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Research Paper

An Approximate Solution of the Bending of Beams on a Nonlinear Elastic Foundation with the Galerkin Method

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Abstract. The analysis of beam deformation on elastic foundation is very important in engineering applications, and studies of beams on linear elastic foundations are abundant and accurate. For practical problems, it is always demanded that the influence of the nonlinear effect of the foundation on the analytical methods and results must be considered. This study treats the static bending problem of an elastic beam resting on the nonlinear elastic foundation by solving the nonlinear differential equations with the Galerkin method, converting the nonlinear differential equations to a system of nonlinear algebraic equations for approximate solutions. The nonlinear equations are solved with a series expansion of the deflection satisfying the boundary conditions, and coefficients of the series are obtained with usual techniques including the iterative method. The accuracy of the approximate solution with the Galerkin method is verified through examples from earlier studies. The procedure and results show that the Galerkin method is effective in solving static nonlinear differential equations in addition to the nonlinear vibrations with the extended Galerkin method in earlier studies.

Keywords: Nonlinear foundation, Beam, Deflection, Galerkin method.

1. Introduction

The elastic beams resting on an elastic foundation have a wide range of applications in engineering fields including civil, mechanical, and many others. A typical Winkler foundation consists of countless closely spaced and unconnected linear springs with each is uniquely defined by the foundation modulus for an easy treatment mathematically [1]. There are plenty of textbooks and literature on the basic theory and method for the analysis of beams resting on elastic foundations [2-4]. Other types of foundation models have also been extensively studied as an extension, such as beams on viscoelastic foundations [5] and two-parameter foundations [6].

With the linear theory of elasticity, it should also be noted that the nonlinear behavior of the reaction force and the displacement of beams will lead to a large difference between the analytical results and the results from the linear theory. Kondner [7] observed experimentally that the load-deflection relationship of the soil has the shape of a general hyperbolic curve, and that this nonlinear relationship makes the problem more difficult to solve analytically. On this basis, Reaufait and Hoadley [8] used the hyperbolic function of deflection to approximate the reaction force of soil and solved the differential equation of a beam on a nonlinear foundation by the finite difference method (FDM) using an iterative method. Similarly, Frydrysek et al. [9-11] approximated the nonlinear relationship of the foundation through data fitting with the third- or fifth-degree polynomials based on beam deflections and solved the nonlinear problem again using the finite difference method. Nevertheless, experimental data have limitations and may not fully represent the properties of soil foundations. For this reason, Soldatos and Selvadurai [12] introduced a nonlinear factor to model the hyperbolic nonlinearity of the foundation and used the regression method to transform the nonlinear differential equations into a set of linearized differential equations thus solving the problem of beam deflection. With the advances in research on nonlinear problems and mathematical techniques, novel approximation methods have been developed, and the homotopy analysis method (HAM) proposed by Liao [13-15], as a new approximation method has been widely accepted and applied to the solution of practical problems [16-19]. Mohammerodpour et al. [20] applied the HAM to solve the nonlinear differential equations governing the deflection of beams on a hyperbolic nonlinear foundation. Jang et al. [21] established a nonlinear integral equation relating beam deflection to the nonlinear foundation and proposed a more convenient and concise iterative method to solve the problem.

For dynamic and static analyses of structures, a variety of analytical or approximate analysis methods are available to help engineers. Duong et al. [22] employed a higher-order shear-normal deformation theory coupled with analytical techniques to explore the static behavior and stress concentration effects in cylindrical shells made of functionally graded carbon nanotube-reinforced composites under different boundary conditions. Tien et al. [23] integrated nonlocal theory with various shear strain

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theories to formulate the equilibrium equations for organic nanoplates utilizing Hamilton's principle, and they obtained analytical solutions based on Navier's solution approach to study the static bending and free vibration characteristics of these nanoplates. The Galerkin method is a popular method for approximate analyses of linear and nonlinear problems such as the structural analysis, vibrations, and wave propagation. This method, which can be interpreted as a projection of the equations onto a linear space [24], allows obtaining relatively accurate solutions of nonlinear differential equations in a simple procedure [25], and the use of appropriate trial functions for the solution that satisfies the basic boundaries and initial conditions allows making the computational process more efficient, which in turn accelerates the convergence of the solution and improves its accuracy [26]. Ipek et al. [27] applied the Galerkin method to solve the fundamental equations of moderately-thick nanocomposites on a Winkler elastic foundation in thermal conditions, obtaining closed-form solutions for uniaxial and biaxial critical compressive loads. Sofiyev used superposition and Galerkin method to transform the basic equation of interaction between an orthogonal anisotropic cylinder and a nonlinear Winkler foundation into a nonlinear ordinary differential equation, and combined with the semi-inverse method or homotopy perturbation method to obtain nonlinear dynamic response [28, 29]. The Galerkin method has also been used in the analytical solution of beams and plates on elastic foundations, all of which yield well approximation results [30, 31]. As an illustration of utilizing the approximate analysis methodology, this study addresses the deformation of a beam resting on a hyperbolic Winkler foundation under the influence of concentrated loads, with both ends rigidly clamped. The Galerkin method is used to approximate the solution of linear differential equations of beams resting on nonlinear foundations. After transforming the nonlinear differential equation into a system of nonlinear algebraic equations of coefficients of the deflection solution, an iterative procedure is utilized to solve for the unknown coefficients. The results from both methods are discovered to be in close agreement. The approximate results of the Galerkin method are quite accurate and the computational process is concise and efficient, and this procedure is also applicable to the problem of bending deformation of a finite beam supported on a hyperbolic nonlinear foundation subjected to an arbitrary form of external force. This study provides a successful example of utilizing the Galerkin method in dealing with a wide variety of nonlinear differential equations of which there are no analytical or other approximate solutions available.

2. The Flexural Deflection of a Beam on a Nonlinear Elastic Foundation

Considering the elastic beam as a slender structure with a length significantly exceeding its width and height, and disregarding shear deformation, its flexural behavior is described by the Euler-Bernoulli beam theory. It is assumed that the beam maintains close contact with the foundation, ensuring synchronized deformation between the beam and the foundation, with no gaps present. The foundation's support is represented by a series of springs, as depicted in Fig. 1.

Fundamentally, for a linearly elastic beam of width b subjected to a transverse load $p(x)$, the Euler-Bernoulli equation can be written as [20]:

$$EI\frac{\mathrm{d}^4 w(x)}{dx^4} = bp(x) \tag{1}$$

where E, I, and w are the Young's modulus of the material, the moment of inertia of the beam, and deflection of the beam, respectively.

To thoroughly examine the influence of foundation on the beam deformation, it is essential to formulate the nonlinear characteristics of the foundations into a mathematical expression and integrate this into the governing equation. Ensuring the versatility of these equations across various soil conditions necessitates the introduction of a parametric representation for the foundation's reactive forces. This approach will allow for a more precise modeling of the interaction between the beam and its supporting foundation. For elastic foundations, a nonlinear relation between the deflection w and the resistance pressure q in the contact surface in the hyperbolic form is [12]:

$$q(x) = \frac{kw(x)}{1+\mu w(x)} \tag{2}$$

where at any point x of the contact surface, k is the linear modulus of reaction force of the foundation, and μ is a nonlinear factor which represents the nonlinear feature of the elastic foundation and gives rise to the key parameter of the nonlinear differential equation of the foundation. Obviously, with $\mu = 0$ it degrades to a linear elastic foundation.

With the fourth-order differential equation of flexural deformation, the classical Euler-Bernoulli beam theory and the combination of the resistance from the nonlinear foundation give the nonlinear differential equation for a beam resting on a nonlinear foundation as [12]:

$$EI\frac{d^4 w(x)}{dx^4} + \frac{kbw(x)}{1+\mu w(x)} = bp(x) \tag{3}$$

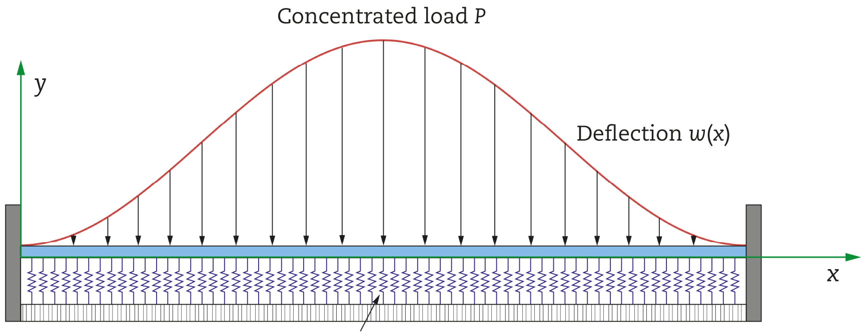

Fig. 1. A clamped beam on a nonlinear elastic foundation under a concentrated load.

Introducing new variables, the dimensionless form of Eq. (3) becomes:

$$\frac{d^4w(z)}{dz^4}+\frac{4w(z)}{1+\mu w(z)}=\frac{4}{k}p(z), z=\frac{x}{\lambda} \tag{4}$$

where $\lambda=(4EI/kb)^{1/4}$, with the dimension of length, therefore $z=x/\lambda$ is a non-dimensional spatial coordinate, and $w(z)$ is a dimensionless variable denoting the deflection of the beam at coordinate z.

The beam is subjected to a point load P as in Fig. 1. In this condition, due to discontinuity in stress distribution, the external stress distribution $p(z)$ can be expressed by the Dirac delta function [32]:

$$p(z)=\overline{P}\delta(z-\alpha),\ \overline{P}=\frac{P}{\lambda} \tag{5}$$

where α is the coordinate of the point where the concentrated load acts on the beam. In this case, the $\delta-$ function is given as an expansion of a sinusoidal Fourier series [20], then the specific form of the line load in Eq. (5) can be rewritten as:

$$\overline{P}\delta(z-\alpha)=\frac{2\overline{P}}{\beta}\sum_{m=1}^{M}\sin\frac{m\pi\alpha}{\beta}\sin\frac{m\pi z}{\beta}, M=1,2,3... \tag{6}$$

where $\beta=L/\lambda$ is a dimensionless parameter indicating the relative length of the beams and M is the number of terms in the approximation.

In Eq. (6), the concentrated force is described as the Dirac delta function, and then the function is further expanded into the form of a trigonometric series, so that the concentrated force P can be considered as a distributed force within a small area. In this context, the notion of a generalized load function is employed, conceived as the limiting case of a sequence of ordinary functions. The outcomes derived from the application of this transformed load function are validated and accurate [33]. In theory, optimal outcomes are achieved as M approaches infinity, yet this ideal is challenging to attain in practical computations. Consequently, finite values are often selected for calculations, necessitating the verification of the convergence of the results across various values to ensure accuracy. Considering the series expansion form of $p(z)$, substituting Eq. (6) into Eq. (4) yields:

$$\frac{d^4w(z)}{dz^4}+\frac{4w(z)}{1+\mu w(z)}=\frac{8\overline{P}}{\beta k}\sum_{m=1}^{M}\sin\frac{m\pi\alpha}{\beta}\sin\frac{m\pi z}{\beta} \tag{7}$$

Equation (7) represents the nonlinear differential equation of deflection of an elastic beam subjected to a concentrated load on a nonlinear elastic foundation. The focus of this study is to solve this equation to obtain an approximate solution of the deflection using the Galerkin method.

3. The Galerkin Method

Galerkin has been shown to provide excellent approximate solutions to linear differential equations, so it is safe to assume that nonlinear equations can be solved in a similar manner with equally accurate solutions [31]. In order to realize the process of solving nonlinear differential equations by the Galerkin method, this paper demonstrates the basic assumptions and associated procedures and techniques involved in this approximate method to nonlinear equations.

Assuming that the solution of the nonlinear differential equation Eq. (7) is a trigonometric series with essential orthogonality, the deflection takes the form of:

$$w(z)=\sum_{n=1}^{\infty}A_n\sin(nz) \tag{8}$$

where A_n are constants as amplitudes of the series.

For the case shown in Fig. 1, the beam is clamped at both the left and right ends, the boundary conditions are symmetric, and the beam is subjected to a concentrated load in the transverse direction at the midpoint of the beam so that the deflection curve of the beam can be considered as symmetric. In initial simulations, it is found that the results using only trigonometric functions as trial functions were not satisfactory close to the end positions of the beam, and the deflection values obtained were somewhat larger than the real situation. A simple but effective remedy to this problem is to multiply the trigonometric function by a factor that is used to control the accuracy of the deflection values near the endpoints of the beam and must not affect the symmetry of the deflection curves and the feature that the deflection is maximum at the midpoint of the beams. Therefore, a simple form of this factor is a parabola, i.e., $z(\beta-z)$, whose axis of symmetry coincides with the axis of symmetry of the deflection curve of the beam and whose value tends to zero near 0 and β. This is able to control the values of deflection of the beam at the endpoints very well, which makes the approximate result more accurate, and furthermore, it will lead to a faster convergence. With the above considerations, Eq. (8) is further rewritten as:

$$w_N(z)=\sum_{n=1}^{N}A_n\varphi_n(z)=\sum_{n=1}^{N}A_n z(\beta-z)\sin\left(\frac{2n-1}{\beta}\pi z\right),\ N=1,2,3... \tag{9}$$

Clearly, Eq. (9) satisfies the boundary conditions $w(0)=w(\beta)=0, w'(0)=w'(\beta)=0,$ which may be a combination of optimal solutions. Since $w_N(z)$ is only an approximate solution, the residual obtained by substituting Eq. (9) into Eq. (7) is not exactly zero, that is:

$$R_N(z)=\frac{d^4w_N(z)}{dz^4}+\frac{4w_N(z)}{1+\mu w_N(z)}-\frac{8\overline{P}}{\beta k}\sum_{m=0}^{M}\sin\frac{m\pi\alpha}{\beta}\sin\frac{m\pi z}{\beta}\neq 0 \tag{10}$$

Applying the Galerkin method, the weighted integral of the residual $R_N(z)$ in the interval $[0,\beta]$ is forced to vanish, and the weight function is the basis function $\varphi_n(z)$ in Eq. (8), so that the approximate solution $w_N(z)$ can be close to the exact solution. The variation of beam displacement in Eq. (9) is:

$$\delta w_N(z) = \sum_{n=1}^{N} \delta A_n \varphi_n(z) \tag{11}$$

Applying the Galerkin method to Eq. (10) as:

$$\int_0^{\beta} R_N(z)\delta w_N(z)dz = \sum_{n=1}^{N}\int_0^{\beta} R_N(z)\left[\delta A_n \varphi_n(z)\right]dz = 0 \tag{12}$$

By the Galerkin method, the arbitrary δA_n implying the vanish of the weighted integral, or the equation is obtained with:

$$\int_0^{\beta} R_N(z)\varphi_n(z)dz = 0, n = 1,2,3,\ldots,N \tag{13}$$

After the completion of the integral in Eq. (13), the nonlinear differential equation in Eq. (7) is now transformed into a set of N nonlinear algebraic equations with respect to the coefficients A_n. With the value of N is large enough, the approximate solution obtained is more accurate, and the difficulty of solving the equation will increase accordingly. With the known function $\varphi_n(z)$, the integration in Eq. (13) can be done only with a numerical procedure. Since the objective of the analysis is to obtain the coefficients of the series expansion of the deflection, which is acceptable if the evaluation of the integrals is done with a numerical procedure without expressions for efficient calculation.

4. The Solution Procedure with the Galerkin Method

By considering a beam with relative length 5, i.e., $\beta = 5$, and assuming the nonlinear parameter $\mu = 6$. The load factor $\overline{P}/k$ is taken to be 0.5 in order to make the displacements reasonable and to avoid excessive deformation.

4.1. Series solutions

In Eq. (13), taking $N = 1$ and $w_1(z) = A_1 z(\beta - z)\sin(\pi z/\beta)$ as the first-order approximation, then the application of the Galerkin method implies:

$$\int_0^{\beta} R_1(z)\varphi_1(z)dz = \int_0^{\beta} R_1(z)z(\beta - z)\sin\left(\frac{\pi z}{\beta}\right)dz = 0 \tag{14}$$

Note that M in $p(z)$ in Eq. (6) is taken to be 100 here, and the results will be discussed later regarding the convergence of the values of M. The calculation of the integral in Eq. (14) is done by Mathematica software. The integration yields a quadratic algebraic equation with respect to the coefficients A_1:

$$-12.50002025614 - 102.95763561059A_1 + 2041.98684351288A_1^2 = 0 \tag{15}$$

Clearly, Eq. (15) is a simple quadratic equation and its two roots are $A_1 = -0.056911$ and $A_1 = 0.1074114$. The coefficients of the first-order approximation A_1 play a dominant role for the flexural deformation, and there is only one solution greater than zero for the coefficients of the first-order approximation for the linear case of $\mu = 0$. Based on the above reasons, the positive root $A_1 = 0.1074114$ is chosen. Thus, the coefficients A_1 have been found and the first-order approximate solution is:

$$w_1(z) = 0.1074114z(5 - z)\sin\left(\frac{\pi z}{5}\right) \tag{16}$$

and the result is plotted in Fig. 2(a).

Taking $N = 2$, the displacement has the form of:

$$w_2(z) = A_1 z(\beta - z)\sin\left(\frac{\pi z}{\beta}\right) + A_2 z(\beta - z)\sin\left(\frac{3\pi z}{\beta}\right) \tag{17}$$

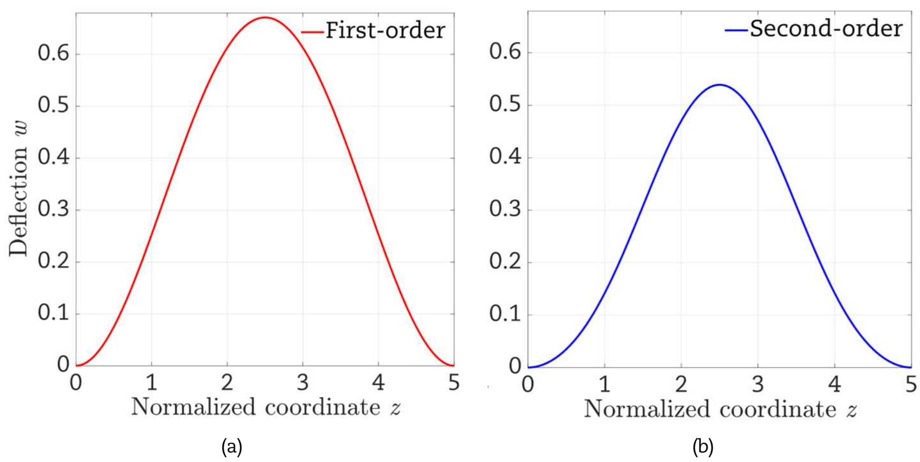

Fig. 2. (a) The first-order and (b) second-order approximation of the deflection ($\mu = 6$).

as the second-order approximation, and substituting it into Eq. (13) yields:

$$\int_0^{\beta} R_2(z)z(\beta-z)\sin\left(\frac{\pi z}{\beta}\right)dz=0,$$
$$\int_0^{\beta} R_2(z)z(\beta-z)\sin\left(\frac{3\pi z}{\beta}\right)dz=0. \tag{18}$$

The deflection function $w_2(z)$ has two terms, so the integration of Eq. (18) yields a system of two equations as:

$$\begin{aligned}&-12.500020256140738-102.95763561059823A_1+2041.986843512897A_1^2\\&-11819.8023728922273A_1A_2+346.8810188339998A_2+29578.127251313737A_2^2=0\end{aligned} \tag{19a}$$

$$\begin{aligned}&12.499939134309342+846.1650987775035A_2-1122.7804942798984A_1^2\\&+30701.12020249196A_1A_2+346.88101883400026A_1-6714.98176290169A_2^2=0\end{aligned} \tag{19b}$$

Here A_1 and A_2 has a total of 4 sets of solutions, but the proper value of A_1 here and the first-order approximation should be close, and the absolute value of coefficient A_2 should be less than A_1, as A_1 is the primary factor influencing the magnitude of deflection. Additionally, coefficients including A_2 and A_3, and so on, are meticulously adjusted to enhance the precision of the approximate solution. With this choice, acceptable solutions of A_1 and A_2 are easily chosen and the second-order approximation solution is written as:

$$w_2(z)=0.0763155z(5-z)\sin\left(\frac{\pi z}{5}\right)-0.00996098z(5-z)\sin\left(\frac{3\pi z}{5}\right) \tag{20}$$

and the deflection is shown in Fig. 2(b), illustrating the difference from the first-order approximation.

Similarly, with N = 3, the third-order approximation can be written as:

$$w_3(z)=A_1z(\beta-z)\sin\left(\frac{\pi z}{\beta}\right)+A_2z(\beta-z)\sin\left(\frac{3\pi z}{\beta}\right)+A_3z(\beta-z)\sin\left(\frac{5\pi z}{\beta}\right) \tag{21}$$

Substituting Eq. (21) into Eq. (10) for the calculation of $R_3(z)$ in Eq. (13) yields:

$$\int_0^{\beta} R_3(z)z(\beta-z)\sin\left(\frac{\pi z}{\beta}\right)dz=0,$$
$$\int_0^{\beta} R_3(z)z(\beta-z)\sin\left(\frac{3\pi z}{\beta}\right)dz=0, \tag{22}$$
$$\int_0^{\beta} R_3(z)z(\beta-z)\sin\left(\frac{5\pi z}{\beta}\right)dz=0.$$

With the completion of integrations in Eq. (22), a system of nonlinear algebraic equations with respect to the coefficients is:

$$\begin{aligned}&-12.5000202561+2041.9868435129A_1^2+29578.1272513137A_2^2+346.881018834A_2-99276.9004653819A_2A_3\\&-478.2789933087A_3+172871.366994499A_3^2-102.957635611A_1-11819.802372893A_1A_2+6362.9342791741A_1A_3=0\end{aligned} \tag{23a}$$

$$\begin{aligned}&12.4999391343-1122.7804942799A_1^2-6714.9817629019A_2^2+846.1650987775A_2+115207.2894456083A_2A_3\\&-467.5717160185A_3-43756.5540969554A_3^2+346.881018834A_1+30701.120202492A_1A_2-74572.7748167817A_1A_3=0\end{aligned} \tag{23b}$$

$$\begin{aligned}&-12.5001017682+203.7866019455A_1^2+9857.5881036624A_2^2-467.5717160184A_2-43950.8699361A_2A_3\\&+6050.5780609353A_3-4383.979971857A_3^2-478.2789933087A_1-26488.0486226053A_1A_2+174028.09056151A_1A_3=0\end{aligned} \tag{23c}$$

Again, solving the above system of equations and choosing the appropriate coefficients based on the previous approximation results, the specific form of the third- order approximation result is obtained as:

$$w_3(z)=0.084221595z(5-z)\sin\left(\frac{\pi z}{5}\right)-0.0061698381z(5-z)\sin\left(\frac{3\pi z}{5}\right)+0.0016361175578z(5-z)\sin\left(\frac{5\pi z}{5}\right) \tag{24}$$

Adhering to the same procedure above, the fourth-order approximation is given at $N=4$ in the form of:

$$w_4(z)=A_1z(\beta-z)\sin\left(\frac{\pi z}{\beta}\right)+A_2z(\beta-z)\sin\left(\frac{3\pi z}{\beta}\right)+A_3z(\beta-z)\sin\left(\frac{5\pi z}{\beta}\right)+A_4z(\beta-z)\sin\left(\frac{7\pi z}{\beta}\right) \tag{25}$$

and the specific form of the fourth-order approximation after solving the equations is given as:

$$\begin{aligned}w_4(z)=&\,0.081802593z(5-z)\sin\left(\frac{\pi z}{5}\right)-0.007701390724z(5-z)\sin\left(\frac{3\pi z}{5}\right)\\&+0.000595163214z(5-z)\sin\left(\frac{5\pi z}{5}\right)-0.00055738521z(5-z)\sin\left(\frac{7\pi z}{5}\right)\end{aligned} \tag{26}$$

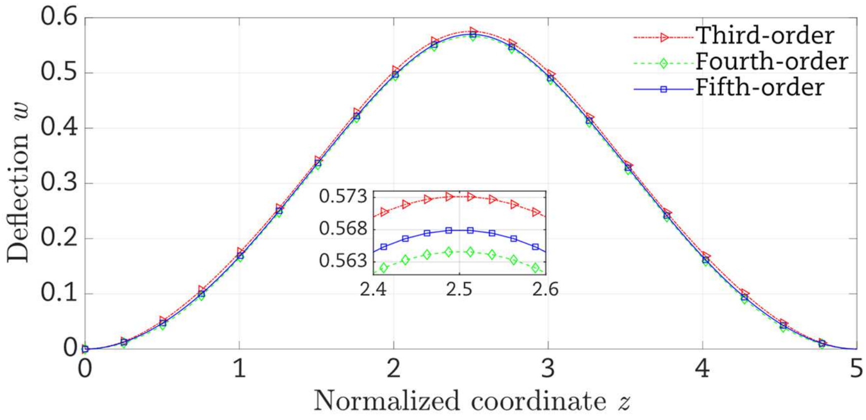

Fig. 3. The third-, fourth-, and fifth-order approximations of deflection ($\mu = 6$).

Finally, letting $N = 5$, the fifth-order approximation takes the form:

$$w_5(z) = A_1 z(\beta - z)\sin\left(\frac{\pi z}{\beta}\right) + A_2 z(\beta - z)\sin\left(\frac{3\pi z}{\beta}\right) + A_3 z(\beta - z)\sin\left(\frac{5\pi z}{\beta}\right) + A_4 z(\beta - z)\sin\left(\frac{7\pi z}{\beta}\right) + A_5 z(\beta - z)\sin\left(\frac{9\pi z}{\beta}\right) \tag{27}$$

and the solution is:

$$\begin{aligned} w_5(z) &= 0.0826236887 z(5-z)\sin\left(\frac{\pi z}{5}\right) - 0.00709333908 z(5-z)\sin\left(\frac{3\pi z}{5}\right) \\ &+ 0.001063602596 z(5-z)\sin\left(\frac{5\pi z}{5}\right) - 0.00019926489 z(5-z)\sin\left(\frac{7\pi z}{5}\right) + 0.0002106375488 z(5-z)\sin\left(\frac{9\pi z}{5}\right) \end{aligned} \tag{28}$$

Since the third-, fourth-, and fifth-order results are close, they are plotted in Fig. 3 to exhibit the subtle differences and improvements of higher-order solutions.

Apparently, higher-order solutions can be obtained in a similar manner with the above procedure. With a proper rule of cut-off, the solutions can be obtained through solving the nonlinear algebraic equations of coefficients as it is demonstrated above or through an iterative procedure below. Either way, a rarely utilized procedure based on the Galerkin method has been shown as a simple and efficient technique for approximate solutions of certain nonlinear differential equations. The successive approximations, as detailed in Fig. 3, demonstrate a trend towards convergence, with the fifth-order results positioned between the third and fourth orders. It should be pointed out that the feature of the convergence may depend upon the differential equation.

From Table 1, it can be clearly seen that the results of the fifth-order and sixth-order approximations are very close to each other, which can well illustrate the convergence of the approximate solutions. In addition, the convergence of the solution shows that it is feasible to use the fifth-order approximation as the approximate solution in this paper. Now the convergence of approximate solutions with the number of terms M for the concentrated load given in Eq. (6) can be examined with the details in Table 2.

Upon a detailed examination of Table 2, we observe that the approximate results for $M = 50$, $M = 100$, and $M = 150$ show minimal differences, which are only evident in the fifth decimal place. Consequently, selecting $M = 100$ for the computational work presented in this paper is deemed a reasonable choice. With the convergence of the approximation results verified, the results obtained with $N = 5$ and $M = 100$ are adopted as the approximations in this paper. For comparison, the results by Mohammerodpour et al. [20], which utilized both the homotopy analysis method (HAM) and the finite difference method (FDM), are presented in Table 3.

Table 1. Verification of convergence of approximate results (M = 100).

z	Normalized deflection $w(z)$			
	N = 3	N = 4	N = 5	N = 6
1	0.174545387	0.165152019	0.167538088	0.167537623
2	0.502356220	0.491988181	0.494590227	0.494589581
2.5	0.575172192	0.566603326	0.569940830	0.569940162
3	0.502356220	0.491988181	0.494590227	0.494589581
4	0.174545387	0.165152019	0.167538088	0.167537623

Table 2. Convergence validation of the number of terms of the line load expansion (N = 5).

z	Normalized deflection $w(z)$			
	M = 10	M = 50	M = 100	M = 150
1	0.166171236	0.167530682	0.167538088	0.167537017
2	0.493033485	0.494580058	0.494590227	0.494588754
2.5	0.567758890	0.569930303	0.569940830	0.569939306
3	0.493033485	0.494580058	0.494590227	0.494588754
4	0.166171236	0.167530682	0.167538088	0.167537017

Table 3. A comparison of solutions with different approximate methods.

z	Normalized deflection $w(z)$		
	Galerkin method	HAM [20]	FDM [20]
1	0.167538088	0.1670509	0.1670520
2	0.494590227	0.4939046	0.4938936
2.5	0.569940830	0.5698224	0.5698091
3	0.494590227	0.4939046	0.4938936
4	0.167538088	0.1670509	0.1670520

As another comparison with known results, the approximate results obtained in this paper with the Galerkin method are very close to those obtained by Mohammerodpour et al. [20] using the homotopy analysis method (HAM), shown a relative error within 0.3% at $z=1,4$. It should be mentioned that the nonlinear differential equation degenerates to a linear case when the nonlinear factor $\mu=0$. The Galerkin method is also applicable to differential equations for different values of μ, as shown in Fig. 4.

The parameter μ represents the nonlinear response of the foundation due to the stress effect of the beam deflection and acts as a resistance factor that prevents the beam from deforming under stress. From Eq. (2), as the value of μ increases, the foundation stress decreases and the resistance factor decreases, so the beam will deform more, as exhibiting in Fig. 4. It was also found during the calculation that the far larger the μ value, the stronger the nonlinear effect, the more deflection trial function terms are required for the calculation to reach a convergence, and the more difficult the calculation process becomes.

4.2. Iterative solutions

The previous solution procedure shows that when solving the sixth-order approximation, a system of equations consisting of 6 nonlinear algebraic equations needs to be solved, which is computationally challenging when dealing with such a problem and it may take a considerable amount of time and resources to obtain the result. For this reason, the use of iterative methods to gradually improve the final results of the approximate solution can be more preferable. The iterative procedure is easy to implement in the solution algorithm with less demands on the computing power.

The basic idea of the iterative method is to set one or a group of initial values based on an iterative relationship to solve the equations iteratively until the result converges to a fixed value. For a system of nonlinear equations with five unknown coefficients, let $A_2=A_3=A_4=A_5=0$, a substitution into the first equation can simplify the coupled equation to the standard quadratic one as:

$$-12.500020256-102.9576356106A_1+2041.986843513A_1^2=0 \tag{29}$$

For a quadratic equation in Eq. (29), a simple iterative relation is used for the solution. With a reasonable choice of the initial value for A_1, it is easy to solve the equation iteratively. Next, with the first solution $A_1^{(1)}$ known, continue by letting $A_3=A_4=A_5=0$, then substituting them into the second equation of the system of equations for another simpler form of the quadratic equation with respect to the coefficients A_2, enabling iterative value of A_2 be obtained using the same iterative relationship as before. Similarly, the coefficients A_3, A_4, and A_5 can be iterated following the same steps of the previous calculation.

After all the first-round iterations are completed, the second-round iteration can be completed by repeating the iteration according to the above process using the first iteration values $A_1^{(1)},A_2^{(1)},A_3^{(1)},A_4^{(1)},A_5^{(1)}$, and repeating until the iteration is considered to have converged when the difference between the two iteration values is less than the error tolerance. Other iterative methods such as the Newton iteration method can also be employed in this context, albeit with a different iterative scheme. Certainly, the number of iterations required to achieve convergence using the methods in this paper will vary due to the relative complexity of their iterative relationships. Following 94 iterations, the outcomes are deemed to have reached convergence, and the resulting iterative solution for the fifth-order approximation coefficients are presented below:

$$A_1=0.082623658, A_2=-0.00709334265, A_3=0.00106360158, A_4=-0.0001992652242, A_5=0.000210637468958 \tag{30}$$

With the iterative solution above, it is found that when the number of iterations is sufficient, the results obtained by the iterative method are close enough to the results obtained by solving the system of equations directly, and the accuracy and convergence of the method have been verified in Table 4. In this paper, as an example, the calculation of the sixth-order approximation is done by using the iterative method outlined above, and the result is detailed in the last column of Table 1.

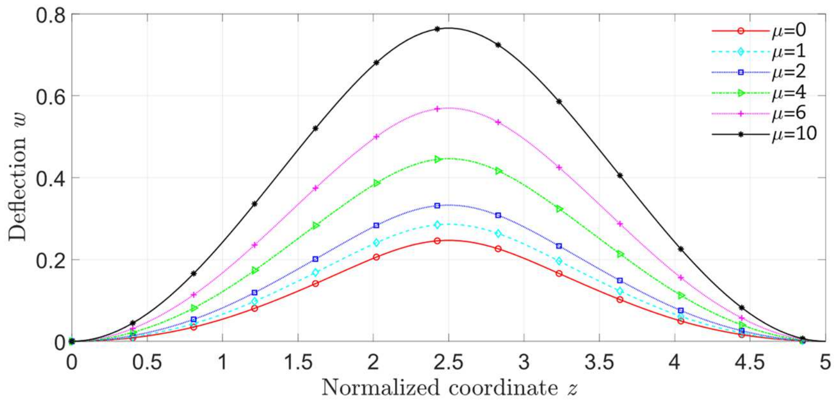

Fig. 4. Variation of deflection profile with nonlinear factors μ ($\mu=0,1,2,4,6,10$).

Table 4. Convergence verification of the iterative method.

z	Normalized deflection $w(z)$		
	93th-iteration	94th -iteration	5th-order
1	0.167537772	0.167538003	0.167538088
2	0.494589540	0.494590063	0.494590227
2.5	0.569940088	0.569940655	0.569940830
3	0.494589540	0.494590063	0.494590227
4	0.167537772	0.167538003	0.167538088

5. Conclusion and Discussion

As it was shown above, the Galerkin method was successfully applied to the differential equation of elastic beams resting on nonlinear elastic foundations, and symbolic mathematical software tools were used to transform the nonlinear differential equation into a system of nonlinear algebraic equations and solved them for accurate results. For the highly nonlinear equation, the Galerkin method is effective in obtaining the accurate solutions for the nonlinear equations which were solved with the finite difference method in general before, and the current procedure is simple to implement. The application of the Galerkin method to nonlinear equations can be more preferable, especially with the help of powerful symbolic mathematical software. In addition, for similar nonlinear problems, the combination of the Galerkin method and iterative method may present a smoother and less extensive process in the numerical procedure. Of course, the results were also in good accuracy as shown in the examples above. It is expected that the method can be utilized for the analysis of more nonlinear differential equations arising from engineering and science fields as an alternative technique in addition to many classical and tradition approaches.

Author Contributions

Conception, revision, and submission of the paper were made by J. Wang. The mathematical formulation, procedure, derivation, calculation, and check were made by J.L. Jiang with help from J. Wang, H.M. Jing, C.C. Lian, B.C. Meng and X. Fang. Reviews and discussions of the study and drafting were also completed with the participation of J. Wang and J.L. Jiang. All authors have read and agreed to the published version of the manuscript.

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Conflict of Interest

The funders and institutions had no role in the design of the study, in the collection, analysis, or interpretation of data, in the writing of the manuscript, or in the decision to publish the results.

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Data Availability Statements

The data of this study are available from the authors with a reasonable request.

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