the obtained systems of differential equations of motion.

The load during the operation of hoisting machines is suspended on ropes and its swinging is observed. This swinging causes both uneven movement of the motion mechanisms of cranes, bogies and other devices, and additional load on individual elements, thus creating inconvenience in their operation. It has been established that about 80% of the failures in modern hoisting machines are associated with dynamic loads, which cause increased wear of the friction elements, occurrence of unacceptable residual deformations, fatigue failure of load-bearing metal structures and parts of mechanism elements, etc. [27, 28]. Therefore, it is necessary to be able to assess the effect of swinging the rope with load during the design calculations of the hoisting machines.

The aim of this paper is to present an approach based on the application of approximate analytical methods in order to investigate the vibrations of a heavy chain with a mass on the end in hoisting machines in a nonstationary regime of the motion mechanism, assuming that its motor or brake torque is a linear function of the speed.

2. Theory and Methods

2.1. Dynamical Model

The analysis of the natural frequencies of the vibrations of bridge and other cranes with standard parameters moving along the rail track showed that the frequencies of the pendulum vibrations of the chain with a mass on the end relative to the crane were significantly lower than the frequencies of the elastic vibrations of the crane metal structure and transmission of the motion mechanism. Even with a small chain length of no more than 2-3 m., the frequency of the load vibrations did not exceed 2-2.6 s^{-1}, whereas the frequency of the elastic vibrations of cranes was several times or ten times greater [27, 28]. Therefore, the pendulum load vibrations are practically independent of the elastic vibrations of the crane; hence, it is possible to use a dynamical model in which the metal structure and transmission of the motion mechanism are considered absolutely rigid.

Based on the above, it is possible to adopt the generalized model of the motion and the hoisting devices shown in Fig. 1(a), including:

m_2 – the mass of the load; m_3 – the mass of the moving parts of the trolley or crane and the motion mechanism determined by reduction to the trolley or crane; l – the chain length in a stable vertical position; F_W – the resistance force of the motion of the crane or trolley; T_p – the motor or brake torque to the driving wheels; r – the hoisting drum radius; and θ – the chain deflection angle.

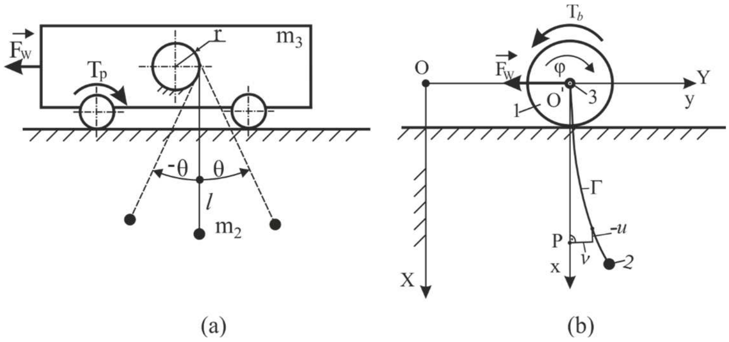

Fig. 1. The motion and hoisting devices: (a) Generalized model; (b) Dynamical model.

The dynamics of a crane trolley with a load in a braking regime was discussed in [29]. There, the chain was presented as massless and inextensible, of constant length, wound on a hoisting drum. Thus, at small deviations of the chain, i.e. $\sin\theta \approx \theta$, it is possible to accept with rather high accuracy that the chain is fixed in the center of the hoisting drum. Hence, the dynamical model shown in Fig. 1(b) is adopted to assess the influence of the chain motion with a load on the dynamics of the motion mechanism. The following assumptions and designations are also accepted:

- Disk 1 (with mass m_1 and radius R) is a homogeneous solid cylinder which rolls without sliding on the horizontal plane, $\varphi = \varphi(t)$ being the angle of rotation. The mass and the geometric parameters of the disk are determined by reduction of the parts of the trolley, crane or motion mechanisms with non-translational motion to the drive wheels;
- The concentrated mass m_3 is equal to the mass of the parts of the crane, trolley or motion mechanisms with translational motion;
- F_W – the resistance force of the motion of the crane or trolley;
- T_b – the motor torque determined by reduction to the drive wheels or the brake torque. In classic schemes of the crane drive, hoisting trolleys use a phase-wound rotor induction motor, which makes it possible to employ linear dependence $T_b = a_b - b_b\dot{\varphi}$ with sufficient accuracy for engineering practice [28], where $a_b > 0$ when starting and $a_b < 0$ when braking;
- We consider a flexible, inextensible, homogeneous, heavy chain of length l. The end O' is fixed and a particle of mass m_2 is attached at the other end of the chain. The position of a generic point P is given by the abscissa $O'P = a$ in the equilibrium position. We consider the planar vibration of the chain and denote the projections of point P on axes $O'x$ and $O'y$ by u and ν, respectively. Functions $u(x = a,t)$ and $\nu(x = a,t)$ determine both motion of any point of the chain (for a fixed a) and form Γ of the chain (for a fixed t). The analysis is limited to small deflections of the chain from the vertical axis $O'x$, i.e. we assume that u,ν and their derivatives with respect to a and t are small. In [1,5], it was proved that if ν and the derivatives $\partial\nu/\partial x$ and $\partial\nu/\partial t$ were quantities of the first order of smallness, then $\partial u/\partial x$ and $u(x,t)$ were second-order quantities. In fact, this is the basis for assuming that the position of point P is currently t determined by coordinates x and $\nu(x,t)$.

The differential equations of motion, taking into account that the chain is a body with a distributed mass, were obtained according to the Hamilton-Ostrogradsky principle. The variational problem derived from the Hamilton-Ostrogradsky principle, in the case of non-potential forces, is stated as follows: among all continuous functions $\nu(x,t)$, $\varphi(t)$ having continuous derivatives with respect to x and t for $0 \le x \le l$ and $t > 0$ satisfying the boundary condition, it is necessary to find the relationship (analogous to the Hamilton-Ostrogradsky principle) in the form:

$$\delta' R = \delta S + \int_{t_0}^{t_1} \delta' W\, dt = \int_{t_0}^{t_1} \left(\delta T - \delta \Pi + \delta' W\right) dt = 0,$$

where δS – the variation in Hamilton's action, $\delta' W$ – the elementary work of non-potential forces, T, Π – the kinetic and the potential energy, respectively.

The dynamical model has the form [30]:

$$\begin{aligned} &MR^2\ddot{\varphi} + m_2R\ddot{\nu}(l,t) + \gamma R\int_0^l \ddot{\nu}(x,t)\,dx = T_b - F_W R, \\ &m_2 g\nu''(x,t) + \gamma g\frac{\partial}{\partial x}\left[(l-x)\nu'(x,t)\right] = \gamma\ddot{\nu}(x,t) + \gamma R\ddot{\varphi}, \end{aligned} \tag{1}$$

where $M = 1{,}5m_1 + m_2 + m_3 + \gamma l$, and γ is the mass of a unit length of the chain.

The solution of the system of differential equations Eq. (1) must satisfy the following initial and boundary conditions:

$$t = 0 \;\Rightarrow\; \varphi(0) = \psi_0,\ \dot{\varphi}(0) = \omega_0,\ \nu(x,0) = f(x),\ \dot{\nu}(x,0) = g(x), \tag{2}$$

$$\nu(0,t) = 0,\quad \ddot{\nu}(l,t) + g\,\nu'(l,t) = -R\ddot{\varphi}, \tag{3}$$

where $f(x)$, $g(x)$ are continuous functions, such that $f(0) = 0$, $g(0) = 0$.

We apply below a new approach (algorithm) to the solution of the model in Eq. (1).

2.2. Algorithm for Solution of the Model Eq. (1)

The first differential equation in Eq. (1) can be written in the form:

$$\ddot{\varphi} + b\dot{\varphi} = a + \varepsilon\left[-m_2R\ddot{\nu}(l,t) - \gamma R\int_0^l \ddot{\nu}(x,t)\,dx\right], \tag{4}$$

where: $b = b_b/MR^2$, $a = (a_b - F_W R)/MR^2$ and $\varepsilon = 1/MR^2$ is a small positive parameter.

The solution of Eq. (4) can be obtained by means of the small parameter method (Poincaré - Lindstedt method) with accuracy to the first power of the small parameter. This yields:

$$\varphi(t) = \varphi_0(t) + \varepsilon\varphi_1(t). \tag{5}$$

Substituting Eq. (5) in Eq. (4), we equate to zero the coefficients at various powers. Thus, we obtain the following equations for the unknown functions $\varphi_0(t)$, $\varphi_1(t)$:

$$\begin{aligned} &\ddot{\varphi}_0 + b\dot{\varphi}_0 = a, \\ &\ddot{\varphi}_1 + b\dot{\varphi}_1 = -m_2R\ddot{\nu}(l,t) - \gamma R\int_0^l \ddot{\nu}(x,t)\,dx. \end{aligned} \tag{6}$$

Our first task will be to search for a solution of Eq. (4) when the conditions $t = 0 \Rightarrow \varphi(0) = \psi_0$, $\dot{\varphi}(0) = \omega_0$ are valid. It is essential to note that these conditions are satisfied if functions $\varphi_0(t)$, $\varphi_1(t)$ are found so that:

$$\begin{aligned} &\varphi_0(0)=\psi_0,\ \dot{\varphi}_0(0)=\omega_0, \\ &\dot{\varphi}_1(0)=0,\ \dot{\varphi}_1(0)=0. \end{aligned} \tag{7}$$

Function $\varphi(t)$ (in the first approximation) is a solution of the first equation in Eq. (6), for which it is obtained:

$$\varphi_0=\psi_0+\frac{a}{b}t+\frac{\left(\frac{a}{b}-\omega_0\right)}{b}\left(e^{-bt}-1\right). \tag{8}$$

After double differentiation of function φ_0 with respect to time t, we have:

$$\ddot{\varphi}_0=(a-b\omega_0)e^{-bt}=Ae^{-bt}\quad (A=a-b\omega_0). \tag{9}$$

Now, we consider the second differential equation in Eq. (1). This equation is transformed to:

$$\begin{aligned} &m_2 g\nu''(x,t)+\gamma g\frac{\partial}{\partial x}\left[(l-x)\nu'(x,t)\right]=\gamma\ddot{\nu}(x,t)+\gamma R\ddot{\varphi}, \\ &m_2 g\frac{\partial}{\partial x}\left(\frac{\partial \nu}{\partial x}\right)+\gamma g\frac{\partial}{\partial x}\left[(l-x)\frac{\partial \nu}{\partial x}\right]-\gamma\frac{\partial^2\nu}{\partial t^2}=\gamma R\ddot{\varphi}, \\ &g\frac{\partial}{\partial x}\left[\left(m_2+\gamma(l-x)\right)\frac{\partial \nu}{\partial x}\right]-\gamma\frac{\partial^2\nu}{\partial t^2}=\gamma R\ddot{\varphi}. \end{aligned}$$

Here $m_2/\gamma=l_1=\mu l$, where μ denotes the ratio of the end mass to the chain mass. Finally, the last partial differential equation assumes the form:

$$g\frac{\partial}{\partial x}\left[\left((l+l_1-x)\frac{\partial \nu}{\partial x}\right)\right]-\frac{\partial^2\nu}{\partial t^2}=R\ddot{\varphi}. \tag{10}$$

The partial differential equation Eq. (10) describes the vibration of a hanging chain with a mass on the end under inertial perturbations. Indeed, this is a dynamic model of the motion mechanisms in the hoisting machines. It is easy to observe that the left part is the equation of free vibrations of a fixed chain, the solution of which is well-known [1]. On the other hand, the right part is the intensity of the inertial perturbation forces. According to [5], the right part can also be obtained when the basic differential equation in the vector form of the rope dynamics is projected on the horizontal axis: see Fig. 1(b).

The solution of Eq. (10) is sought by the Fourier method. Function $\nu(x,t)$ is represented by Eigenfunction series $X(t)$ in the case of free vibration [31], i.e. in the form:

$$\nu(x,t)=X_1(x)T_1(t)+X_2(x)T_2(t)+\cdots=X(x)T(t). \tag{11}$$

The right part $R\ddot{\varphi}(t)$ of Eq. (10), which we will define as $q(x,t)=q_0(x)H(t)$, can be represented by:

$$q(x,t)=X_1(x)S_1(t)+X_2(x)S_2(t)+\cdots=X(x)S(t). \tag{12}$$

To determine functions $S_k(t)$, it is necessary to multiply both sides of Eq. (12) by $X_k(t)$, and integrate it along the entire length of the chain. In integration, if the Eigenfunctions are orthogonal to each other with respect to the distributed mass function $m(x)$, all the terms on the right-hand side will be zero except for term k. Thus, a formula for $S_k(t)$ is obtained:

$$S_k(t)=\frac{\int_0^l m(x)q(x,t)X_k(x)dx}{\int_0^l m(x)X_k^{\,2}(x)dx}. \tag{13}$$

Since there is a fixed concentrated mass at the end of the chain, the boundary condition of this end of the chain will include an Eigenfrequency that is not the same for all Eigenfunctions. Hence, $\int_0^l X_n(x)X_j(x)dx\neq 0$ when $m(x)=\gamma=\text{const}$, i.e. there will be no orthogonality of Eigenfunctions [32]. Therefore, we will consider the load as part of a chain that has a free end. We assume that the concentrated mass is evenly distributed along a small section of the chain with a length ε' around its free end. In this setting, the lower end is already free and the proper functions will have the form $X_n(\varepsilon',x)\,(n=1,2,\ldots)$. Since the concentrated mass distributed over an infinitesimal area is a boundary case of the finite mass, then, in accordance with the physical meaning, it follows that:

$$\lim_{\varepsilon'\to 0}X_n(\varepsilon',x)=X_n^{\,*}(x),$$

where $X_n^*(x)=J_0(\mu_n\sqrt{(l-x)/l})$ is an Eigenfunction in the case of no mass at the end of the chain [1], and J_0, μ_n are Bessel function of the first kind and its roots. Now the Eigenfunctions are orthogonal with respect to the modified function of the distributed mass $m(\varepsilon',x)$, i.e. $n\neq j$, we have:

$$\int_0^l m(\varepsilon',x)X_n(\varepsilon',x)X_j(\varepsilon',x)dx=0, \tag{14}$$

when $0 < x < l-\varepsilon' \Rightarrow m(\varepsilon',x) = \gamma$, and when $l-\varepsilon' < x < l \Rightarrow m(\varepsilon',x) = \gamma + (\mu\gamma l)/\varepsilon'$.

For determination of functions $S_k(t)$ (see Eq. (13)), we use Eq. (14), where we find its limit at $\varepsilon' \to 0$. It is worth noting that this follows from the orthogonality of Eigenfunctions obtained. The numerator $q(x,t) = q_0(x)H(t) = R\ddot{\varphi}(t)$ is only a function of t, therefore:

$$\int_0^l m(\varepsilon',x)q(x,t)X_k(\varepsilon',x)dx = R\ddot{\varphi}\left[\int_0^{l-\varepsilon'} m(\varepsilon',x)X_k(\varepsilon',x)dx + \int_{l-\varepsilon'}^{l} m(\varepsilon',x)X_k(\varepsilon',x)dx\right] =$$
$$= \gamma R\ddot{\varphi}\left[\int_0^{l-\varepsilon'} X_k(\varepsilon',x)dx + \left(1+\frac{\mu l}{\varepsilon'}\right)\int_{l-\varepsilon'}^{l} X_k(\varepsilon',x)dx\right].$$

Let ε' be a small parameter. Thus, the integrand in the second integral must be a continuous function, and applying the mean value theorem, it turns out that:

$$\int_0^l m(\varepsilon',x)q(x,t)X_k(\varepsilon',x)dx = \gamma R\ddot{\varphi}\left[\int_0^{l-\varepsilon'} X_k(\varepsilon',x)dx + (\varepsilon'+\mu l)X_k(\varepsilon',\xi)\right], (l-\varepsilon' < \xi < l).$$

The limit of the resultant expression at $\varepsilon' \to 0$ is obtained in the form:

$$\lim_{\varepsilon'\to 0}\int_0^l m(\varepsilon',x)q(x,t)X_k(\varepsilon',x)dx = \lim_{\varepsilon'\to 0}\left\{\gamma R\ddot{\varphi}\left[\int_0^{l-\varepsilon'} X_k(\varepsilon',x)dx + (\varepsilon'+\mu l)X_k(\varepsilon',\xi)\right]\right\} =$$
$$= \gamma R\ddot{\varphi}\left[\lim_{\varepsilon'\to 0}\int_0^{l-\varepsilon'} X_k(\varepsilon',x)dx + \lim_{\varepsilon'\to 0}\left((\varepsilon'+\mu l)X_k(\varepsilon',\xi)\right)\right] = \gamma R\ddot{\varphi}\left[\int_0^l X_k^*(x)dx + \mu l X_k^*(l)\right] = \gamma R\ddot{\varphi}\left[\int_0^l J_0\mu_k\sqrt{\frac{l-x}{l}}dx + \mu l.1\right]. \tag{15}$$

Analogically, for the denominator of Eq. (13) we obtain:

$$\lim_{\varepsilon'\to 0}\int_0^l m(\varepsilon',x)X_k^2(\varepsilon',x)dx = \gamma\left[\int_0^l (X_k^*(x))^2 dx + \mu l(X_k^*(l))^2\right]. \tag{16}$$

Taking into account Eq. (9), Eq. (15) and Eq. (16), for functions $S_k(t)$ we have:

$$S_k(t) = RA\frac{[2l^2\mu_k^{-1}J_1(\mu_k)+\mu l]}{l^2 J_1^{\,2}(\mu_k)+\mu l}e^{-bt} = A_k e^{-bt}, \tag{17}$$

where $A_k = RA[2l^2\mu_k^{-1}J_1(\mu_k)+\mu l]/[l^2J_1^{\,2}(\mu_k)+\mu l]$.

It should be noted that according to [33], $\int_0^l J_0(\mu_k\sqrt{(l-x)/l})dx = 2l^2\mu_k^{-1}J_1(\mu_k)$, $\int_0^l (X_k^*(x))^2 dx = l^2 J_1^{\,2}(\mu_k)$.

Consider functions $T_k(t)$. Their definition is based on the fact that each term in the right part of Eq. (12) causes motion, determined by the corresponding term in Eq. (11). Therefore, substituting Eq. (11) and Eq. (12) into Eq. (10), we get:

$$g\frac{\partial}{\partial x}\left[(l+l_1-x)\frac{\partial}{\partial x}(X(x)T(t))\right] - \frac{\partial^2}{\partial t^2}(X(x)T(t)) = X(x)S(t),$$

which, after some transformations, has the form:

$$g\frac{1}{X(x)}\frac{\partial}{\partial x}\left[(l+l_1-x)\frac{\partial X(x)}{\partial x}\right] = \frac{\ddot{T}(t)}{T(t)} + \frac{S(t)}{T(t)}.$$

Since the left part depends on x only and the right part on t only, equality can only exist if each part is equal to a constant, which we denote with $-\omega^2$. Hence, we obtain:

$$\ddot{T}_k(t) + \omega_k^{\,2}T(t) = -S_k(t),$$

or, taking into account Eq. (17):

$$\ddot{T}_k(t) + \omega_k^{\,2}T(t) = -A_k e^{-bt}. \tag{18}$$

The general solution of Eq. (18) can be written as a sum of the solution of the corresponding homogeneous equation obtained with initial conditions and the particular solution obtained with zero initial conditions from Duhamel's integral, i.e. in the form:

$$T_k = a_k\cos(\omega_k t) + b_k\sin(\omega_k t) + \int_0^t \frac{1}{\omega_k}\sin\omega_k(t-\tau)(-A_k)e^{-b\tau}d\tau$$

or

$$T_k = a_k\cos(\omega_k t) + b_k\sin(\omega_k t) - \frac{A_k}{\omega_k^{\,2}+b^2}\left[\frac{b}{\omega_k}\sin(\omega_k t) - \cos(\omega_k t)\right] - \frac{A_k}{\omega_k^{\,2}+b^2}e^{-bt}. \tag{19}$$

The general solution of Eq. (10) in the first approximation is obtained in the form:

$$\nu(x,t) = \sum_{k=1}^{\infty}X_k(x)T_k(t) = \sum_{k=1}^{\infty}X_k(x)\left\{a_k\cos(\omega_k t) + b_k\sin(\omega_k t) - \frac{A_k}{\omega_k^2+b^2}\left[\frac{b}{\omega_k}\sin(\omega_k t) - \cos(\omega_k t)\right] - \frac{A_k}{\omega_k^{\,2}+b^2}e^{-bt}\right\}, \tag{20}$$

where for Eigenfunctions in [1], the following is obtained:

$$X_k(x) = Y_0(\eta_k) J_0\left(\eta_k\sqrt{1-\frac{x}{l+l_1}}\right) - J_0(\eta_k) Y_0\left(\eta_k\sqrt{1-\frac{x}{l+l_1}}\right).$$

Moreover, the natural frequencies ω_k are defined as the roots of the following transcendental equation:

$$J_0(\eta_k)\left[Y_0'(\eta_k^*) + 0{,}5\eta_k^* Y_0(\eta_k^*)\right] - Y_0(\eta_k)\left[J_0'(\eta_k^*) + 0{,}5\eta_k^* J_0(\eta_k^*)\right] = 0,$$

where $\eta_k = 2\omega_k\sqrt{(l+l_1)/g}$, $\eta_k^* = \eta_k\sqrt{l_1/(l+l_1)} = \eta_k\sqrt{\mu/(l+\mu)}$, and Y_0 is Bessel function of the second kind.

Constants a_k and b_k are determined in [1] by means of the initial conditions:

$$a_k = \frac{1}{N_k}\left[\int_0^l f(x)X_k(x)dx + l_1 f(l)X_k(l)\right], \qquad b_k = \frac{2}{\eta_k N_k}\sqrt{\frac{l+l_1}{g}}\left[\int_0^l g(x)X_k(x)dx + l_1 g(l)X_k(l)\right],$$

where $N_k = \int_0^l X_k^{\,2}(x)dx + l_1 X_k^{\,2}(l)$.

We transform the second equation of Eq. (6), using the results obtained for $\nu(x,t)$, as follows:

$$\ddot{\varphi}_1 + b\dot{\varphi}_1 = -m_2 R\ddot{\nu}(l,t) - \gamma R\int_0^l \ddot{\nu}(x,t)dx = -m_2 R\sum_{k=1}^{\infty}\ddot{T}_k(t)X_k(l) - \gamma R\sum_{k=1}^{\infty}\ddot{T}_k(t)\int_0^l X(x)dx = \sum_{k=1}^{\infty}\ddot{T}_k(t)\left\{-m_2 R X_k(l) - \gamma R\left[2l^2\mu_k^{-1}J_1(\mu_k) + \mu l\right]\right\}.$$

Let us represent the above equation in the following form:

$$\ddot{\varphi}_1 + b\dot{\varphi}_1 = \sum_{k=1}^{\infty} D_k\left[C_{1k}\cos(\omega_k t) + C_{2k}\sin(\omega_k t) + B_k e^{-bt}\right], \tag{21}$$

where

$$D_k = m_2 R X_k(l) + \gamma R\left[2l^2\mu_k^{-1}J_1(\mu_k) + \mu l\right],\ \ C_{1k} = \omega_k^2\left(a_k + \frac{A_k}{\omega_k^2 + b^2}\right),\ \ C_{2k} = \omega_k^2\left(b_k - \frac{b}{\omega_k}\frac{A_k}{\omega_k^2 + b^2}\right),\ \ B_k = A_k\frac{b^2}{\omega_k^2 + b^2}.$$

Taking into account zero initial conditions from Eq. (7), a solution of Eq. (21) is found by means of the Duhamel's integral in the form:

$$\begin{aligned}\varphi_1(t) &= \int_0^t \frac{1}{b}\left(1 - e^{-b(t-\tau)}\right)\sum_{k=1}^{\infty} D_k\left[C_{1k}\cos(\omega_k\tau) + C_{2k}\sin(\omega_k\tau) + B_k e^{-b\tau}\right]d\tau = \\ &= \varphi_{11}(t) + \varphi_{12}(t) = \sum_{k=1}^{\infty} F_k\left[\frac{1}{b^2} - \frac{1}{b}\left(\frac{1}{b} + t\right)e^{-bt}\right] + \sum_{k=1}^{\infty} D_k\left[F_{1k} + F_{2k}\cos(\omega_k t) + F_{3k}\sin(\omega_k t) + F_{4k}e^{-bt}\right],\end{aligned} \tag{22}$$

where

$$F_k = D_k A_k\frac{b^2}{\omega_k^2 + b^2} = \frac{b^2}{\omega_k^2 + b^2}\frac{R(a - b\omega_0)\left[2l^2\mu_k^{-1}J_1(\mu_k) + \mu l\right]\left\{m_2 R X_k(l) + \gamma R\left[2l^2\mu_k^{-1}J_1(\mu_k) + \mu l\right]\right\}}{l^2 J_1^{\,2}(\mu_k) + \mu l},\quad F_{1k} = \frac{1}{b\omega_k^2}C_{2k},$$

$$F_{2k} = -\frac{1}{b}\left[\frac{b}{\omega_k^2 + b^2}C_{1k} + \left(\frac{1}{\omega_k} - \frac{\omega_k}{\omega_k^2 + b^2}\right)C_{2k}\right],\ \ F_{3k} = \frac{1}{b}\left[-\frac{b}{\omega_k^2 + b^2}C_{2k} + \left(\frac{1}{\omega_k} - \frac{\omega_k}{\omega_k^2 + b^2}\right)C_{1k}\right],\ \ F_{4k} = \frac{1}{b\left(\omega_k^2 + b^2\right)}\left(bC_{1k} - \omega_k C_{2k}\right).$$

Function $\varphi_{11}(t)$ is the solution of that part of the Duhamel's integral which takes into account the effect of purely forced vibrations of the chain with a mass on the end. On the other hand, for function $\varphi_{12}(t)$ the effect of free vibrations, including those caused by the swinging of the load, is taken.

Finally, for function $\varphi(t)$ in the second approximation we have:

$$\varphi(t) = \varphi_0(t) + \varepsilon\varphi_1(t) = \psi_0 + a_0 t + \frac{(a_0 - \omega_0)}{b}\left(e^{-bt} - 1\right) + \varepsilon\left\{\sum_{k=1}^{\infty} F_k\left[\frac{1}{b^2} - \frac{1}{b}\left(\frac{1}{b} + t\right)e^{-bt}\right] + \sum_{k=1}^{\infty} D_k\left[F_{1k} + F_{2k}\cos(\omega_k t) + F_{3k}\sin(\omega_k t) + F_{4k}e^{-bt}\right]\right\}, \tag{23}$$

and for the angular velocity $\omega(t)$ the following form is obtained:

$$\omega(t) = \dot{\varphi}(t) = a_0 - (a_0 - \omega_0)e^{-bt} + \varepsilon\left\{\sum_{k=1}^{\infty} D_k\left[\omega_k\left(F_{3k}\cos(\omega_k t) - F_{2k}\sin(\omega_k t)\right) - bF_{4k}e^{-bt}\right] + t\sum_{k=1}^{\infty} F_k e^{-bt}\right\}. \tag{24}$$

Function $\omega(t)$ obtained in Eq. (24) allows us to determine the time when the stationary speed of motion is reached at startup or the time of stopping in the brake regime of the motion mechanism.

The partial differential equation Eq. (10) describes the vibrations of a chain with a mass on the end under inertial perturbation. Taking into account Eq. (23), this equation has the form:

$$g\frac{\partial}{\partial x}\left[\left((l + l_1 - x)\frac{\partial\nu}{\partial x}\right)\right] - \frac{\partial^2\nu}{\partial t^2} = R\left[\left(A + \varepsilon\sum_{k=1}^{\infty} F_k\right)e^{-bt} - \varepsilon bt\left(\sum_{k=1}^{\infty} F_k\right)e^{-bt}\right]. \tag{25}$$

Here, only purely forced vibrations are determined by function $\varphi_{11}(t)$ from Eq. (22) and are taken into account.

Let us apply the Fourier method. Hence, we represent function $\nu(x,t)$ and the right part of Eq. (25) in series by the Eigenfunctions for free vibrations (from Eq. (11) and Eq. (12)), and by applying the approach used in solving a partial differential equation in the first approximation, we obtain:

$$S_k(t) = A_k' e^{-bt} - A_k'' e^{-bt}, \tag{26}$$

where

$$A_k' = R\left(A + \varepsilon \sum_{m=1}^{\infty} F_m\right) \frac{2l^2 \mu_k^{-1} J_1(\mu_k) + \mu l}{l^2 J_1^{\,2}(\mu_k) + \mu l}, \qquad A_k'' = R\varepsilon b \sum_{m=1}^{\infty} F_m \frac{2l^2 \mu_k^{-1} J_1(\mu_k) + \mu l}{l^2 J_1^{\,2}(\mu_k) + \mu l}.$$

Functions $T_k(t)$ are the solution of differential equation Eq. (18), which has the following form in the second approximation:

$$\ddot{T}_k(t) + \omega_k^{\,2} T(t) = -A_k' e^{-bt} + A_k'' t e^{-bt}. \tag{27}$$

The general solution, as the sum of the solution of a homogeneous equation and a partial integral defined by the Duhamel's integral, is obtained as:

$$\begin{aligned} T_k &= a_k \cos(\omega_k t) + b_k \sin(\omega_k t) - \frac{A_k'}{\omega_k^{\,2} + b^2}\left[\frac{b}{\omega_k}\sin(\omega_k t) - \cos(\omega_k t)\right] - \frac{A_k'}{\omega_k^{\,2} + b^2} e^{-bt} + \\ &+ \frac{A_k''}{\omega_k\left(\omega_k^{\,2} + b^2\right)}\left\{\left(\omega_k t + \frac{2b\omega_k}{\omega_k^{\,2} + b^2}\right) e^{-bt} + \frac{1}{\omega_k^{\,2} + b^2}\left[\left(b^2 - \omega_k^2\right)\sin(\omega_k t) - 2b\omega_k \cos(\omega_k t)\right]\right\}. \end{aligned} \tag{28}$$

Finally, function $\nu(x,t)$ that describes the vibrations of a chain with a mass on the end, in the second approximation, is obtained in the form:

$$\begin{aligned} \nu(x,t) = \sum_{k=1}^{\infty} X_k(x) T_k(t) = \sum_{k=1}^{\infty} X_k(x)\Big\{ & a_k \cos(\omega_k t) + b_k \sin(\omega_k t) - \frac{A_k'}{\omega_k^{\,2} + b^2}\left[\frac{b}{\omega_k}\sin(\omega_k t) - \cos(\omega_k t)\right] - \frac{A_k'}{\omega_k^{\,2} + b^2} e^{-bt} + \\ & + \frac{A_k''}{\omega_k\left(\omega_k^{\,2} + b^2\right)}\left\{\left(\omega_k t + \frac{2b\omega_k}{\omega_k^{\,2} + b^2}\right) e^{-bt} + \frac{1}{\omega_k^{\,2} + b^2}\left[\left(b^2 - \omega_k^2\right)\sin(\omega_k t) - 2b\omega_k \cos(\omega_k t)\right]\right\}\Big\}, \end{aligned} \tag{29}$$

where the Eigenfunctions are defined by Eq. (20).

The solution of the system of differential equations Eq. (1) is obtained in two approximations and is determined by Eq. (23) and Eq. (29). In the next section, the numerical analysis is based on the results obtained in Eq. (23) and Eq. (29).

3. Numerical Results

In order to investigate numerically the motion of the crane trolley in the braking regime, we use the formulas of the approximate analytical solution of the system of differential equations of motion Eq. (1). To complete the analysis, we assume that the braking of the crane trolley results from an electric motor with reverse-current braking action. Moreover, we accept that the crane trolley's mass is equal to 6200 kg, and its speed is 25 m/min. After a reduction of the crane trolley to the dynamic model shown in Fig. 1(b), the corresponding numerical values for the model parameters are:

$$m_1 = 250{,}8\,\text{kg}, \quad m_3 = 5949{,}2\,\text{kg}, \quad \gamma = 2{,}274\,\text{kg/m}, \quad R = 0{,}16\,\text{m}, \quad l = 16\,\text{m}, \quad b_b = 4{,}83\,\text{Nms}, \quad F_W = 1216\,\text{N}.$$

In Fig. 2, the change in the angular velocity ω (during braking of the motion mechanism) as a function of parameter μ (the ratio of the end mass to the chain mass) is shown. The case when the mass of the load is ten times as great as the mass of the chain is shown with a solid line, i.e. at $\mu = 10$, and with a dashed line at $\mu = 1$, i.e. both masses are equal. It should be noted that the swinging effect of the rope with a load on the braking process can be observed more clearly when the value of μ increases, which corresponds to an increase in the mass of the load. It is established that with an increase in the value of μ, the unevenness of the angular velocity of the motion mechanism increases. Several sections of local extremes of $\omega(t)$ are highlighted, where acceleration changes will also be observed. The analytical formula of function $\varphi(t)$ obtained in this work allows us to determine the acceleration of the motion mechanism and, as a result, the additional dynamic load on its elements generated by load swinging.

The positions of the chain at various moments of stopping the motion mechanism when $\mu = 10$ and $\mu = 1$ are shown respectively in Fig. 3(a) and Fig. 3(b). Zero initial conditions at the initial moment of stopping are assumed for the chain, since this state corresponds to a large extent to the steady regime of motion of the trolley. The positions of the chain corresponding to 0.25; 0.5; 0.75 of braking time are shown by numbers 1; 2; 3, respectively, and at the moment of stopping, by number 4. It should be noted that these graphs are obtained when we take into account only the first three Eigenfrequencies. The position of the chain at the moment of stopping when the initial speed of the crane trolley is equal to 3,4 m/min is shown with a bold dashed line in Fig. 3(a). The data required for plotting Fig. 2, and Fig. 3 was obtained using a program created in the Visual Fortran based on the results in this work. The figures demonstrate that the curve describing the shape of the chain (Γ in Fig. 1(a)) approaches a straight line as the mass of the load increases. It is also evident that with an increase in the mass of the suspended load, the amplitude of the pendulum vibrations decreases. At a higher speed of the motion mechanism, a greater swinging of the load should be expected, which is confirmed by the placed on the chain at the moment of stopping, shown by a dashed line in Fig. 3(a).

The change in the angular velocity of the motion mechanism and the position of the chain without mass on the end, determined under the same initial conditions, are shown in Figs. 4(a) and 4(b) [26]. The positions of the chain corresponding to 0.25; 0.5; 0.75 of braking time are shown by numbers 1; 2; 3, respectively, and at the moment of stopping, by number 4. Note that these graphs are obtained when we take into account only the first three Eigenfrequencies. A smoother change in the angular velocity is observed. The difference in the shape of the chain is clearly noticeable when it is loaded and unloaded. When the motion mechanism is braking, the chain is deflected in only one direction, whereas in the presence of a load, motion occurs in both directions.

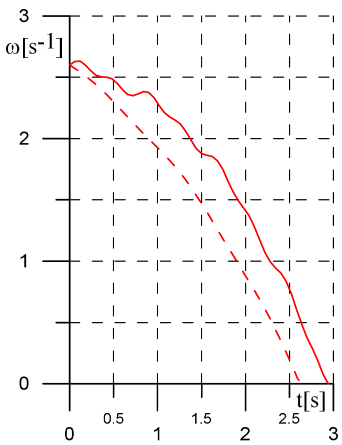

Fig. 2. Graphs of angular velocity ω as a function of parameter μ. The dashed line is for $\mu = 1$, and the solid line for $\mu = 10$.

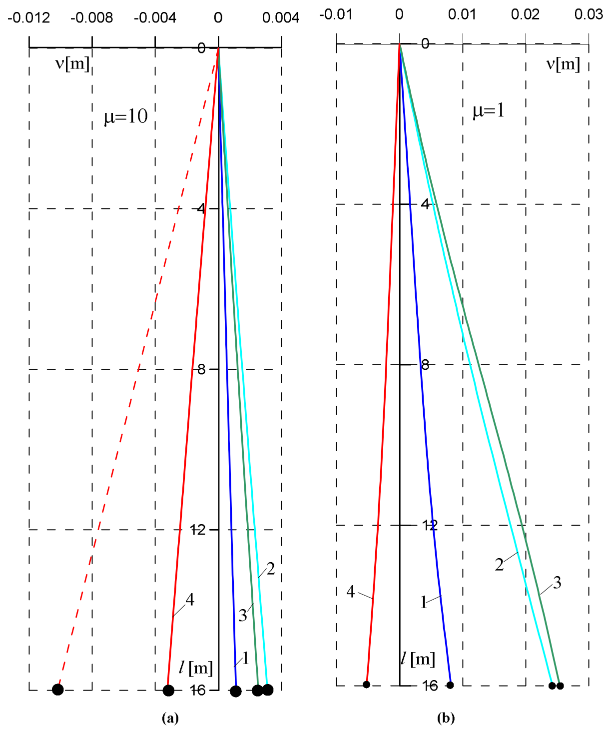

Fig. 3. Family of positions of the chain for various moments of stopping: (a) $\mu = 10$, (b) $\mu = 1$. For details, refer to the text.

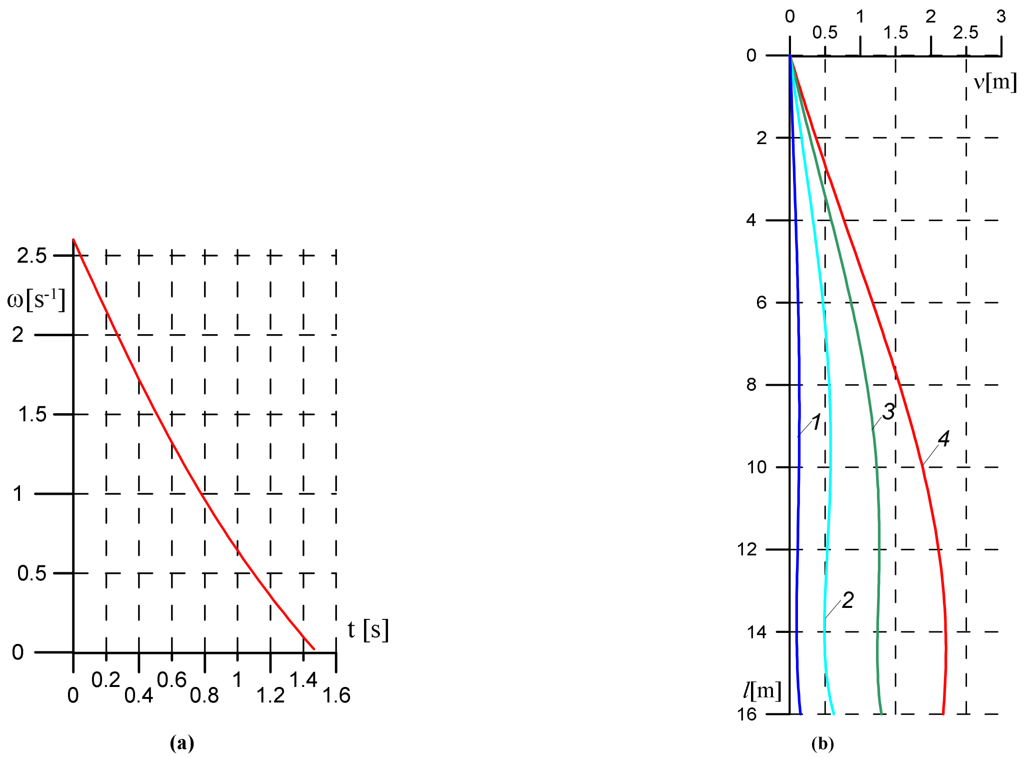

Fig. 4. Numerical results for: (a) angular velocity changes and (b) position of a chain with a missing mass on the end.

4. Conclusion

This paper presented an approach based on the application of approximate analytical methods which can be used to investigate the plane vibrations of a heavy chain with a mass in hoisting machines. The dynamical model of the motion mechanism of hoisting machines, such as a crane trolley, was presented. In the construction of the model, it was assumed that: (1) the chain is flexible, inextensible, homogeneous and heavy, and (2) the motor or brake torque is a linear function of the speed of the motion mechanism. The analytical solution of the differential equations of motion derived from the Hamilton-Ostrogradsky principle was obtained by two iterations based on the Fourier and small parameter methods, the representation of the motion of any point of the chain, and the intensity of the inertial perturbation forces by Eigenfunction series of the free vibration task. The load, considered as a concentrated mass, was represented as a uniformly distributed mass over a small portion of the chain around its free end, thereby ensuring the orthogonality of the Eigenfunctions. The analytical results thus obtained were illustrated with an example based on the technical characteristics of the crane trolley that the Visual Fortran computer program was created for this purpose.

The main conclusions from the present paper can be summarized as follows: From a theoretical point of view, we have presented an approach to an asymptotic analytical solution of a system of differential equations of the motion of mechanical systems with a heavy chain with a mass at the end, when the motion mechanisms of the hoisting machines are in a non-stationary regime of motion. The approach presented is an extension of the existing methods of dynamical investigation of mechanical systems with suspended loads. With this approach in hand and a computer program based on it in the Visual Fortran environment, we can continue with the further practical investigation of various properties of the flexible hanging chain with a mass on the end: calculation of suspended load deviation; determination of the rope form when the mechanism hoisting machines is in a start/stop regime, additional dynamic loads on the elements of the motion mechanisms caused by the pendulum movements of the load, etc. The analytical approach presented can also be considered a basis for the study of the spatial vibrations of heavy chains generated by the combination of the straight-line motion of the motion mechanisms and the rotation of the hoisting machines.

Author Contributions

All results presented in the work are equally owned by both authors. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

Nomenclature

F_W	Resistance force of the motion of the crane or trolley [N]
J_0, μ_n	Bessel function of the first kind and its roots (n = 1, 2, 3, …)
l	Chain length [m]
m_1, R	Mass and radius of the disk [kg], [m]
m_2	Mass of the load [kg]
m_3	Mass of the parts of the crane, trolley or motion mechanisms with translational motion [kg]
$m(x)$	Distributed mass function [kg]
T, Π	Kinetic and potential energy [J]
T_b	Motor torque determined by reduction to the drive wheels or the brake torque [Nm]
$u(x,t)$, $\nu(x,t)$	Functions determining motion of any point of the chain in $O'xy$ coordinate system at time t [m]
X_i	Eigenfunction (i = 1, 2, 3, …) [m]
X_i^*	Eigenfunction (i = 1, 2, 3, …) in the case of no mass at the end of the chain [m]
Y_0	Bessel function of the second kind
S	Hamilton's action [J.s]
$\delta' W$	Elementary work of non-potential forces [J]
γ	Mass of a unit length of the chain [kg/m]
ε	Small positive parameter
ε'	Length of the section on which it is distributed the concentrated mass [m]
φ	Angle of rotation function of the disk [rad]
μ	Ratio of the end mass to the chain mass
ω	Angular velocity function of the disk [s^{-1}]
ω_k	Natural frequencies (k = 1, 2, 3, …) [s^{-1}]

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