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Research Paper

Critical Strain Energy Levels Criterion for Structures with Lumped Parameters

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Abstract. The paper discusses the theory of critical strain energy levels for structures with lumped parameters. The theoretical assumptions and profs for common case are presented. The idea of external actions and strain energy field separation leads to the minimum strain energy principle. It has the self-stress of the structure physical sense. In the general case, a structure's extremal values of parameters are determined from an eigenvalue problem. The critical levels criterion means the self-stress state change. The strain energy consists of two parts: strain energy, which equilibrates the action work, and residual strain energy, which does not allow a deformable body to collapse. This allows for the total and residual strain energy to be calculated. The traditional problem formulation does not give us that option. The proposed theory is illustrated on a rod system, which explains the change in the self-stress state of the structure in a simple manner. The static matrix and stiffness matrix are obtained for the three-bar structure. The eigenvalue problem allows us to obtain the principal values of the nodal reactions and displacements of the structure. New formulations of structural design and structural analysis tasks are given. The results are compared with classical methods of solution. The formulations of weak link problems and progressive limit state problems are given. A structure's residual load capacity is evaluated by the residual strain energy.

Keywords: Energy methods, self-stress state, strain energy, critical energy criterion, matrix methods, lumped parameters.

1. Introduction

The most popular approach to the formulation and solution problems of structural mechanics remains the minimum total potential energy principle. For brevity, it will be called the Lagrange approach. Most problems in the statics and dynamics of structures [1-8] is based on this formulation of the problem. Numerical procedures which are popular among designers, including FEM (finite element method) and other numerical methods [9-13], are implemented based on the mentioned approach. As a result, a reaction of the structure depends on the actions prescribed by designer. After the element's parameters selection in the cross-section, the same parameters are assigned to other elements as well, to fulfill the process requirements. The structure is given a load-bearing capacity reserve that is difficult to estimate. As a result of the inspection of the building, it is required to assess the residual bearing capacity of its structures, after the appearance of defects from operational actions. When renovating a building, it is necessary to assess the residual load-bearing capacity of structural elements and the entire building to select a structural reinforcement.

Problems with tracking loads, temperature, and similar effects, where essentially nonlinear behavior of the structure is investigated, cannot be solved with satisfactory results in the Lagrange formulation, which has been repeatedly expressed by a number of scientists. Indeed, having obtained the equilibrium equations, continuity conditions, and surface conditions, from the variational principal the researcher cannot conclude whether the system has lost its carrying capacity or not. The concept of limit state, which is mathematically and physically unrelated to the Lagrange formulation, is introduced. As a result of solving the problem, the values of the parameters corresponding to a given load are obtained. Therefore, there is no way to estimate the maximum value of the load-carrying capacity of the structure. But then it is not possible to estimate the residual load-carrying capacity of the designed structure either. In this case even in linear statement the problem of designing structures gets iterative (in the sense of trial-and-error methodology) manner.

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Many theories of progressive collapse of structures [14-27] is explained by the fact that it is impossible to construct a theory based on the Lagrange principle that consistently excludes structural elements from operation. The structural optimization theory is developed to reduce the amount of residual load-bearing capacity [28-30]. But it possesses optimization for the objective function, which in many cases non guarantees eliminating residual strain energy. One of the ways to reduce the residual load-bearing capacity could be the theory of sensitivity of structural systems [31-33]. However, there are problems in solving nonlinear structures behavior tasks, and derivation calculations procedure.

The paper is devoted to the development of the theory of critical energy levels for structures with lumped parameters, which uses a new variational principle of minimum potential deformation energy [34]. Since the deformed state of a structure depends on its geometric parameters, mechanical characteristics, and support conditions, it is possible to calculate the self-stress parameters of the structure for any level of loading (including its absence). Based on the described above reasons, the idea of separating the fields of external influences and the field of deformations of the structure arises.

2. Materials and Methods

Considering the historically established approach in structural mechanics of setting the problem based on the minimum total potential energy principle of the structure, the suggested approach is based on this centuries-tested method. And we must know what mathematical model derive from the Lagrange formulation after separation strain field and action field.

2.1. Separation of the energies of the actions and strain fields

Let the domain of existence of the total strain energy functional of a solid deformable body Π can be divided into the areas of external $\eta \in \Omega_1$ and internal $\chi \in \Omega_1$ parameters. Then, according to the Bellman optimality principle [35], the extremum of the initial functional can be found as:

$$\underset{\chi,\eta \in \Omega}{\text{extr}}\, \Pi = \underset{\chi \in \Omega_1}{\text{extr}} \;\; \underset{\eta \in \Omega_1}{\text{extr}}\, \Pi. \tag{1}$$

This means that extremal of functional on external actions:

$$\delta \Pi(\chi,\eta) = \delta \Pi(\chi,c) = 0. \tag{2}$$

where c is the domain of external parameters defining the extremum.

Then, determining the extremum of the functional on the domain of internal parameters, from the condition of orthogonality of the extremals of external and internal parameters one can write:

$$-\delta \Pi(\chi) + \Pi(c) = 0. \tag{3}$$

Here $\Pi(\chi) = U(\chi)$ is the strain energy of the structure, and $\Pi(c)$ can be assumed as a constant, because the internal parameters vary only. The sign was chosen considering that the internal energy varies.

According to [36], the task (3) can be written as:

$$\delta U(\chi) = 0. \tag{4}$$

under the condition, that for all *j* extremals:

$$\delta U(\chi) = \Pi(c), \quad j = 1,2,\ldots,n, \tag{5}$$

or in the normalized to unit form:

$$\sum_j U_j^*(\chi) = 1. \tag{6}$$

Expression (6) is a completeness condition of the eigenfunctions of objective (4). According to [36, 37], this requirement corresponds to the condition of equality to zero and nonnegativity of the second variation of the functional. The connection between the first and second variations of the functional and the properties of global and local invariance of the functional is well known [37].

2.2. Self-stressing state of the structure and limiting state

Here we say a few words about the mechanical meaning of criterion (4) and (6). The proposed criterion consists in investigating the first variation of the strain energy functional of the deformable body. Values of external parameters at the domain of variation of internal parameters at extremals are constant. Constancy of external parameters does not exclude absence of external actions. We call this states the critical strain energy levels, including the initial unstressed state.

The normalization condition (6) makes it possible to describe any limit state in a uniform mathematical form, both for ULS (ultimate limit state) and SLS (serviceability limit state). The homogeneity property of the strain energy function allows us to remove the smallness restriction on the change in the generalized parameters of the problem. And the condition of orthonormality of generalized parameters of the problem allows us not to care about the requirement of their change smallness, which is the main problem of the overwhelming majority of step and iteration procedures in nonlinear problems of structural mechanics. It follows from the above that ordinary function (functional) (4) has physical meaning of self-stress state of the structure.

Rzhanitsyn [38] connected the normalization condition of generalized parameters (6) characterizing the strain energy with the state of self-stressing of the structure. He called self-stressed state as a state of the unloaded statically indeterminate structure. In structural mechanics terms deformable body consist of the bounds (links), pins and discs, when may be a bound. It means that the structure has a framework which resist to external actions, and guarantee against dividing into separate parts. If we remove one of the bonds and replace it with a unit of internal forces, we can calculate the self-stress state.

It is worth mentioning the Saint-Venant principle, according to which external influences sharply reduce their influence on the stress-strain state of the structure as they move away from the point of their application. Therefore, both the structure and the continuum will deform according to the laws prescribed by their shape and geometric dimensions, as well as by the support conditions and mechanical characteristics of the materials. At the same time, external influences are subject to the laws of the fields that caused them. But they are balanced by the internal reactions of the structure, so they do not work at the critical level.

When a structure is at a critical strain level, it is one of the possible self-stressed states that differs from the initial state by a constant multiplier. The new state can be obtained by increasing the external load, which will be balanced by internal forces if the limit state condition is not violated. Or we can consider possible forms of deformed state of the body when the load is constant or absent. When the critical level of strain energy is reached, the limit state of the body is approached. Then the deformation energy of the structure consists of two parts at any impact levels:

$$U_{extr} = U_F + U_S. \tag{7}$$

Here U_{extr} is the strain energy at a limiting state of the structure, U_F is strain energy part, which balanced work of action, U_S is the strain energy part, on which self-stress state transformation after limit state exceedance depends.

A further change in the energy of the system is possible due to the release of the residual value of the internal strain energy. A small perturbation of the strain energy field leads to a subsequent change in the equilibrium form of the system. But if the limit state condition is violated, the structure loses its bonds, the design model of the structure and the self-stress state changes.

The limit state of the structure is not violated if:

$$U_{extr} \le U_{cr}, \chi_{extr} \le \chi_{cr}. \tag{8}$$

The structure is stable and resists increasing impacts due to the residual part of the deformation energy of the structure. On the critical level of strain energy residual part of strain energy has self- balanced (self-stressed), as on the initial, non-loaded level. By this means, at critical energy levels, the type of loading and the magnitude (if normalized load values are meant) do not matter. It is important that under given conditions there is a self-balanced state of the system, and further actual work will be done by variations in internal forces or deformations fields.

2.3. Critical energy levels of structures with lumped parameters

In classical mechanics, it is stated that the choice of a phase variables and its rate of change is sufficient to describe any processes [37]. In our tasks more convenient use the vectors of generalized displacement ξ and generalized forces Φ, which arises as a variable conjugate to the vector of generalized displacements. Then strain energy takes a form $U(\chi) = U(\xi, \Phi)$.

The strain energy function $\tilde{U}(\tilde{\xi}, \tilde{\Phi})$ can be represented as a Taylor series expansion to quadratic terms on the areas of generalized forces $a \le \tilde{\Phi} \le b$ and generalized displacements $c \le \tilde{\xi} \le d$ as:

$$\tilde{U}(\tilde{\xi},\tilde{\Phi}) = \tilde{U}(\xi_0,\Phi_0) + \left.\frac{\partial \tilde{U}}{\partial \tilde{\xi}}\right|_{\tilde{\xi}=\xi_0} (\tilde{\xi} - \xi_0) + \left.\frac{\partial \tilde{U}}{\partial \tilde{\Phi}}\right|_{\tilde{\Phi}=\Phi_0} (\tilde{\Phi} - \Phi_0) + 2\left.\frac{\partial^2 \tilde{U}}{\partial \tilde{\Phi}\partial \tilde{\xi}}\right|_{\substack{\tilde{\xi}=\xi_0 \\ \tilde{\Phi}=\Phi_0}} (\tilde{\xi} - \xi_0)(\tilde{\Phi} - \Phi_0) + \left.\frac{\partial^2 \tilde{U}}{\partial \tilde{\xi}^2}\right|_{\tilde{\xi}=\xi_0} (\tilde{\xi} - \xi_0)^2 + \left.\frac{\partial^2 \tilde{U}}{\partial \tilde{\Phi}^2}\right|_{\tilde{\Phi}=\Phi_0} (\tilde{\Phi} - \Phi_0)^2. \tag{9}$$

Then in the nearness of the initial point ξ_0, Φ_0, it can be written in the form:

$$U(\xi,\Phi) = A\xi^2 + 2B\xi\Phi + C\Phi^2, \tag{10}$$

where $\xi = \tilde{\xi} - \xi_0$, $\Phi = \tilde{\Phi} - \Phi_0$, and the constants are denoted as:

$$A = \left.\frac{\partial^2 \tilde{U}}{2\partial \tilde{\xi}^2}\right|_{\tilde{\xi}=\xi_0}, \; B = 2\left.\frac{\partial^2 \tilde{U}}{\partial \tilde{\Phi}\partial \tilde{\xi}}\right|_{\substack{\tilde{\xi}=\xi_0 \\ \tilde{\Phi}=\Phi_0}}, \; C = \left.\frac{\partial^2 \tilde{U}}{2\partial \tilde{\Phi}^2}\right|_{\tilde{\Phi}=\Phi_0}. \tag{11}$$

At the stationarity point, the conditions are fulfilled:

$$\left.\frac{\partial \tilde{U}}{\partial \tilde{\xi}}\right|_{\tilde{\xi}=\xi_0} = 0, \quad \left.\frac{\partial \tilde{U}}{\partial \tilde{\Phi}}\right|_{\tilde{\Phi}=\Phi_0} = 0. \tag{12}$$

The Lagrange multipliers method are used to investigate the extreme problem:

$$\min U(\xi,\Phi) > 0, \xi^2 + \Phi^2 = 1. \tag{13}$$

The Lagrange function of the problem has the form:

$$L(\xi,\Phi) = A\xi^2 + 2B\xi\Phi + C\Phi^2 - \lambda(\xi^2 + \Phi^2). \tag{14}$$

The stationarity conditions of the Lagrange function give a system of equations that has a solution under the condition:

$$\begin{vmatrix} A-\lambda & B \\ B & C-\lambda \end{vmatrix} = 0. \tag{15}$$

Solving the quadratic equation and using the normalization condition we obtain a partial solution in the form:

$$\xi_1 = -\frac{B}{\sqrt{B^2 + (A-\lambda_1)^2}}, \qquad \Phi_1 = \frac{A-\lambda_1}{\sqrt{B^2 + (A-\lambda_1)^2}}. \tag{16}$$

The second pair of solutions is defined in the same way when replaced λ_1 by λ_2. If the roots of the characteristic equation are different, then one of them gives the maximum and the other the minimum value of the quadratic form $U(\xi,\Phi)$.

The state of a system with lamped parameters at critical energy levels is described by the problem of eigenvalues of the form Eqs. (13) and (15) or as:

$$U_{ik}^{\xi}\xi_k = \Lambda_{ik}^{\xi}\xi_k, k=1,...,n; i=1,...,n;$$
$$U_{ik}^{\Phi}\Phi_k = \Lambda_{ik}^{\Phi}\Phi_k, k=1,...,n; i=1,...,n. \tag{17}$$

The geometric meaning of the solution of the eigenvalue problem (17) is the transition from some arbitrary energy axes to the principal axes in the direction these axes energy is extreme. The eigenvalues of the problem correspond to the critical energy levels, when the behavior of the body changes [36, 37].

2.4. Matrix formulation of critical strain energy problems

To calculate the self-stress state of the deformable structure we must varying the general forces or general displacements in nodes at DOF (degree of freedoms) directions. If we examine nodal reaction of the structure on the kinematic action, then Eq. (16) may be that:

$$[L]\{\delta\Phi\} = [\lambda^L]\{\delta\Phi\}. \tag{18}$$

Here $[L]$ is the flexibility matrix of the structure; $[\lambda^L]$ is the eigenvalue diagonal matrix as:

$$[L] = \begin{bmatrix} \delta_{11} & \delta_{12} & ... & \delta_{1n} \\ \delta_{21} & \delta_{22} & ... & \delta_{2n} \\ ... & ... & ... & ... \\ \delta_{n1} & \delta_{n2} & ... & \delta_{nn} \end{bmatrix}, [\lambda^L] = \begin{bmatrix} \lambda_1^L & 0 & ... & 0 \\ 0 & \lambda_2^L & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n^L \end{bmatrix}. \tag{19}$$

where δ_{ij} are coefficients, denotes displacement in the i-direction from unit force in the j-direction. General reactive nodal vector of forces $\{\delta\Phi\} = [n^L]\{R\}$ consists from direction cosine matrix and reaction forces vector of the structure.

If nodal displacements are found out, stimulated by external actions, then Eqs. (17) lead to:

$$[K]\{\delta\xi\} = [\lambda^K]\{\delta\xi\}. \tag{20}$$

Here $[K]$ is the stiffness matrix of the structure; $[\lambda^K]$ is the eigenvalue diagonal matrix:

$$[K] = \begin{bmatrix} r_{11} & r_{12} & ... & r_{1n} \\ r_{21} & r_{22} & ... & r_{2n} \\ ... & ... & ... & ... \\ r_{n1} & r_{n2} & ... & r_{nn} \end{bmatrix}, [\lambda^K] = \begin{bmatrix} \lambda_1^K & 0 & ... & 0 \\ 0 & \lambda_2^K & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n^K \end{bmatrix}. \tag{21}$$

where r_{ij} are coefficients, denotes reaction in the i-direction from unit displacement in the j-direction. General reactive nodal vector of displacements $\{\delta\xi\} = [n^K]\{Z\}$ consists from direction cosine matrix and reaction displacements vector of the structure.

Some well-known formulae of structural mechanics let us calculate internal forces vector $\{N\}$ in structure elements:

$$\{N\} = [B]^{-1}[A]^T[L]\{\delta\Phi\}, \tag{22}$$

elements deformations $\{\varepsilon\}$:

$$\{\varepsilon\} = [A]^T[L][A]\{N\}, \tag{23}$$

and structure total energy U_t:

$$U_t = \{N\}^T[L]\{N\}/2. \tag{24}$$

Here $[B]$ is an internal flexibility matrix, and $[A]$ is static matrix of structure, T denotes transposition.

We are able not only to estimate the maximum and minimum possible generalized forces of the system, but also to determine from them the maximum possible strain energy of the system. Residual strain energy of the structure may derive as odds beside maximum strain energy of the structure and strain energy balanced external actions work:

$$U_{res} = U_{extr} - U_F. \tag{25}$$

The residual carryon capacity of the structure is the main goal of the design procedure. But nowadays methods used probabilistic basis to estimation which in construction not always justified.

2.5. The weak link problem and progressive limit state problem

One of the goals of structural mechanics is to determine the displacements and forces what allow to find the section and point of the structure in order to write down the corresponding limit state condition. The formulation of the limit state criteria of the system is not considered in structural mechanics, but in the regulatory documents. This issue is also related to the evaluation of strength properties of materials.

The left part of the inequalities describing the limiting state of the structure contains the extreme values of the structure parameters found by the methods of structural mechanics (8), as $U(\xi_{extr},\Phi_{extr})$ is extremal value of strain energy; ξ_{extr} are extremal values of general displacements; Φ_{extr} are extremal values of general reactions. In the right part of the inequalities, there are experimental data, which are documented by normative documents, as U_{cr} is limit value of strain energy; ξ_{cr} are limit values of general displacements; Φ_{cr} are limit values of general forces:

$$U(\xi_{extr}, \Phi_{extr}) \le U_{cr};\ \xi_{extr} \le \xi_{cr};\ \Phi_{extr} \le \Phi_{cr}. \tag{26}$$

It is well known that the limit states of a structure of the first group are closely related to the appearance of extreme values of the deformation energy of the body. The limit states of the second group can be put only as constraints on the design parameters and serviceability parameters of the structure, and in some cases, the energy of the system at a certain moment of deformation. In mathematics, only the apparatus of the theory of problems on eigenvalues, which allows us to consider the restrictions of the above types in a unified formulation, is known. It seems that all the theories describing the destruction of the system from the point of view of structural mechanics will eventually lead to a formalization related to the term of removal of bonds. Thus, for a geometrically stable structural system which has received displacements and deformations as a result of external actions, the condition for reaching the limit state can be represented as the removal of redundant bonds (transformation of the system into a statically determined one) and then the removal of one of the conditionally necessary bonds. The weak link will be understood as the link to which the internal forces (deformations) reach the limit value first. Obviously, these are the maximum values of the generalized forces or displacements that can be achieved in the system [39]. If we construct the theory of sequential elimination of links, it is necessary to obtain a method of determining the link in the system, which will start the process of removal of links, due to the onset of the limiting state in them [40, 41]. Residual strain energy value is determinate by subtraction of the external actions work from total strain energy.

3. Results

Let us consider some problems to illustrate methodology of structure critical strain energy study. The static matrix, flexibility matrix, eigenvalues matrix and eigenvectors were obtained in [40, 41].

3.1. Statically indeterminate structure with two DOF and one node

Consider the statically indeterminate system shown in Fig. 1, where the values of the stiffnesses of the rods are the same $\eta_1 = \eta_2 = \eta_3 = 1$, and the inclined rods are located at equal angles $\alpha = \beta = \pi / 4$, $l = 2$ m, the same areas of rods of round cross section $A = 0.785 \times 10^{-4}$ m^2, diameters $d = 10$ mm, elastic modulus $E = 2.1 \times 10^5$ MPa, yield stress $\sigma_t = 240$ MPa. Check the bearing capacity of the system from the load applied along the horizontal and vertical axes, respectively $F_1 = 10$ kN, $F_2 = 20$ kN.

3.2. External loads and nodal reactions comparison for the structure carrying capacity determination

Nodal reactions of the structure obtained as a result of solving the problem on eigenvalues form an ellipse of limiting reactions of the structure to external actions [40]. Without calculating the stresses in the bars of the structure, the load carrying capacity of the structure can be estimated by comparing the magnitude of the external load and the reactive nodal response of the structure.

Let's find the resultant of external forces, the angle of inclination to the horizontal axis and the normalized value of the load: $F = 22.36$ kN, $\tan(\varphi) = 20 / 10 = 2$, $\varphi = 63.32^\circ$, $\bar{F} = F / 22.36 = 1.0$.

The radii of the limit load ellipse for the acting force are determined [40]. The maximum and minimum radii of the reaction ellipse of the structure from the unit kinematic displacements are calculated by the formula:

$$\{R_{max}\} = [\lambda^K_{max}]\{n^K_{max}\}\{\bar{Z}\} = 1.707\Delta EA / l,$$
$$\{R_{min}\} = [\lambda^K_{min}]\{n^K_{min}\}\{\bar{Z}\} = 0.7072\Delta EA / l. \tag{27}$$

A reactive force balancing the external action occurs in the structure node. The limit value of this reaction is defined as follows:

$$\{R_{min}\} = 0.7072\Delta EA / l = \sigma_t A. \tag{28}$$

Then displacement from limit action is $\Delta = \sigma_t l / (0.7071E)$, and ellipse radii from reactive nodal forces turns out that:

$$\{R_{max}\} = 1.707\sigma_t A / 0.7072 = 2.414\sigma_t A,\ R_{min} = \sigma_t A \tag{29}$$

Find the dimensionless value of the permissible load in the same direction as the given load:

$$\bar{F}_{cr} = R_{max}R_{min} \Big/ \sqrt{R^2_{max}(\sin\varphi)^2 + R^2_{min}(\cos\varphi)^2} = 2.205, \tag{30}$$

where $\bar{F}_{cr} = F_{cr} / \sigma_t A$.

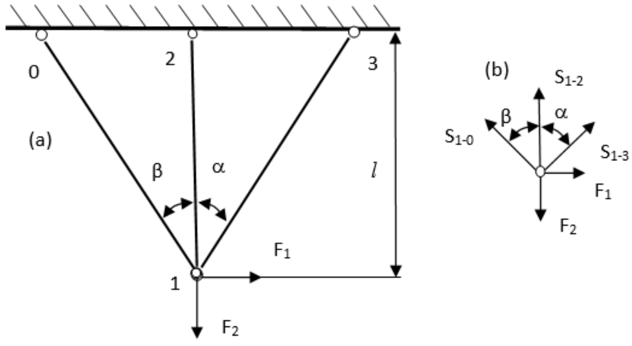

Fig. 1. Two DOF structure: (a) model of structure, (b) node 1.

If we get cross section one of the structure bars $A = A_1$, the largest value of the reactive nodal force generated by the system in the direction of the acting external load is as follows $F_{cr} = \sigma_t A_1 = 2.05 \times 240 \times 0.785 \times 10^2 = 38.62$ kN.

The value of the specified external load is less than the limit reaction value obtained. The structure does not lose its load-bearing capacity from the given load. This means that no yield stresses occur in any of the bars of the system. The structure has a load safety margin (or residual load-bearing capacity) and can be subjected to an optimization procedure to reduce the residual load-bearing capacity.

3.3. Comparison of the forces in the bars from a given normalized load and the forces detected during self-stressing of the system

It is considered that the limit state of the structure occurs when the limit value of the force in one or more rods is reached simultaneously. We normalize the vectors of projections of external loads on DOF directions and obtain it in the form of $\{\bar{F}\} = \{F\} / 22.36 = \{0.4472; 0.8945\}^T$.

Determining the internal forces in the rods of the structure from the normalized vector of external load:

$$\left[\bar{S}^P\right] = \begin{vmatrix} 0.5782 \\ 0.524 \\ -0.05423 \end{vmatrix}. \tag{31}$$

Finding the vector of internal forces from the nodal reaction vector of the system:

$$\left[\bar{S}^F\right] = \begin{vmatrix} 1.0 \\ 0.5858 \\ -0.4141 \end{vmatrix}. \tag{32}$$

From the results obtained, it is observed that the largest force from the external load occurring in rod 0-1. It is less than the self-stressing force in the same rod $\bar{S}^P_{0-1} > \bar{S}^F_{0-1}$. Consequently, this rod should not lose strength from a given load. Therefore, the specified rod should not lose strength from a given load. The compared forces can be represented in dimensional units. Internal force from the load in rod 0-1: $S^P_{0-1} = 0.5782 \times F = 0.5782 \times 22.36 = 12.93$ kN. Magnitude of internal force in the same rod from the nodal reaction of the structure $S^F_{0-1} = \sigma_t A = 240 \times 10^6 \times 0.785 \times 10^{-4} = 18.84$ kN. The permissible maximum force in the rod from the magnitude of the system reaction is greater than the force from a given load by 5.91 kN. The structure does not lose its load-bearing capacity from a given load. This means that no yield stresses occur in any of the rods of the system.

3.4. Comparison of stresses in bars from a given load and allowable stresses (design resistances) according to traditional methods

Determining the internal forces in the bars of the structure from the external load {F} = {10; 20} (kN):

$$\{S\} = \begin{vmatrix} 12.93 \\ 11.72 \\ -1.213 \end{vmatrix}. \tag{33}$$

$$\sigma^{cr}_{1-0} = \frac{S^{cr}_{1-0}}{A} = \frac{12.93}{0.785 \times 10^{-4}} = 16.47 \times 10^4 \text{ kN/m}^2 = 164.7 \text{ MPa}. \tag{34}$$

which is less than the yield stress of the rod material. That is, the rod, and therefore the system, will not lose its load-bearing capacity.

3.5. Residual value of stress energy of the structure

One of the most important applications of the theory of structural mechanics in practical calculations is to find the residual strength of a structure. In most cases, a probabilistic expression of a structure's ability to carry the load (residual life) or increase the load in the future is formulated.

The theory of critical energy levels allows us to calculate the residual strain energy in deterministic form. If we know what maximum strain energy a structure can possess, it is easy to calculate the residual strain energy. To do this, it is necessary to subtract from the maximum value of the deformation energy of the structure the value of the potential deformation energy equal to the work of external forces (25).

To calculate the strain energy, we will use expressions (24):

$$U^{\Phi}_{\max} = 0.7071(\sigma_t A)^2 l / EA. \tag{35}$$

The magnitude of the energy from the external load can be calculated through the forces in the bars of the structure caused by the external load:

$$U_F = 0.3758(\sigma_t A)^2 l / EA. \tag{36}$$

The residual potential energy of deformation of the structure is:

$$U_{res} = 0.3313(\sigma_t A)^2 l / EA. \tag{37}$$

The reserve of bearing capacity of the structure is due to the fact that the load applied to the structure under the problem condition is not ultimate. Only 53.1% of the bearing capacity is exhausted.

3.6. Problems "weak link" and "progressive limit state" of a structure

Knowing the residual load-bearing capacity of a structure is not the only goal of structural design. It is equally important to know which of the structure's bars will be the first to fail under load due to the onset of the limit state in it. Such a rod will be called a "weak link", and the problem of finding it will be called the problem of the weak link of the structure.

If the structure does not lose its load-bearing capacity after the weak link is removed, the limit state can occur in the next rod and so on. The load may remain constant or increase further until the structure becomes geometrically changeable. The problem of successive failure of system elements will be called the problem of "progressive limit state".

Returning to the previous problem in Section 3.5, we notice that the remaining part of the structure strain energy is in a self-stressing state (like the unloaded structure). Due to this fact, it is possible to load the structure by increasing the external load components. Since further loading of the structure will be at the expense of the residual strain energy, which is in a state of self-stress, it is possible to use the results (31) to identify in which of the rods the first limit state will occur. Once the limit load is exceeded, the rod will no longer carry the load, and its further increase will be absorbed by the remaining rods of the system.

In order to obtain the limit state of the structure already at the first stage of the structural calculation, for the problem given in Section 3.1, we assume the cross-sectional area of the rods $A_2 = 0.1963\times10^{-4}$ m^2 (circular cross-section d = 5 mm). This variant could occur due to rod corrosion or other reasons of abuse operation.

From the solution of the problem in Section 3.3, it follows that the force in rod 0-1 is $S^{P}_{0-1} = 12.93$ kN, and the stress is $\sigma = 658.7$ MPa. The force in the same rod from the reactive nodal force (self-stressing force) is equal to $S^{\delta\Phi m}_{0-1} = \sigma_t A = 4.711$ kN. That is, rod 0-1 will be the first to lose strength and no longer work under load. At the same time, the structure remains geometrically unchanged and can withstand the load. Recall that the value of strain energy at any critical level is determined by multiplying the value of energy at the first (initial) critical level of strain energy by a constant [39].

Because of the complete symmetry of the structure, the flexibility matrix is diagonal, indicating that the main diagonal contains the eigenvalues of the flexibility matrix:

$$[L] = [\lambda^L] = \frac{l}{EA}\begin{vmatrix} 1.414 & 0 \\ 0 & 0.5858 \end{vmatrix}. \quad (38)$$

It is not difficult to write out the eigenvectors of the flexibility matrix. Two vectors form a vector matrix:

$$[\vartheta^L] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad (39)$$

that is, the originally chosen axes are the principal axes of the limiting energy of the system.

It should be noted that since the quantity of the components of the flexibility matrix [units of length / units of force], the components of the eigenvector (22) have the quantity [units of force], then the multiplication $Z_{\max,\min} = [\lambda^L]\{\vartheta^L_{\max,\min}\}$ has the dimension of quantity displacement, and shows the values of projections of nodal displacement on the originally selected coordinate axes.

The stiffness matrix of the system is also diagonal and contains eigenvalues:

$$[K] = [\lambda^K] = \frac{EA}{l}\begin{vmatrix} 0.7072 & 0 \\ 0 & 1.707 \end{vmatrix}. \quad (40)$$

with their corresponding eigenvectors:

$$[\vartheta^K] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}. \quad (41)$$

Let's calculate the forces for the possible states of self-stress. By varying the values of the system response in the direction of the first axis shown in Fig. 1, we obtain the internal forces in the rods of the structure for the first state of self-stressing. Thus, for the first direction of the unit vector of the internal reaction $\delta\Phi^1_{\text{in}} = 1$, according to (21), we have:

$$S_{1-2} = 0,\ S_{1-0} = -S_{1-3} = 0.7071. \quad (42)$$

That is the first state of self-stressing corresponds to the compression of the right inclined rod and the stretched left one. There is no force in the vertical middle rod. The values obtained from the unit forces have the quantity [units of force].

For the second state of self-stressing, in the case of variation of the compliance in the direction of the second eigenvector $\delta\Phi^2_{\text{in}} = 1$, we have the following force distribution:

$$S_{1-2} = -0.5858,\ S_{1-0} = S_{1-3} = 0.2929, \quad (43)$$

when the two inclined rods are tensed by the same force and the vertical rod is compressed. The forces from the combined action will be equal to the sum of the received separately in Eqs. (28) and (29).

The limiting state of the structure is choosing for the maximum value of the internal forces $S^{cr}_{1-0} = 0.7071P^{cr}$, obtained at the first stage. Here it should be mentioned that the sign of the forces is important only for choosing the type of the limit state: loss of tensile (compression) strength or stability of the rod. If the limiting state corresponds to the elastic phase of the material, then $P^{cr} = A\sigma_I$. Where A is the cross-sectional area of the rod, and equivalent stress according to the first strength hypothesis is σ_I. Then the critical stress be:

$$\sigma^{cr}_{1-0} = \frac{S^{cr}_{1-0}}{A} = 0.7071\sigma_I. \quad (44)$$

If the elastic-plastic operation of the material is allowed, the ultimate stress σ_t, and the magnitude of the ultimate load $P^{cr} = A\sigma_t$. Similarly, for the cases of rod stability, crack theory, and other types of stress state.

The selected symmetric design scheme is close to ideal with respect to the loads acting. Therefore, the calculated energy of the structure according to (23) is maximum for the first axes and is equal to $\bar{U}_t = 1,\ \bar{U}_t = U_t / [(\sigma_I A)^2 l / EA]$. For the first load (self-stressed state) will be $\bar{U}^1_t = 0.707$, and for the second load (self-stressed state) $\bar{U}^2_t = 0.293$.

We should pay attention to the obvious fact that the total strain energy of the structure is greater than in the other two cases of self-stressing. If we assume that the structure has been designed for a vertical load, it has residual a load carrying capacity equal to $\bar{U}^2_{res} = 0.707$.

Let's determine which rod will lose its bearing capacity next. In the second step, we obtain the system shown in Fig. 2 without the remote rod.

It is easy to see that $\mathrm{Det}[A] \neq 0,$ and hence the system is unchangeable.

The matrix of flexibility coefficients (internal flexibility) of the rod system taking into account the notations $\eta_2 = EA / E_2A_2$, $\eta_3 = EA / E_3A_3$, has the form:

$$[B] = \frac{l}{EA}\begin{vmatrix} \eta_2 & 0 \\ 0 & \eta_3 / \cos\alpha \end{vmatrix}. \tag{45}$$

The flexibility matrix (external flexibility) of the rod system is written as:

$$[L] = \frac{l}{EA}\begin{vmatrix} 3.8285 & 1 \\ 1 & 1 \end{vmatrix}. \tag{46}$$

Now we have non symmetric structure. The matrix of eigenvalues takes the form:

$$[\lambda^L] = \frac{l}{EA}\begin{vmatrix} 4.1463 & 0 \\ 0 & 0.68217 \end{vmatrix}. \tag{47}$$

The eigenvectors of the system corresponding to the eigenvalues are represented in the following vector matrix:

$$[\vartheta^L] = \frac{l}{EA}\begin{vmatrix} 0.953 & -0.3029 \\ 0.3029 & 0.953 \end{vmatrix}. \tag{48}$$

For the three types of self-stress the internal forces in the rods for the unit vectors of maximum eigenvalues are:

$$S^t_{1-2} = 2,\ S^t_{1-3} = -1.414,\ S^1_{1-2} = 1,\ S^1_{1-3} = -1.414,\ S^2_{1-2} = 1,\ S^2_{1-3} = 0. \tag{49}$$

The result means that the ultimate load in the vertical rod of the two-rod system is higher in absolute value than in the inclined rod. If we assume that the material of the system resists tension and compression equally, then the system becomes geometrically variable from the loss of tensile strength of the vertical rod.

The algorithm discussed above illustrates a new formulation of the problem, which we will call the progressive limit state problem [37].

For unsymmetrical structure at second stage (Fig. 2) we have energy $\bar{U}_t = 3.414$ for two unit variation action, and $\bar{U}^1_t = 0.5$ for the horizontal unit variation action (first self-stress), and for the vertical unit variation action (second self-stressed state) $\bar{U}^2_t = 1.914$. As one would expect, the energy for the two cases of self-stress is not equal to the energy from the two unit variation action applied simultaneously. This is a consequence of the violation of the principle of superposition of works in the case of the action of several forces. However, for the main energy axes, the superposition principle is satisfied because there are no summands of multiplication parameters as in Eqs. (8) and (16).

In two rods structure we have three principal self-stressed states with the internal forces:

$$\begin{gathered} S^{pt}_{1-2} = 5.651,\ S^{pt}_{1-3} = -5.296,\ S^{p1}_{1-2} = 5.207, \\ S^{p1}_{1-3} = -5.588,\ S^{p2}_{1-2} = 0.4435,\ S^{p2}_{1-3} = 0.2922. \end{gathered} \tag{50}$$

Here indices "*pt*" denote the internal forces from the simultaneous action of two principal reaction vectors in the nodes of the structure. Indexes "*p*1" denote the internal forces in the rods from the maximum principal reaction vector in the nodes of the structure, and "*p*2" denote the internal forces from the minimum principal reaction vector in the nodes of the structure.

Strain energy for each case of loading (35) is:

$$\bar{U}_{pt} = 35.8,\ \bar{U}_{p1} = 35.64,\ \bar{U}_{p2} = 0.1587. \tag{51}$$

As one can see, the principal superposition of works is satisfied for the main nodal reactions. Construction has two critical levels of strain energy, before becoming unstable. The strain energy increases as the structure loses its bonds. The internal forces in the remaining bonds increase to carry the applied load.

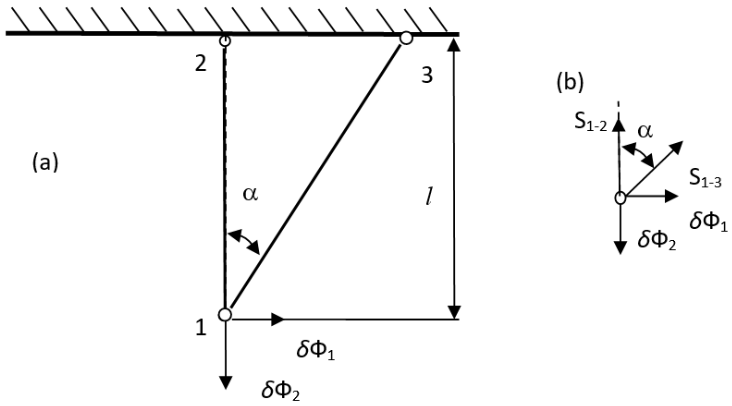

Fig. 2. Three-rod structure: (a) without remote rod 0-1, (b) node 1.

4. Limitations and Future Research

The goal of the paper was to provide a mathematical basis for the hypothesis on the existence of such levels of internal energy of the continuum after passing which its physical model and consequently the properties of the internal energy field changes. In this case, the properties of the field determine the presence of critical energy levels, and external influences only reveal this fact, but not cause it.

Since the authors' area of interest is structural mechanics and the mechanics of a solid deformable body, the meaning of the proposed variational principle is to preserve the continuity (bearing capacity) of the load-bearing structural system of a deformable body. Then the variation criterion for destruction (loss of load-bearing capacity, violation of the conditions of the limit state of the structure) is to overcome a sufficient number of critical energy levels. In the language of structural mechanics, the criterion for the loss of load-bearing capacity (strength, stability, etc.) is the loss of a sufficient number of connections correlated with the degrees of freedom of the structural system.

The traditional approach to setting structural mechanics problems excludes the possibility of determining the residual load-bearing capacity of a structure. Existing probabilistic approaches to determining residual life do not make it possible to calculate the residual energy of a structure after removing the connection before losing its load-bearing capacity, and the total energy of deformation of the structure. Therefore, further efforts will be aimed at developing a methodology for determining the total and residual deformation energy of a structure.

Existing methods of progressive collapse have a large number of implementation options. This indicates that the model of the phenomenon is insufficiently developed. It is impossible to understand from which element the destruction of the structure will begin under various influences and what is the general criterion for collapse. The proposed variational principle and criterion provide a theoretical justification for the model in the understanding of the limit state of a structure accepted from structural mechanics. Therefore, one of the directions is to study the sequence of occurrence of the limit state in structural elements (progressive limit state).

The classical formulation of the strength of materials problems implies obtaining design parameters for a particular case of load action, then checking the compliance of the obtained parameters with the hypothesis of the limit state of a dangerous structural element. Therefore, it is impossible to avoid step-by-step solutions to structural mechanics problems, even in static problems. The proposed method allows to avoid a step-by-step solution, since the extreme possible values of the parameters are immediately found. Therefore, a new technology for designing structural load-bearing systems will be built. Thus, it is impossible to solve the optimal design problem in general form, and not for a given type of loading. Problems with tracking loading and a number of temperature problems can also be solved effectively based on the proposed variational principle. In the authors' opinion, the main problem of strength calculations of nanostructures is the inability to formulate surface loads, which is not decisive in the proposed approach. It is possible to list other areas of research in the mechanics of solid deformable bodies, but the scope of the article does not allow this.

5. Conclusion

The separation of the fields of external actions and the field of strain energy allows us to understand that the limit state of a structure depends on the level of strain energy and its state of self-stressing. There-fore, the criterion of the limit state is the change in the self-stressing of the structure, which occurs at the critical level of strain energy. The self-stress energy of the structure changes from the initial level of the unstressed structure to the critical level in the domain of self-reciprocal function of parameters of the strain energy field in proportion to a constant value (similarly). The proposed criterion and methodology of structural analysis allows to estimate the maximum carrying capacity possible for the existing structure. The classic approach allows to find a design for the load of a particular type and magnitude. As a consequence, determining the maximum load capacity becomes a task with an infinitely large number of variants. The residual strain energy of the construction is determined by the difference between the maximum strain energy of the deformable structure and the energy equal to the work of external forces to deform the structure. This energy is in a self-balanced form until a small variation of the strain field parameters at a critical level does not lead to a change in the design model or the appearance of irreversible defects in the material of the structure. The residual energy of deformation of a construction is the most important value that allows us to evaluate the residual bearing capacity of the structure in service. The eigenvalue model of design problem allows from a unified methodological point of view to represent the limit states as a condition of not exceeding by the internal parameters the values established from the most different assumptions of the designer. This is one of the main differences between the proposed weak link and progressive limit state theory and the progressive collapse methodology. Many variants of formulation of problems of progressive collapse and their solutions testify to the weakness of the concept and insufficient development of theoretical foundations. One of more important issue is the result of calculating the total and residual strain energy, which allows us to estimate the total and residual bearing capacity of the structure.

Author Contributions

L. Stupishin: conceptualization and supervision, formal analysis, supervision; M. Moshkevich: formal analysis and writing—original draft; M. Rynkovskaya: formal analysis and writing—original draft. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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