J. Appl. Comput. Mech., 11(3) (2025) 568-584
DOI: 10.22055/jacm.2024.47592.4747

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

Higher-Order Singular BVP Emerging in the Semiconductor Industry: Analysis and Computation of Solutions

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Received July 30 2024; Revised September 07 2024; Accepted for publication September 12 2024.
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Abstract. In this work, we consider a class of non-linear, singular, non-self-adjoint and fourth-order BVP arising in the semiconductor industry. We investigate the theoretical and numerical results of solutions for the governing model corresponding to different boundary conditions by considering the domain as a disk of any radius $T > 0$. Problems are difficult to analyze the solutions due to their characteristics. Moreover, it has no exact solutions. Additionally, there are two solutions to these problems. So, it becomes difficult to apply any discrete approach to obtain both approximate solutions. We develop an iterative method based on boundary conditions that gives numerical approximations. The convergence analysis of the technique is discussed. Some numerical examples are given to show the efficiency and accuracy of the method.

Keywords: Fourth Order; Singular Boundary Value Problem; Dual Solutions; Non-Self-Adjoint Operators; Iterative Method.

1. Introduction

Numerous physical applications include nonlinear singular boundary value problems (SBVP) [1]. The applications in applied mathematics, physics, and physiological science, as well as applications in engineering, such as transport processes, atomic structure, nuclear physics, thermal explosion, magneto hydrodynamics, electro-hydrodynamics, gas dynamics, etc. are the source of these types of problems [1]. Solving various singular BVPs generally a challenging task. Some of these physical phenomena that occur in the actual world can be solved analytically. However, in many cases, an exact analytical solution is typically difficult or impossible to find. As a result, to find an approximate solution to such problems, we need a precise and efficient numerical approach. Still, very few works are done in numerical computation of higher-order SBVP. Swati et al. [2] proposed a nonlinear third-order Emden–Fowler type equation with some initial conditions as well as boundary conditions. They have given a numerical algorithm based on the uniform Haar wavelet collocation method and discussed the convergence analysis. Rufai et al. [3] extended the literature of Swati et al. [2] by developing a one-step hybrid block Nyström-type method to obtain numerical approximation of solutions. Pandit et al. [4] proposed the Haar wavelets collocation method for the numerical solution of a nonlinear SBVP of order $n = 4, 6, 8, 10$. Also, they have shown that numerical approximation converges to the exact solution. Hasan et al. [5] considered a class of SBVP of order $n+1$, and approximated the solution by using the modified Adomian decomposition method.

We have seen that the main objective of their studies is to compute the numerical approximation of a unique solution. Later, Pandit et al. [6-9] computed the numerical approximations of fourth-order singular BVP, which has two solutions by considering the equation as:

$$\frac{1}{z}\left\{z\left[\frac{1}{z}\left(z\Psi'(z)\right)'\right]'\right\}' = \frac{\Psi'(z)\Psi''(z)}{z} + \lambda, \qquad 0 \le z \le 1. \tag{1}$$

They have shown both numerical solutions exist for $0 \le \lambda < \lambda_c$, where λ_c is the critical value of λ. Before that, the existence of dual solutions to the model (1) for $0 \le \lambda < \lambda_c$ is discussed in [10]. Later, Pandit et al. [11] extended the numerical results of [10-13] for the following class of fourth-order singular BVP:

$$\frac{1}{z^\alpha}\left\{z^\alpha\left[\frac{1}{z^\alpha}\left(z^\alpha\Psi'(z)\right)'\right]'\right\}' = \frac{\nu'\left(\Psi'(z)\right)^2 + 2\nu\Psi'(z)\Psi''(z)}{2z^\alpha} + \lambda G(z), \qquad 0 \le z \le 1, \tag{2}$$

where $\alpha \in \mathbb{Z}^+$ such that $\alpha \ge 1, \nu = kz^{2\alpha-2}, k \in \mathbb{R}^+, \lambda$ is the speed of particle and uniform deposition rate $G(z)$ i.e., $G(z) = 1$. After that, for $\alpha \ge 1$, equation (2) is discussed in [1] with respect to different types of boundary conditions. They showed the existence of

Published online: September 22 2024

at least one solution with the help of monotone iterative technique. By using the transformation $h(z) = z^{\alpha}\Psi'(z)$, from equation (2), we have:

$$z^2 h''(z) - \alpha z h'(z) = \frac{\kappa h^2(z)}{2} + z^2 \int_0^z \lambda t^{\alpha} G(t)dt\,, z \in (0,1]. \tag{3}$$

In this work, we try to fill the gap in the literature by extending the work of Pandit et al. [11]. We study the theoretical and numerical results of the solutions of a class of fourth-order singular differential equation (2) by considering the function $G(z)$ as variable deposition rate such that $0 < b_1 \le G(z) \le b_2$ and the domain as $0 \le z \le T$ instead of $[0,1]$ with respect to the following boundary conditions:

$$\Psi'(0) = 0, \Psi(T) = 0, \Psi'(T) = 0, \lim_{z\to 0} z^{\alpha}\, \Psi'''(z) = 0, \tag{4}$$

$$\Psi'(0) = 0, \Psi(T) = 0, \Psi'(T) + T\ \Psi''(T) = 0, \lim_{z\to 0} z^{\alpha}\, \Psi'''(z) = 0, \tag{5}$$

$$\Psi'(0) = 0, \Psi(T) = 0, \Psi''(T) = 0, \lim_{z\to 0} z^{\alpha}\, \Psi'''(z) = 0. \tag{6}$$

For $\alpha = 1$, $\kappa = 1$ and $G(z) = 1$, the equations (2) and (1) are identical. Equation (1) is the radial form [8, 10] of fourth-order elliptic partial differential equation, which is given by:

$$\Delta^2 f = \det(D^2 f) + \lambda G(x), x \in \tau \subset R^2, \tag{7}$$

where $f\colon \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ is the height of the growing interface at the spatial point $x \in \mathbb{R}^2$ and time $t \in \mathbb{R}^+$, D is the Hessian matrix, $G(x)$ is deposition rate and λ is the speed of the incoming particle. In the semiconductor sector [10], the model (7) is becoming more prevalent. Under the continuum hypothesis, one can easily obtain equation (7) from the following stochastic partial differential equation (SPDE):

$$\frac{\partial f}{\partial t} + K_2 \nabla^4 f = 2K_1 \det(D^2 f) + \xi(x,t). \tag{8}$$

If we put $K_0 = 0 = K_3$ in the following Kardar–Parisi-Zhange (KPZ) equation [6, 10, 12], then we can easily derive the equation (8):

$$\frac{\partial f}{\partial t} = K_0 \nabla^2 f + 2K_1 \det(D^2 f) - K_2 \nabla^4 f - \frac{1}{2} K_3 \Delta(\Delta f)^2 + \xi(x,t). \tag{9}$$

To know further background on the semiconductor industry's significance and the use of mathematical models in this field reader can read the references [12, 13].

Again, the physical interpretation of the boundary conditions (4), (5) and (6) can be found in [7, 10]. Now, by using $h(z) = z^{\alpha}\Psi'(z)$, the boundary conditions (4), (5) and (6) are reduced to:

$$h'(0) = 0, h(T) = 0, \tag{10}$$

$$h'(0) = 0, (\alpha - 1)h(T) = Th'(T), \tag{11}$$

$$h'(0) = 0, \alpha h(T) = Th'(T), \tag{12}$$

respectively. The study of the numerical solutions of problem (3) with respect to the boundary conditions (10), (11) and (12) are difficult due to the following characteristics:

- It is higher-order, non-linear, non-self-adjoint and singularity present at the origin.
- The behavior of both solutions varies depending on the parameter λ.
- The exact solutions are not known.
- Also, the problem has two solutions for $0 \le \lambda < \lambda_c$. Therefore, many discrete methods may not be capable to compute both the solutions with desired accuracy.

In this paper, first, we investigate the characteristics of the solutions. Then we construct an iterative scheme by using equation (3) and the corresponding boundary conditions to compute the two solutions with desired accuracy. To check the validity of the scheme, we discuss the convergence analysis. We demonstrate approximated value of the parameter λ for various values of α. We compare the numerical results with existing data and place the order of convergence to assess performance and robustness of the proposed approach. We also verify the numerical results with theoretical results.

To investigate the characteristic of the solutions, we transformation equation (3) by using $s = \frac{z^{1+\alpha}}{1+\alpha}$ and $h(z) = v(s)$ into the following form:

$$v''(s) = \frac{\kappa v^2(s)}{2(1+\alpha)^2 s^2} + (1+\alpha)^{-\tilde{\gamma}} s^{-\tilde{\gamma}} \int_0^{L(s)} \lambda t^{\alpha} G(t)dt, \tag{13}$$

where $\tilde{\gamma} = 2\alpha\tilde{\delta}$, $\tilde{\delta} = \frac{1}{1+\alpha}$ and $L(s) = (1+\alpha)^{\tilde{\delta}} s^{\tilde{\delta}}$. Similarly, the boundary conditions (10), (11) and (12) are reduced to:

$$\lim_{s\to 0} s^{\alpha\tilde{\delta}}\, v'(s) = 0, v\left(T^{1+\alpha}\tilde{\delta}\right) = 0, \tag{14}$$

$$\lim_{s\to 0} s^{\alpha\tilde{\delta}}\, v'(s) = 0, (\alpha - 1)v\left(T^{1+\alpha}\tilde{\delta}\right) = T^{1+\alpha}v'\left(T^{1+\alpha}\tilde{\delta}\right), \tag{15}$$

$$\text{and } \lim_{s\to 0} s^{\alpha\tilde{\delta}}\, v'(s) = 0, \alpha v\left(T^{1+\alpha}\tilde{\delta}\right) = T^{1+\alpha}v'\left(T^{1+\alpha}\tilde{\delta}\right), \tag{16}$$

Remark 1.1 *Several researchers [7, 8, 10] have examined the equation (1) by treating the domain as* $[0,1]$ *rather than* $[0,T]$*, where* $T>0$ *with regard to various boundary conditions. To the best of our knowledge, no research has been done in the literature on equation (1) over the domain* $[0,T]$ *for any values of* $\alpha>1$*. The key advantage of the proposed scheme is its ability to accurately compute the numerical approximations of the dual solutions for a class of singular BVPs (3) within the domain* $[0,T]$ *for* $\alpha\geq 1$ *and* $T>0$*, subject to boundary conditions (10), (11) and (12).*

Now, we consider the space as $E=C^2_{loc}\left(\left(0,T^{1+\alpha}\tilde{\delta}\right],\mathbb{R}\right)$, where:

$$C^2_{loc}\left(\left(0,T^{1+\alpha}\,\tilde{\delta}\right],\mathbb{R}\right)=\left\{v\colon\left(0,T^{1+\alpha}\tilde{\delta}\right]\to\mathbb{R}\middle|\, v\in\mathcal{C}^2([a,b],\mathbb{R})\ \text{ on every compact set } [a,b]\subset\left(0,T^{1+\alpha}\tilde{\delta}\right]\right\}. \tag{17}$$

Below, we discuss some previous works which are relevant to the proposed methodology. Second-order boundary value problems can be solved by using various numerical techniques such as the variational iterative method [8], homotopy perturbation method [9], Adomian's decomposition method [6, 14, 15] etc. Earlier, in 2008, Mittal et al. [16] considered a class of singular boundary value problems $\left(p(t)y'(t)\right)'=q(t)f\left(t,y(t)\right)$ with boundary conditions $y'(0)=0$ and $y(1)=B$. They applied the Adomian decomposition method to find the numerical approximations. In 2009, Pandey et al. [17], considered the same class of singular boundary value problems with boundary conditions $y'(0)=0$ and $\alpha y(1)+\beta y'(1)=\gamma$. They applied the finite difference method and approximated the exact solution. Later, Singh et al. [14], considered a class of strongly nonlinear singular boundary value problems $u''(x)+\frac{\alpha}{x}u'(x)=f\left(x,u(x)\right)$ with Neumann and Robin boundary conditions $u'(0)=0, au(1)+bu'(1)=c$ and developed an improved decomposition method which provides better results than Adomian decomposition method. They showed the convergence of the proposed scheme. In 2010, Kanth et al. [18] considered a class of singular boundary value problems $u''=\frac{\alpha}{x}y'+f(x,y)=0$ subject to boundary conditions $y(0)=A$ and $y(1)=C$. They demonstrated the applicability of He's variational iteration method to this problem. In 2018, Roul and Thula [19] considered a class of nonlinear SBVP $u''(x)+\left(\frac{\alpha}{x}+\frac{p'(x)}{p(x)}\right)u'(x)=g\left(x,u(x)\right)$ with Neumann and Robin boundary conditions and applied optimal quartic B-spline collocation method to find the approximate solution. After that, Roul et al. [20] solved a second-order singular boundary value problem, which is related to the electro hydrodynamic flow of a fluid in a circular cylindrical by using a sixth-order numerical method. In 2019, He et al., [21] applied the Taylor series method to find a solution of Lane-Emden type equation. To know more about the literature reader can read the review paper [2] and the references [3, 4].

The remaining work is organized as follows. In Section 2, we provide the characteristics of the solutions of the equation (3). In Section 3, we build an iterative technique to find the approximate solutions of equation (3). We demonstrate that the computed series solutions converge to the exact solutions in Section 4. We provide the numerical results of the proposed method in Section 5. Ultimately, in Section 6, we have concluded the work.

2. Characteristics of the Solutions

In this section, we demonstrate some basic qualitative characteristics of the solutions $v\in E$ corresponding to $\lambda\geq 0$ and $\alpha\geq 1$, which satisfies the following inequality:

$$v''(s)\geq\frac{\kappa v^2(s)}{2(1+\alpha)^2s^2}+(1+\alpha)^{-\tilde{\gamma}}s^{-\tilde{\gamma}}\int_0^{L(s)}\lambda t^{\alpha}G(t)dt\,, s\in\left(0,T^{1+\alpha}\tilde{\delta}\right]. \tag{18}$$

For $T=1$ and $\alpha>1$, Pandit et al. [11] analyzed the solutions of the equation (2) with respect to different types of boundary conditions. Below, we have presented the results to extend the literature for $T>1$ and $\alpha\geq 1$. Due to the similarity of the proof, we refer to the reference [11], corresponding to few lemmas.

Lemma 2.1 *Consider a function* $v(s)$ *in* E*. If* $v(s)$ *satisfies the condition* $\lim_{s\to 0}s^{\alpha\tilde{\delta}}v'(s)=0$ *and inequality (18) then:*

$$\lim_{s\to 0^+}v(s)=0. \tag{19}$$

Proof. The proof is similar to the derivation of Lemma 2.0.1 in [11].

Lemma 2.2 *Consider a function* $v(s)$ *in* E*. If* $v(s)$ *satisfies the condition (14), inequality (18) and equation (19) then* $v(s)\leq 0$ *for all* $s\in\left(0,T^{1+\alpha}\tilde{\delta}\right]$.

Proof. By similar analysis as in Lemma 2.0.1 in [11], we get the result.

Lemma 2.3 *Consider a function* $v(s)$ *in* E*. If* $v(s)$ *satisfies the condition (15), inequality (18) and equation (19) then* $v(s)\leq 0$ *for all* $s\in\left(0,T^{1+\alpha}\tilde{\delta}\right]$.

Proof. At first, we need to show $v\left(T^{1+\alpha}\tilde{\delta}\right)\leq 0$. Let us assume $v\left(T^{1+\alpha}\tilde{\delta}\right)>0$. According to equation (19), there exists a $s_0\in\left(0,T^{1+\alpha}\tilde{\delta}\right]$ such that $v(s_0)<v\left(T^{1+\alpha}\tilde{\delta}\right)\left[1-(\alpha-1)\tilde{\delta}+\frac{(\alpha-1)s_0}{T^{1+\alpha}}\right]$. Now, we have:

$$\frac{v\left(T^{1+\alpha}\tilde{\delta}\right)-v(s_0)}{T^{1+\alpha}\tilde{\delta}-s_0}=v'(\xi),\ \text{ where }\ \min\{T^{1+\alpha}\tilde{\delta},s_0\}\leq\xi\leq\max\{T^{1+\alpha}\tilde{\delta},s_0\}. \tag{20}$$

Since $v'(s)$ is an increasing function, therefore, equation (20) can be written as:

$$\frac{v\left(T^{1+\alpha}\tilde{\delta}\right)-v(s_0)}{T^{1+\alpha}\tilde{\delta}-s_0}\leq v'\left(T^{1+\alpha}\tilde{\delta}\right)=\frac{(\alpha-1)v\left(T^{1+\alpha}\tilde{\delta}\right)}{T^{1+\alpha}}. \tag{21}$$

Thus, we get,

$$v(s_0)\geq\left[1-\frac{(\alpha-1)\left(T^{1+\alpha}\tilde{\delta}-s_0\right)}{T^{1+\alpha}}\right]v\left(T^{1+\alpha}\tilde{\delta}\right)\geq\left[1-(\alpha-1)\tilde{\delta}+\frac{(\alpha-1)s_0}{T^{1+\alpha}}\right]v\left(T^{1+\alpha}\tilde{\delta}\right). \tag{22}$$

This leads to a contradiction. Therefore, $v\left(T^{1+\alpha}\tilde{\delta}\right)\leq 0$. Again, $v(s)$ is a convex function, which implies $v'(s)$ is increasing in $\left(0,T^{1+\alpha}\tilde{\delta}\right]$. So, $v(s)$ is a decreasing function in $\left(0,T^{1+\alpha}\tilde{\delta}\right]$. Therefore, from $\lim_{s\to 0^+}v(s)=0$ and $v\left(T^{1+\alpha}\tilde{\delta}\right)\leq 0$, we have the result.

Lemma 2.4 *Consider a function $v(s)$ in E. If $v(s)$ satisfies the condition (16), inequality (18) and equation (19), then $v(s) \leq 0$ for all $s \in \left(0, T^{1+\alpha}\tilde{\delta}\right]$.*

Proof. The proof follows from Lemma 2.3.

Lemma 2.5 *Consider a function $v(s)$ in E. If $v(s)$ satisfies $v(s) \leq 0$, inequality (18) and equation (19), then:*

$$\lim_{s \to 0^+} s^{\alpha\tilde{\delta}} v'(s) = 0 \text{ if and only if } \lim_{s \to 0^+} \frac{v(s)}{s^{\tilde{\delta}}} = 0. \tag{23}$$

Proof. By similar analysis as in Corollary 2.0.1 in [11], we can proof the results.

Lemma 2.6 *Consider a function $v(s)$ in E. If $v(s)$ satisfies $\lim_{s \to 0^+} \frac{v(s)}{s^{\tilde{\delta}}} = 0$. and $v(s) \leq 0$, then the parameter λ follows the inequality:*

$$0 \leq \lambda \leq \frac{(1+\alpha)^4(3+\alpha)\kappa\pi^2}{\kappa b_1 T^{(3+\alpha)}}. \tag{24}$$

Proof. To find the range of the parameter λ, we first rewrite equation *(13)* as:

$$\left(sv'(s) - v(s)\right)' = \frac{\kappa v^2(s)}{2(1+\alpha)^2 s} + (1+\alpha)^{-\tilde{\gamma}} s^{-\tilde{\gamma}+1} \int_0^{L(s)} \lambda t^{\alpha} G(t)dt. \tag{25}$$

Here, we put $w(s) = -\frac{v(s)}{s}$. Also, $w(s) \geq 0$ for all $s \in \left(0, T^{1+\alpha}\tilde{\delta}\right]$. Integrating both sides on equation (25) from 0 to s, we get:

$$\int_0^s \left(-t^2 w'(t)\right)' dt = \int_0^s \frac{\kappa t w^2(t)}{2(1+\alpha)^2} dt + \int_0^s (1+\alpha)^{-\tilde{\gamma}} t^{\tilde{\eta}} \int_0^{L(t)} \lambda y^{\alpha} G(y) dy\, dt, \tag{26}$$

where $\tilde{\eta} = -\tilde{\gamma} + 1$. Since $v(s) \leq 0$, from equation (26) we get:

$$-s^2 w'(s) = \frac{\kappa s^2 w^2(s)}{4(1+\alpha)^2} - \int_0^s \frac{\kappa t^2 w(t) w'(t)}{2(1+\alpha)^2} dt + \int_0^s (1+\alpha)^{-\tilde{\gamma}} t^{\tilde{\eta}} \int_0^{L(s)} \lambda y^{\alpha} G(y) dy\, dt. \tag{27}$$

As we have $0 < b_1 \leq G(z)$, therefore:

$$s^2 w'(s) \leq -\frac{\kappa s^2 w^2(s)}{4(1+\alpha)^2} + \int_0^s \frac{\kappa t^2 w(t) w'(t)}{2(1+\alpha)^2} dt - \int_0^s \lambda\, b_1 (1+\alpha)^{-\tilde{\gamma}} t^{\tilde{\eta}+1} dt. \tag{28}$$

Now, we can write inequality (28) as:

$$s^2 w'(s) \leq -\frac{\kappa s^2 w^2(s)}{4(1+\alpha)^2} - \frac{\lambda b_1 (1+\alpha)^{-\tilde{\gamma}} s^{\tilde{\eta}+2}}{\tilde{\eta}+2}. \tag{29}$$

So, from inequality (29), we get:

$$s^2 w'(s) \leq -\frac{\kappa s^2 w^2(s)}{4(1+\alpha)^2} - \frac{\lambda b_1 (1+\alpha)^{\tilde{\eta}} s^{(3+\alpha)\tilde{\delta}}}{3+\alpha}. \tag{30}$$

Since $(3+\alpha)\tilde{\delta} > \tilde{\eta}$, we obtain:

$$\begin{aligned} w'(s) &\leq -\frac{\kappa w^2(s)}{4(1+\alpha)^2} - \frac{\lambda b_1 (1+\alpha)^{\tilde{\eta}} s^{\tilde{\eta}}}{3+\alpha}, \\ &\leq -\tfrac{\kappa w^2(s)}{4(1+\alpha)^2} - \tfrac{\lambda b_1 T^{(1-\alpha)}}{3+\alpha}, \forall\ \mathrm{s} \in \left(0, \mathrm{T}^{1+\alpha}\tilde{\delta}\right]. \end{aligned} \tag{31}$$

Thus, we have:

$$\frac{-w'(s)}{w^2(s) + \dfrac{4(1+\alpha)^2 \lambda b_1 T^{1-\alpha}}{(3+\alpha)\kappa}} \geq \frac{\kappa}{4(1+\alpha)^2}. \tag{32}$$

Integrating both sides of inequality (32) from s to $T^{1+\alpha}\tilde{\delta}$, we get:

$$\int_s^{T^{1+\alpha}\tilde{\delta}} \frac{-w'(t)}{w^2(t) + \dfrac{4(1+\alpha)^2 \lambda b_1 T^{1-\alpha}}{(3+\alpha)\kappa}} dt \;\; \geq \;\; \int_s^{T^{1+\alpha}\tilde{\delta}} \frac{\kappa}{4(1+\alpha)^2} dt\,. \tag{33}$$

Equivalently, we can write inequality (33) as:

$$-\int_{w(s)}^{w\left(T^{1+\alpha}\tilde{\delta}\right)} \frac{du}{u^2 + \dfrac{4(1+\alpha)^2 \lambda b_1 T^{1-\alpha}}{(3+\alpha)\kappa}} \;\; \geq \;\; \frac{\kappa}{4(1+\alpha)^2} \left(T^{1+\alpha}\tilde{\delta} - s\right). \tag{34}$$

Applying $\lim_{s \to 0^+}$ to both sides of inequality (34) we get:

$$\int_0^{\infty} \frac{du}{u^2 + \dfrac{4(1+\alpha)^2 \lambda b_1 T^{1-\alpha}}{(3+\alpha)\kappa}} \geq \lim_{s \to 0^+} \int_{w\left(T^{1+\alpha}\tilde{\delta}\right)}^{w(s)} \frac{du}{u^2 + \dfrac{4(1+\alpha)^2 \lambda b_1 T^{1-\alpha}}{(3+\alpha)\kappa}} \;\; \geq \;\; \frac{\kappa T^{1+\alpha}}{4(1+\alpha)^3}. \tag{35}$$

Solving equation (35), we obtain:

$$\lambda \leq \frac{(1+\alpha)^4(3+\alpha)\pi^2}{\kappa b_1 T^{(3+\alpha)}}. \tag{36}$$

Lemma 2.7 *If the BVPs (3) with respect to boundary conditions (10), (11) and (12) can be solved for some $\lambda_c \geq 0$, then these BVPs can also be solved for each $0 \leq \lambda \leq \lambda_c$.*

Proof. The proof is similar to the derivation of Lemma 4.0.2 in [11].

In the following section, we develop an iterative scheme to find numerical approximations of the solutions to equation (3).

3. Construction of the Scheme

As equation (3) is non-self-adjoint, therefore we convert equation (3) to a self-adjoint form for constructing the scheme. Now, from equation (3), we have:

$$\frac{h''(z)}{z^\alpha} - \frac{\alpha h'(z)}{z^{\alpha+1}} = \frac{\kappa h^2(z)}{2z^{\alpha+2}} + \frac{1}{z^\alpha}\int_0^z \lambda t^\alpha G(t)dt\,, \qquad z \in (0,T]. \tag{37}$$

Equivalently, equation (37) can be written as:

$$\frac{d}{dz}\left(\frac{h'(z)}{z^\alpha}\right) = \frac{\kappa h^2(z)}{2z^{\alpha+2}} + \frac{1}{z^\alpha}\int_0^z \lambda t^\alpha G(t)dt. \tag{38}$$

Now, integrating both side of equation (38) from 0 to z, we get:

$$\frac{h'(z)}{z^\alpha} - \lim_{z\to 0}\frac{h'(z)}{z^\alpha} = \frac{\kappa}{2}\int_0^z \frac{h^2(t)}{t^{\alpha+2}}dt + \int_0^z \frac{1}{t^\alpha}\int_0^t \lambda y^\alpha G(y)dy\,dt. \tag{39}$$

First, we have to show the value of $\lim_{z\to 0}\frac{h'(z)}{z^\alpha}$ exists and finite. As $\alpha \geq 1$ and singularity present at $z=0$, therefore the situation is not in hand immediately. Now, on account of equation (37), we have:

$$z^{1-\alpha}h''(z) - \frac{\alpha h'(z)}{z^\alpha} = \frac{\kappa h^2(z)}{2z^{\alpha+1}} + z^{1-\alpha}\int_0^z \lambda t^\alpha G(t)dt. \tag{40}$$

By taking limit at $z \to 0$ on both sides of equation (40) , we obtain:

$$\lim_{z\to 0} z^{1-\alpha}\,h''(z) - \alpha \lim_{z\to 0}\frac{h'(z)}{z^\alpha} = \lim_{z\to 0}\frac{\kappa h^2(z)}{2z^{\alpha+1}} + \lim_{z\to 0}\ z^{1-\alpha}\int_0^z \lambda t^\alpha G(t)dt. \tag{41}$$

Now, we impose the initial condition $h'(0)=0$ and use the result of Lemma 2.1 that is $\lim_{z\to 0} h(z)=0$. Therefore, by using L' Hospital's rule, from equation (41), we get:

$$\lim_{z\to 0} z^{1-\alpha}\,h''(z) - \alpha \lim_{z\to 0}\ \frac{h''(z)}{\alpha z^{\alpha-1}} = \frac{\kappa}{2}\lim_{z\to 0}\ \frac{2h(z)h'(z)}{(\alpha+1)z^\alpha} + \lim_{z\to 0}\ \frac{\lambda z^\alpha G(z)}{(\alpha-1)z^{\alpha-2}}. \tag{42}$$

So, in view of equation (42), we can assume $\lim_{z\to 0} z^{1-\alpha}\,h''(z) = a$ (a finite value). Consequently, we have:

$$\lim_{z\to 0}\frac{h'(z)}{z^\alpha} = \lim_{z\to 0}\frac{h''(z)}{\alpha z^{\alpha-1}} = \lim_{z\to 0}\frac{z^{1-\alpha}h''(z)}{\alpha} = \frac{a}{\alpha}. \tag{43}$$

Hence, by using equations (43) and (39), we obtain:

$$h'(z) = \frac{az^\alpha}{\alpha} + \frac{\kappa z^\alpha}{2}\int_0^z \frac{h^2(t)}{t^{\alpha+2}}dt + z^\alpha\ \int_0^z \frac{1}{t^\alpha}\int_0^t \lambda y^\alpha G(y)dy\,dt. \tag{44}$$

Again, integrating both sides on equation (44) from 0 to z, we get:

$$h(z) = \frac{az^{\alpha+1}}{\alpha(\alpha+1)} + \frac{\kappa}{2}\int_0^z t^\alpha \int_0^t \frac{h^2(y)}{y^{\alpha+2}}dydt + \int_0^z x^\alpha \int_0^x \frac{1}{t^\alpha}\int_0^t \lambda y^\alpha G(y)dy\,dtdx. \tag{45}$$

Now, we assume $h(z) \in \mathcal{C}^2(0,T]$. By using second fundamental theorem of calculus (Theorem 34.3, p. 294, [25]) to equation (45) we can conclude that $h(z) \in \mathcal{C}^2[0,T]$. Therefore, equation (45) can be expressed as follows:

$$h(z) = c + L^{-1}[N(h(z))], \tag{46}$$

where $c = \frac{az^{\alpha+1}}{\alpha(\alpha+1)} + \int_0^z x^\alpha \int_0^x \frac{1}{t^\alpha}\int_0^t \lambda y^\alpha G(y)dy\,dtdx$, $N[h(z)] = h^2(z)$ and:

$$L^{-1}\{\cdot\} = \int_0^z t^\alpha \int_0^t \frac{\kappa}{2y^{\alpha+2}}\{\,\cdot\}dydt. \tag{47}$$

Now, we use $\sum_{i=0}^\infty h_i(z)$ to approximate $h(z)$. Consequently, we can write the non-linear term $N[h(z)]$ as [26, 27]:

$$N[h(z)] = N\left(\sum_{i=0}^\infty h(z)\right) = \left(\sum_{i=0}^\infty h_i(z)\right)^2 \approx \sum_{i=0}^\infty A_i(h_0,h_1,\cdots,h_i), \tag{48}$$

where $A_i(h_0,h_1,\cdots,h_i)$ can be defined by:

$$A_i(h_0,h_1,\cdots,h_i)=\frac{1}{i!}\frac{d^i}{d\beta^i}N\left(\sum_{j=0}^{i}\beta^j h_j\right)_{\beta=0},i=0,1,2,\cdots. \tag{49}$$

Therefore, we can get:

$$A_n(h_0,h_1,\cdots,h_n)=\sum_{i=0}^{n}h_{n-i}(z)h_i(z),n=0,1,\cdots. \tag{50}$$

From equations (48) and (46), we obtain:

$$\sum_{i=0}^{\infty}h_i(z)\approx c+L^{-1}\left[\sum_{i=0}^{\infty}A_i(h_0,h_1,\cdots,h_i)\right]. \tag{51}$$

Thus, the iterative scheme is defined as follows:

$$h_0(z)=\frac{az^{\alpha+1}}{\alpha(\alpha+1)}+\int_0^z x^\alpha\int_0^x\frac{1}{t^\alpha}\int_0^t\lambda y^\alpha G(y)dy\,dtdx, \tag{52}$$

$$h_1(z)=\int_0^z t^\alpha\int_0^t\frac{\kappa}{2y^{\alpha+2}}A_0(h_0)dydt, \tag{53}$$

$$\vdots$$

$$h_{n+1}(z)=\int_0^z t^\alpha\int_0^t\frac{\kappa}{2y^{\alpha+2}}A_n(h_0,h_1,\cdots,h_n)dydt, \tag{54}$$

and so on. The nth approximation of exact solutions $h(z)$ can be defined as $\sum_{i=0}^{n}h_i(z)$, where $a\in\mathbb{R}$ is the constant. By using boundary condition, we compute two values of the constant a and consequently, two solutions can be achieved. The proposed method's algorithm is shown below for the reader's better understanding.

Algorithm 3.1

Step 1: By using initial condition $h'(0)=0,$ convert the equation (3) into the form $h(z)=c+L^{-1}[N(h(z))]$, where L^{-1} is the integral operator defined by (47), $N(h(z))$ is the nonlinear term and c is the constant term.

Step 2: Use $\sum_{i=0}^{\infty}h_i(z)$ and $\sum_{i=0}^{\infty}A_i(h_0,h_1,\cdots,h_i)$ to approximate the functions $h(z)$ and $N(h(z))$, respectively, where $A_i(h_0,h_1,\cdots,h_i)$ is defined by equation (50).

Step 3: Chose the initial approximation $h_0(z)$ as in equation (52), and compute $h_i(z)$ for $i=1,2,\cdots,n$.

Step 4: Determine the constant's value a in c by using the boundary conditions and find the approximate solutions as $h(z)\approx\sum_{i=0}^{n}h_i(z)$.

Step 5: Calculate the residue error $\mathrm{R(z)}$, and maximum absolute residue error L_∞ with respect to the domain $[0,T]$.

Step 6: To stop the algorithm, fix the stopping criteria as $\mathrm{L}_\infty<\epsilon$, where ϵ is the tolerance.

In the next section, we discuss about the convergence analysis of the series $\sum_{i=0}^{\infty}h_i(z)$ to validate the proposed scheme.

4. Convergence Analysis

We consider a sequence $\{q_n\}$ as n^{th} partial sums of the series $\sum_{i=0}^{\infty}h_i(z)$. Thus, we can write $h(z)=\lim_{n\to\infty}q_n(z)=\lim_{n\to\infty}\sum_{i=0}^{n}h_i(z)$. Now, we have to prove the series $\sum_{i=0}^{\infty}h_i(z)$ is uniformly convergent, and it converge to the exact solution $h(z)$. The series $\sum_{i=0}^{\infty}h_i(z)$ can be expressed as:

$$q_0+(q_1-q_0)+\cdots+(q_n-q_{n-1})+\cdots=q_0+\sum_{i=0}^{\infty}(q_{i+1}-q_i). \tag{55}$$

Therefore, $\{q_n\}$ can be written as:

$$q_n=q_0+(q_1-q_0)+\cdots+(q_n-q_{n-1}). \tag{56}$$

For each sequence $\{q_n\}$, we approximate $N(\sum_{i=0}^{n}h_i(z))$ as follows [26, 27]:

$$N(q_n)=(q_n(z))^2=\left(\sum_{i=0}^{n}h_i(z)\right)^2\approx\sum_{i=0}^{n}A_i(h_0,h_1,\cdots,h_i). \tag{57}$$

Lemma 4.1 *Consider the initial approximation $h_0(z)$ given by equation (52). Assume that $h_n(z)\in\mathcal{C}^2(0,T]$ for all $n\in\mathbb{N}\cup\{0\}$. Then for all $n\in\mathbb{N}\cup\{0\}$, we have:*

(i) $h_n(0)=0,h_n'(0)=0,h_n(z),h_n'(z),h_n''(z)\ and\ A_n are\ bounded\ on\ [0,T].$

(ii) $h_n(z)\in\mathcal{C}^2[0,T].$

Proof. In view of equations (52) to (53), we have $h_n(0)=0$. Now, from equation (52), we obtain:

$$h_0'(z) = \frac{az^\alpha}{\alpha} + z^\alpha \int_0^z \frac{1}{t^\alpha} \int_0^t \lambda y^\alpha G(y) dy\, dt. \tag{58}$$

Thus, from equation (58), we have $\lim_{z\to 0} h_0'(z) = 0$. Now:

$$h_0''(z) = az^{\alpha-1} + \alpha z^{\alpha-1} \int_0^z \frac{1}{t^\alpha} \int_0^t \lambda y^\alpha G(y) dy\, dt + \int_0^z \lambda t^\alpha G(t) dt - z^\alpha \lim_{z\to 0} \frac{\lambda t G(t)}{\alpha}. \tag{59}$$

So, we have $\lim_{z\to 0} h_0''(z) = 0$. Thus, $h_0''(z)$ is bounded. Also, as $\lim_{z\to 0} A_0(h_0) = \lim_{z\to 0} h_0^2(z) = 0$. Thus, $A_0(h_0)$ is bounded on $[0,T]$. By using the definition of big ohh [7], we can write $h_0(z) = \mathcal{O}(z^{\alpha+1})$. Also,

$$|A_0(h_0)| = |h_0^2(z)| \le d_0' z^{2\alpha+2}, \tag{60}$$

where d_0' is a fixed positive constant. Hence, we have $A_0(h_0) = \mathcal{O}(z^{2(\alpha+1)})$. Now, for $n=1$, we get:

$$h_1'(z) = z^\alpha \int_0^z \frac{\kappa}{2y^{\alpha+2}} A_0(h_0) dy. \tag{61}$$

Thus $\lim_{z\to 0} h_1'(z) = 0$. Hence, $h_1'(z)$ is bounded. Also,

$$h_1''(z) = \alpha z^{\alpha-1} \int_0^z \frac{\kappa h_0^2(y)}{2y^{\alpha+2}} dy + \frac{\kappa h_0^2(z)}{2z^2} - z^\alpha \lim_{z\to 0} \frac{\kappa h_0^2(z)}{2z^{\alpha+2}}. \tag{62}$$

Since $h_0(z) = \mathcal{O}(z^{\alpha+1})$ and $\lim_{z\to 0} \frac{\kappa h_0^2(z)}{2z^2} = \lim_{z\to 0} \frac{2\kappa h_0(z) h_0'(z)}{4z} = \lim_{z\to 0} \frac{\kappa\left[h_0(z)h_0''(z) + \left(h_0'(z)\right)^2\right]}{2} = 0$, therefore from equation (62), we get $h_1''(z)$ is bounded. Consequently, we have $A_1(h_0,h_1)$ is bounded. Hence, by using second fundamental theorem of calculus ([25], Theorem 34.3, p. 294), we can conclude that $h_1(z) \in \mathcal{C}^2[0,T]$. Now, from equation (54), we have:

$$|h_1(z)| \le \int_0^z t^\alpha \int_0^t \frac{\kappa}{2y^{\alpha+2}} |A_0(h_0)| dy dt \ \le\ \int_0^z t^\alpha \int_0^t \frac{\kappa d_0}{2y^{\alpha+2}} y^{2\alpha+2} dy dt \ \le\ \frac{\kappa d_0}{2(\alpha+1)} \frac{z^{(2\alpha+2)}}{(2\alpha+2)}. \tag{63}$$

Therefore, we can write $h_1(z) = \mathcal{O}(z^{2(\alpha+1)})$. Again,

$$|A_1(h_0,h_1)| = |2h_0(z)h_1(z)| \le d_1' z^{(3\alpha+3)}, \tag{64}$$

where d_1' is a fixed positive constant. Thus, using properties of big oh, we have $A_1(h_0,h_1) = \mathcal{O}(z^{3(\alpha+1)})$. Let the statement be true for all values of $n = 2,3,\cdots,m$. So, $h_n(z) \in \mathcal{C}^2[0,T], h_n(0) = 0, h_n'(0) = 0, h_n(z), h_n'(z), h_n''(z)$ and $A_n(h_0,h_1,\dots,h_n)$ are bounded, and $h_n(z) = \mathcal{O}(z^{(n+1)(\alpha+1)})$, $A_n(h_0,\cdots,h_n) = \mathcal{O}(z^{(n+2)(\alpha+1)})$ for all values of $n = 2,3,\cdots,m$. Now, for $n = m+1$, from equation (54), we can easily conclude $h_{m+1}(0) = 0$ and:

$$h_{m+1}'(z) = z^\alpha \int_0^z \frac{\kappa}{2y^{\alpha+2}} A_m(h_0,h_1,\dots,h_m) dy. \tag{65}$$

It is easy to observe that $\lim_{z\to 0} h_{m+1}'(z) = 0$ as $A_m(h_0,\cdots,h_m) = \mathcal{O}(z^{(m+2)(\alpha+1)})$. Also,

$$h_{m+1}''(z) = \alpha z^{\alpha-1} \int_0^z \frac{\kappa A_m(h_0,h_1,\dots,h_m)}{2t^{\alpha+2}} dt + \frac{\kappa \sum_{i=0}^m h_{m-i}(z)h_i(z)}{2z^2} - z^\alpha \lim_{z\to 0} \frac{\kappa \sum_{i=0}^m h_{m-i}(z)h_i(z)}{2z^{\alpha+2}}, \tag{66}$$

and

$$|h_{m+1}(z)| \le \int_0^z t^\alpha \int_0^t \frac{\kappa}{2y^{\alpha+2}} |A_m(h_0,h_1,\cdots,h_m)| dy dt \le \frac{\kappa d_m'}{2} \int_0^z t^\alpha \int_0^t \frac{y^{(m+2)\alpha+(m+2)}}{y^{\alpha+2}} dy dt,$$

So,

$$|h_{m+1}(z)| \ \le \frac{\kappa d_m'}{2\big((m+1)\alpha + (m+1)\big)} \frac{z^{(m+2)\alpha+(m+2)}}{\big((m+2)\alpha+(m+2)\big)}. \tag{67}$$

Therefore, we have $h_{m+1}(z) = \mathcal{O}(z^{(m+2)(\alpha+1)})$ and $A_{m+1}(h_0,h_1,\cdots,h_{m+1}) = \mathcal{O}(z^{(m+3)(\alpha+1)})$. By similar analysis we can conclude that $h_{m+1}'(0) = 0$, $h_{m+1}'(z)$, $h_{m+1}''(z)$ and A_{m+1} are bounded on $[0,T]$ and $h_{m+1}(z) \in \mathcal{C}^2[0,T]$. Thus, we get the desired result by mathematical induction.

Lemma 4.2 *Consider $h_n(z) \in \mathcal{C}^2[0,T]$. Let the n^{th} approximation $h_n(z)$ be obtained from equations (53) and (54) for all $n \in \mathbb{N}\cup\{0\}$. Then:*

(i) $q_n(0) - q_{n-1}(0) = 0$ *for each* $n \in \mathbb{N}$.

(ii) $q_n'(z)$ *and* $q_n''(z)$ *are bounded on* $[0,T]$ *for all* $n \ge 0$.

Proof. By using *Lemma 4.1*, we have $h_n(0) = 0$. Now, $q_n(0) - q_{n-1}(0) = h_0(0) + h_1(0) + \cdots + h_n(0) - h_0(0) - h_1(0) - \cdots - h_{n-1}(0) = h_n(0) = 0$. Since $q_n(z) = h_0(z) + h_1(z) + \cdots + h_n(z)$, and $h_n(z) \in \mathcal{C}^2[0,T]$, therefore all terms of $q_n'(z)$ and $q_n''(z)$ are bounded for all $n \in N\cup\{0\}$. Hence we conclude the results.

Lemma 4.3 *Consider $h_n(z) \in \mathcal{C}^2[0,T]$. Let the n^{th} approximation $h_n(z)$ be obtained from equations (53) and (54) for all $n \in \mathbb{N}\cup\{0\}$. Assume $d_i = \sup_{z\in[0,T]} \left|\frac{A_i}{z^{\alpha+2}}\right|$, where $i = 0,1,\cdots$. If there exist a natural number $n_0 \in \mathbb{N}$ such that $d_{i+1} \le r d_i$ for all $i \ge n_0$ and some value of $r < 1$, then there exists a real number $L_1 > 0$ such that:*

$$\left|\frac{q_n(z) + q_{n-1}(z)}{2z^{\alpha+1}}\right| \le L_1, \quad \text{for all } z \in [0,T] \text{ and } n \in \mathbb{N}. \tag{68}$$

Proof. For $n=1$ and by using Lemma 4.1, we have:

$$\left|\frac{q_1(z)+q_0(z)}{2z^{\alpha+1}}\right| = \left|\frac{h_1(z)+2h_0(z)}{2z^{\alpha+1}}\right| \le l_1, \forall\ z\in[0,T]. \tag{69}$$

For $n=2$, we get:

$$\left|\frac{q_2(z)+q_1(z)}{2z^{\alpha+1}}\right| \le \left|\frac{2h_0(z)+h_1(z)}{2z^{\alpha+1}}\right| + \left|\frac{h_1(z)+h_2(z)}{2z^{\alpha+1}}\right| \le l_1 + \left|\frac{h_1(z)+h_2(z)}{2z^{\alpha+1}}\right|. \tag{70}$$

In view of Lemma 4.1, we have $\left|\frac{h_1(z)+h_2(z)}{2z^{\alpha+1}}\right|$ is bounded on $[0,T]$. Therefore, we consider:

$$l_2 = l_1 + \sup_{z\in[0,T]}\left|\frac{h_1(z)+h_2(z)}{2z^{\alpha+1}}\right|.$$

Hence, we have:

$$\left|\frac{q_2(z)+q_1(z)}{2z^{\alpha+1}}\right| \le l_2, \forall\ z\in[0,T]. \tag{71}$$

In general, we have:

$$\left|\frac{q_m(z)+q_{m-1}(z)}{2z^{\alpha+1}}\right| \le l_m, \forall\ z\in[0,T], \tag{72}$$

where,

$$l_m = l_{m-1} + \sup_{z\in[0,T]}\left|\frac{h_{m-1}(z)+h_m(z)}{2z^{\alpha+1}}\right|. \tag{73}$$

Now, for $n=m+1$, we get:

$$\begin{aligned}\left|\frac{q_{m+1}(z)+q_m(z)}{2z^{\alpha+1}}\right| &\le \left|\frac{2h_0(z)+2h_1(z)+\cdots+h_m(z)}{2z^{\alpha+1}}\right| + \left|\frac{h_m(z)+h_{m+1}(z)}{2z^{\alpha+1}}\right|,\\ &\le l_m + \left|\frac{h_m(z)+h_{m+1}(z)}{2z^{\alpha+1}}\right| \le l_{m+1}, \forall z\in[0,T],\end{aligned} \tag{74}$$

where,

$$l_{m+1} = l_m + \sup_{z\in[0,T]}\left|\frac{h_{m+1}(z)+h_m(z)}{2z^{\alpha+1}}\right|. \tag{75}$$

Thus, by using mathematical induction we can write:

$$\left|\frac{q_n(z)+q_{n-1}(z)}{2z^{\alpha+1}}\right| \le l_n, \qquad \forall z\in[0,T], \qquad n\in N, \tag{76}$$

where,

$$l_n = l_{n-1} + \sup_{z\in[0,T]}\left|\frac{h_n(z)+h_{n-1}(z)}{2z^{\alpha+1}}\right|. \tag{77}$$

Equivalently, we can say that there exists a natural number n_0 such that:

$$|l_n - l_{n-1}| = \sup_{z\in[0,T]}\left|\frac{h_n(z)+h_{n-1}(z)}{2z^{\alpha+1}}\right|, \qquad \forall n\ge n_0. \tag{78}$$

Now, it is necessary to show the sequence $\{l_n\}$, defined by equation (78), is an eventually constant sequence. Thus:

$$\begin{aligned}\left|\frac{h_n(z)+h_{n-1}(z)}{2z^{\alpha+1}}\right| &\le \left|\frac{h_n(z)}{2z^{\alpha+1}}\right| + \left|\frac{h_{n-1}(z)}{2z^{\alpha+1}}\right|,\\ &\le \frac{1}{2z^{\alpha+1}}\int_0^z t^\alpha \int_0^t \frac{\kappa}{2y^{\alpha+2}}(|A_{n-1}(h_0,h_1,\cdots,h_{n-1})| + |A_{n-2}(h_0,h_1,\cdots,h_{n-2})|)dydt.\end{aligned} \tag{79}$$

Since $d_i = \sup_{z\in[0,T]}\left|\frac{A_i}{z^{\alpha+2}}\right|$, where $i=0,1,\ldots$, therefore, equation (79) can be written as:

$$\left|\frac{h_n(z)+h_{n-1}(z)}{2z^{\alpha+1}}\right| \le \frac{1}{2z^{\alpha+1}}\int_0^z t^\alpha \int_0^t \frac{\kappa}{2}\ (d_{n-1}+d_{n-2})dydt. \tag{80}$$

By using $d_{i+1}\le rd_i$, from equation (80) we have:

$$\begin{aligned}\left|\frac{h_n(z)+h_{n-1}(z)}{2z^{\alpha+1}}\right| &\le \frac{1}{2z^{\alpha+1}}\int_0^z t^\alpha \int_0^t \frac{\kappa}{2}(1+r)d_{n-2}dydt,\\ &\le \frac{1}{2z^{\alpha+1}}\int_0^z t^\alpha \int_0^t \frac{\kappa}{2}(1+r)rd_{n-3}dydt,\\ &\ \ \vdots,\end{aligned} \tag{81}$$

$$\leq \frac{1}{2z^{\alpha+1}} \int_0^z t^\alpha \int_0^t \frac{\kappa}{2}(1+r)r^{n-2}d_0 dydt,$$
$$\leq \frac{\kappa(1+r)r^{n-2}d_0}{4z^{\alpha+1}} \int_0^z t^\alpha \int_0^t dy\,dt,$$
$$\leq \frac{\kappa(1+r)r^{n-2}d_0 z}{4(\alpha+2)}.$$

Hence, we get:

$$\sup_{z\in[0,T]} \left|\frac{h_n(z)+h_{n-1}(z)}{2z^{\alpha+1}}\right| \leq \sup_{z\in[0,T]} \left|\frac{\kappa(1+r)r^{n-2}d_0 z}{4(\alpha+2)}\right|. \tag{82}$$

Since $r<1$, therefore, taking limit on both side we get:

$$\lim_{n\to\infty} \sup_{z\in[0,T]} \left|\frac{h_n(z)+h_{n-1}(z)}{2z^{\alpha+1}}\right| \leq \lim_{n\to\infty} \frac{\kappa(1+r)r^{n-2}d_0 T}{4(\alpha+2)} = 0. \tag{83}$$

Thus, from equation (78), we deduce that $\{l_n\}$ is an eventually constant sequence, that is, $l_n = l_{n-1}, \forall n \geq n_0$. Hence, the result follows.

Lemma 4.4 *If $\{q_n\}$ is defined as the n^{th} partial sum of the series $\sum_{i=0}^{\infty} h_i(z)$, then $\{q_n\}$ can be written as follows:*

$$q_n = c + L^{-1}[N(q_{n-1})], n \geq 1, \tag{84}$$

where the nonlinear operator N is provided by equation (57) and c is the constant.

Proof. By using equations (52) and (54), we obtain:

$$\begin{aligned} q_n(z) &= h_0(z)+h_1(z)+\cdots+h_n(z),\\ &= c + \int_0^z t^\alpha \int_0^t \frac{\kappa}{2y^{\alpha+2}} \sum_{i=0}^{n-1} A_i(h_0,h_1,\cdots,h_i)dy\,dt,\\ &= c + \int_0^z t^\alpha \int_0^t \frac{\kappa}{2y^{\alpha+2}} N\left(\sum_{i=0}^{n-1} h_i(z)\right) dydt,\\ &= c + \int_0^z t^\alpha \int_0^t \frac{\kappa}{2y^{\alpha+2}} N(q_{n-1}(z))dydt. \end{aligned} \tag{85}$$

This completes the proof.

Theorem 4.1 *Let $h_0(z)$ be the initial approximation of an iterative scheme (54). Assume $\{q_n\}$ represent the n^{th} partial sum of $\sum_{i=0}^{\infty} h_i(z)$. Then the series $\sum_{i=0}^{\infty} h_i(z)$ is uniformly convergent and converge to the exact solution $h(z)$ of the problem (3) in $\mathcal{C}^2[0,T]$.*

Proof. By using equation (84) for $n=1$, we get:

$$|q_1(z)-q_0(z)| \leq \int_0^z \left| t^\alpha \int_0^t \frac{\kappa}{2y^{\alpha+2}}(q_0)^2 dy\right| dt. \tag{86}$$

By using Lemma 4.1, we assume $L_2 = \sup_{t\in[0,T]} \left\{\left|t^\alpha \int_0^t \frac{\kappa}{2y^{\alpha+2}}(q_0)^2 dy\right|\right\}$. Let $L = \max\{L_1, L_2\}$. Thus, we get:

$$|q_1(z)-q_0(z)| \leq L_2 z \leq Lz. \tag{87}$$

Now, for $n=2$, by using Lemma 4.3, we have:

$$|q_2(z)-q_1(z)| \leq \int_0^z t^\alpha \int_0^t \left|\frac{\kappa}{2y^{\alpha+2}}(q_1^2-q_0^2)dy\right| dt. \tag{88}$$

Now, inequality (88) can be written as:

$$|q_2(z)-q_1(z)| \leq \int_0^z t^\alpha \int_0^t \left|\frac{q_1+q_0}{2y^{\alpha+1}}\right| \left|\frac{q_1-q_0}{y}\right| dydt. \tag{89}$$

By using the results of Lemma 4.3 and inequality (87), we get:

$$|q_2(z)-q_1(z)| \leq \int_0^z t^\alpha \int_0^t \kappa L^2\, dydt \leq \kappa L^2 \frac{z^{\alpha+2}}{(\alpha+2)}. \tag{90}$$

Again for $n=3$, we get:

$$|q_3(z)-q_2(z)| \leq \int_0^z t^\alpha \int_0^t \left|\frac{\kappa}{2y^{\alpha+2}}(q_2^2-q_1^2)dy\right| dt \leq \int_0^z t^\alpha \int_0^t \left|\frac{q_2+q_1}{2y^{\alpha+1}}\right| \left|\frac{q_2-q_1}{y}\right| dydt. \tag{91}$$

By using inequality (89), we get:

$$|q_3(z)-q_2(z)| \leq \frac{\kappa^2 L^3}{(\alpha+2)} \int_0^z t^\alpha \int_0^t y^{\alpha+1}\, dydt \leq \frac{\kappa^2 L^3}{(\alpha+2)^2} \frac{z^{2\alpha+3}}{(2\alpha+3)}. \tag{92}$$

Similarly, we get:

$$|q_4(z)-q_3(z)| \le \frac{\kappa^3 L^4}{(\alpha+2)^2(2\alpha+3)}\int_0^z t^\alpha \int_0^t y^{2\alpha+2}\,dydt \;\le \frac{\kappa^3 L^4}{(\alpha+2)^2(2\alpha+3)^2}\frac{z^{3\alpha+4}}{(3\alpha+4)}. \tag{93}$$

Let the statement be true for $n=m$, i.e.,

$$|q_m(z)-q_{m-1}(z)| \;\le \frac{\kappa^{m-1}L^m}{(\alpha+2)^2(2\alpha+3)^2\cdots\big((m-2)\alpha+(m-1)\big)^2}\;\frac{z^{(m-1)\alpha+m}}{\big((m-1)\alpha+m\big)}. \tag{94}$$

Now, for $n=m+1$, we have:

$$\begin{aligned}|q_{m+1}(z)-q_m(z)| &\le \int_0^z t^\alpha \int_0^t \left|\frac{q_m+q_{m-1}}{2y^{\alpha+1}}\right|\left|\frac{q_m-q_{m-1}}{y}\right| dydt,\\ &\le \frac{\kappa^m L^{m+1}}{(\alpha+2)^2\cdots\big((m-2)\alpha+(m-1)\big)^2\big((m-1)\alpha+m\big)}\int_0^z t^\alpha\int_0^t y^{(m-1)\alpha+(m-1)}\,dydt.\end{aligned} \tag{95}$$

This implies:

$$|q_{m+1}(z)-q_m(z)| \le \frac{\kappa^m L^{m+1}}{(\alpha+2)^2\cdots\big((m-2)\alpha+(m-1)\big)^2\big((m-1)\alpha+m\big)^2}\;\frac{z^{m\alpha+(m+1)}}{\big(m\alpha+(m+1)\big)}. \tag{96}$$

Hence, the statement is true for $n=m+1$. Therefore, we have:

$$|q_n(z)-q_{n-1}(z)| \;\le \frac{\kappa^{n-1}L^n}{(\alpha+2)^2(2\alpha+3)^2\cdots\big((n-2)\alpha+(n-1)\big)^2}\frac{z^{(n-1)\alpha+n}}{\big((n-1)\alpha+n\big)},\qquad \forall n\in N. \tag{97}$$

Now, we consider the series:

$$q_0+Lz+\sum_{n=2}^{\infty}\frac{\kappa^{n-1}L^n}{(\alpha+2)^2(2\alpha+3)^2\cdots\big((n-2)\alpha+(n-1)\big)^2}\frac{z^{(n-1)\alpha+n}}{(n-1)\alpha+n}. \tag{98}$$

Thus, we can see that the series (98) is convergent for every $z\in\mathbb{R}$. Thus, for every $z\in[0,T]$, we infer that the series (55) is convergent absolutely and uniformly. Consequently, we have the limit function $h(z)$ in $\mathcal{C}^2[0,T]$. Hence, the proof is complete.

5. Numerical Discussions

In this section, the numerical observations of the solutions $h(z)$ for various values of α and T are shown. Subsection 5.1 contains the numerical findings for $\alpha=1$ and $T>1$. In the Subsection 5.2, $\alpha>1$ and $T=1$ are taken into consideration. We present numerical data corresponding to the approximate solutions $h(z)$ for $\alpha>1$ and $T>1$ in Subsection 5.3.

The residue error of problem (3) is provided by:

$$R(z)=z^2h''(z)-\alpha z h'(z)-\frac{\kappa h^2(z)}{2}-z^2\int_0^z \lambda t^\alpha G(t)dt. \tag{99}$$

The maximum absolute residue error is defined by the following formula:

$$L_\infty=\max_{i=0,\dots,\frac{T}{0.1}}|R(z_0+i\times 0.1)|\,,z_0=0. \tag{100}$$

5.1 Approximate Solutions $\mathbf{h(z)}$ for $\mathbf{T>1}$ and $\boldsymbol{\alpha=1}$

We examine for $G(z)=1$ and $p=1$. With the help of iterations (52) and (54), we have:

$$h_0(z)=\frac{az^2}{2}+\frac{z^4\lambda}{16},\qquad h_1(z)=\frac{a^2z^4}{64}+\frac{az^6\lambda}{768}+\frac{z^8\lambda^2}{24576}, \tag{101}$$

and so forth. For any $n\in N$, the approximate solutions to $h(z)$ is $\sum_{i=0}^n h_i(z)$. Here, the constant a can be calculated with the help of the Mathematica programme.

5.1.1 BCs (10)

We choose $\mathrm{n}=51$ and $\mathrm{T}=1.1$. We calculate the value of the constant a by applying the boundary condition $\mathrm{h(T)}=0$. We notice that there are two solutions for $0\le\lambda\le\lambda_c$. Here, the critical value of λ is denoted by λ_c. Since the value of the constant a is imaginary, we see that there are no approximate solutions for $\lambda>\lambda_c$. We determine 115.27 to be the critical value of λ. Moreover, two solutions are approaching one another as we increase the value of λ. Numerically, we are unable to identify a unique solution for $\lambda=\lambda_c$. We provide L_∞ error corresponding to some positive λ in Table 1 and plot the graph corresponding to $\lambda=110.25$ in Fig. 1.

Remark 5.1 *We have included few values of d_i corresponding to $\lambda=0$ to confirm the condition $d_{i+1}\le rd_i$ of Lemma 4.3, in Table 2. Here, we observe that approximate upper solution $\sum_{i=0}^n h_i(z)$ of problem (3) corresponding to the boundary condition (10) does not converge to the exact upper solution $h(z)$ for $\lambda=0$, as the values do not meet the condition $d_{i+1}\le rd_i$. However, the proposed iterative technique accurately computed both solutions for $\lambda=115.27$.*

5.1.2 BCs (11)

We consider $T=2$ and $n=39$. The constant a is determined by using the boundary condition $(\alpha-1)h(T)=Th'(T)$. We observe the same observations as in Subsection 5.1.1, except the value of λ_c. λ_c has been estimated to be 1.996. For varying values of λ, we have included numerical data In Table 3, Table 4, Fig. 2 and Fig. 3.

Table 1. L_∞ error of approximate solutions $h(z)$.

λ	Lower solution	Upper solution
110.25	5.68×10^{-14}	5.76×10^{-2}

Table 2. The values of $\sup_{z\in[0,T]} \left|\frac{A_i}{z^{\alpha+2}}\right|$ corresponding to approximate upper solution for $\lambda = 0$.

d_i	d_0	d_1	d_2
Upper solution	850.40	3576.40	8773.60

Table 3. L_∞ error of approximate solutions $h(z)$.

λ	Lower solution	Upper solution
0.78	8.88×10^{-16}	1.04×10^{-8}

Table 4. L_∞ error of approximate solutions $h(z)$.

λ	Lower solution	Upper solution
1.37	3.55×10^{-15}	6.16×10^{-12}

Table 5. L_∞ error of approximate solutions $h(z)$.

λ	Lower solution	Upper solution
0.19	2.22×10^{-16}	5.33×10^{-15}

Table 6. L_∞ error of approximate solutions $h(z)$.

λ	Lower solution	Upper solution
0.58	4.44×10^{-16}	3.55×10^{-15}

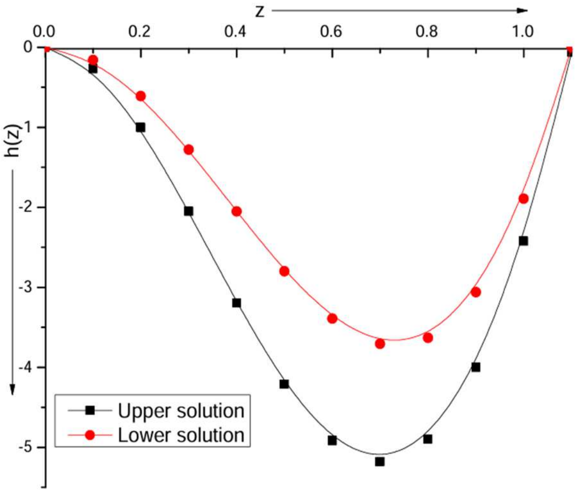

Fig. 1. Approximate solutions $h(z)$ corresponding to $\lambda = 110.25$.

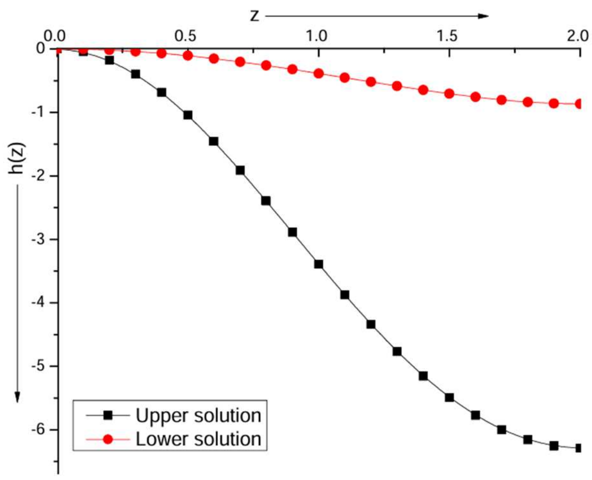

Fig. 2. Approximate solutions $h(z)$ for $\lambda = 0.78$.

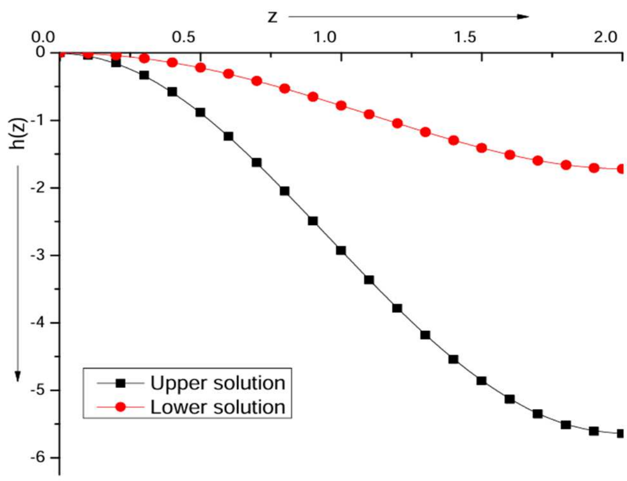

Fig. 3. Approximate solutions $h(z)$ for $\lambda = 1.37$.

5.1.3 BCs (12)

We choose $T = 2$ and $n = 39$. The boundary condition $\alpha h(T) = Th'(T)$ are used to find the value of the constant a. We observe the same observations as in Subsection 5.1.1. We find λ_c as 0.708. For $\lambda = 0.19$ and $\lambda = 0.58$, the maximum absolute residue error is provided in Table 5 and Table 6, respectively. The graphs of approximate solutions are shown in Figs. 4 and 5.

5.2. Approximate Solutions $h(z)$ for $T = 1$ and $\alpha > 1$

Here also we consider $G(z) = 1$ and $p = 1$. Now, by using equations (52) and (54), for $\alpha = 2$ we have $h_0(z) = \frac{az^3}{6} + \frac{z^5\lambda}{30}$, $h_1(z) = \frac{a^2z^6}{1296} + \frac{az^8\lambda}{7200} + \frac{z^{10}\lambda^2}{126000}$ and so on. For $\alpha = 3$, we compute $h_0(z) = \frac{az^4}{12} + \frac{z^6\lambda}{48}$, $h_1(z) = \frac{a^2z^8}{9216} + \frac{az^{1}0\lambda}{34560} + \frac{z^{12}\lambda^2}{442368}$ and so on.

5.2.1. BCs (10)

For $\alpha = 2$ and $\alpha = 3$, we consider the approximate solutions as $h(z) \approx \sum_{i=0}^{51} h_i(z)$. Similar observations to Subsection 5.1.1 are made here. Corresponding to $\alpha = 2$ and $\alpha = 3$, the value of λ_c are approximated as 874 and 2960, respectively. We provide Table 7 and Table 8 regarding numerical data.

Table 7. L_∞ error of approximate solutions $h(z)$ for $\alpha = 2$.

λ	Lower solution	Upper solution
322.73	2.84×10^{-14}	5.11×10^{-9}

Table 8. L_∞ error of approximate solutions $h(z)$ for $\alpha = 3$.

λ	Lower solution	Upper solution
2000	2.84×10^{-13}	2.30×10^{-10}

Table 9. L_∞ error of approximate solutions $h(z)$ for $\alpha = 2$.

λ	Lower solution	Upper solution
91.92	7.11×10^{-15}	2.13×10^{-14}

Table 10. L_∞ error of approximate solutions $h(z)$ for $\alpha = 3$.

λ	Lower solution	Upper solution
137.29	1.07×10^{-14}	8.53×10^{-14}

Table 11. L_∞ error of approximate solutions $h(z)$ for $\alpha = 2$.

λ	Lower solution	Upper solution
23.79	1.33×10^{-15}	7.11×10^{-15}

Table 12. L_∞ error of approximate solutions $h(z)$ for $\alpha = 3$.

λ	Lower solution	Upper solution
47.83	8.88×10^{-15}	1.24×10^{-14}

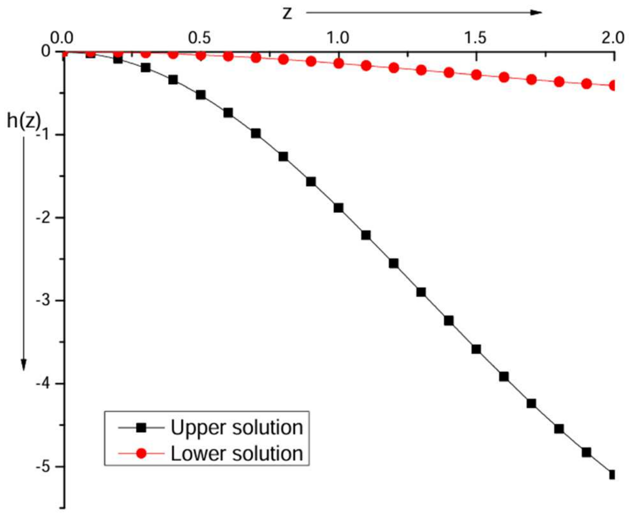

Fig. 4. Approximate solutions $h(z)$ for $\lambda = 0.19$.

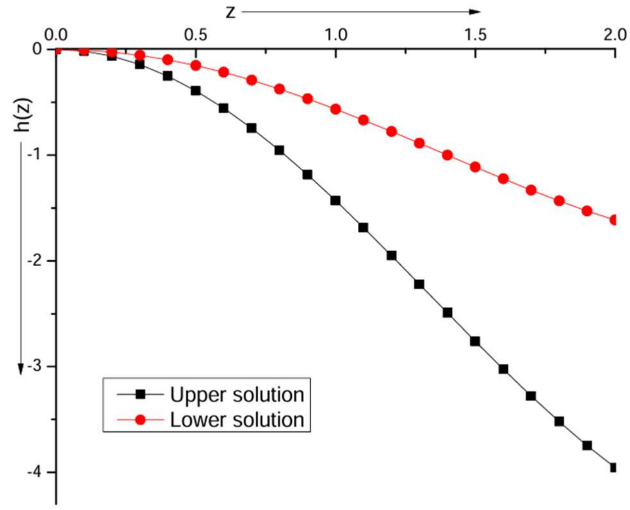

Fig. 5. Approximate solutions $h(z)$ for $\lambda = 0.58$.

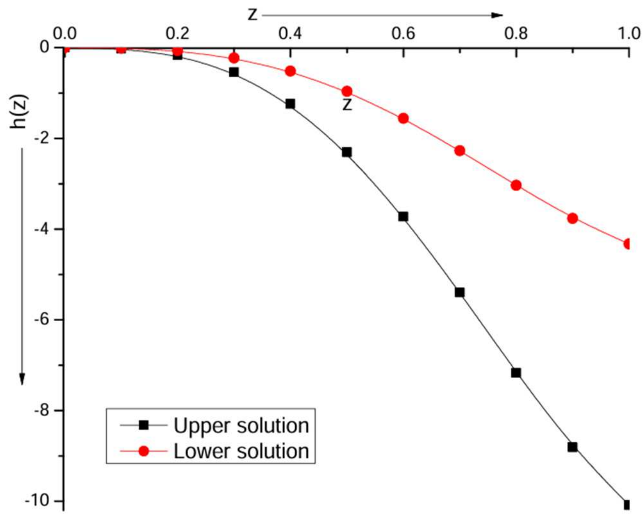

Fig. 6. Approximate solutions $h(z)$ for $\lambda = 91.92$.

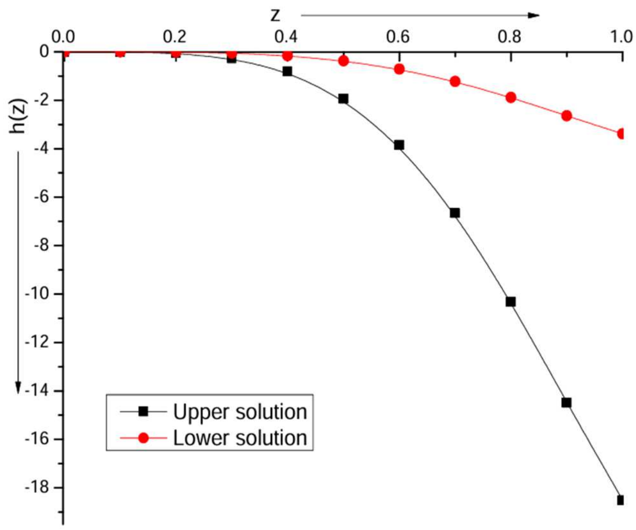

Fig. 7. Approximate solutions $h(z)$ for $\lambda = 137.29$.

Remark 5.2 *It is observed that, when $\lambda = 0$, the proposed scheme (54) is unable to yield the exact upper solution $h(z)$ of problem (3) that corresponds to the boundary condition (10). This is because, for all $n \in \mathbb{N}$ and a certain value of $r < 1$, the sequence $\{d_n\}$ does not satisfy the condition $d_{n+1} \leq r d_n$.*

5.2.2 BCs (11)

We take $n = 39$ into consideration to get the approximate solutions. We observe the same findings as in Subsection 5.1.1. For $\alpha = 2$ and $\alpha = 3$, we have approximated λ_c as 109.92 and 262.19, respectively. For various values of λ, we have placed numerical data in Table 9, Table 10, Figs. 6 and 7.

5.2.3 BCs (12)

By taking into account $h(z) \approx \sum_{i=0}^{39} h_i(z)$, we can derive approximate solutions. Similar remarks are seen as in Subsection 5.1.1. The values of λ_c that correspond to $\alpha = 2$ and $\alpha = 3$, respectively, are estimated as 35.30 and 79.58. Table 11 and Table 12 include the maximum absolute residue error. Figures 8 and 9 show the graphical representation of the solutions.

Table 13. L_∞ error of approximate solutions $h(z)$ for $\alpha = 2$.

λ	Lower solution	Upper solution
233.49	4.26×10^{-14}	3.21×10^{-7}

Table 14. L_∞ error of approximate solutions $h(z)$ for $\alpha = 3$.

λ	Lower solution	Upper solution
1000	1.17×10^{-13}	9.88×10^{-9}

Table 15. L_∞ error of approximate solutions $h(z)$ for $\alpha = 2$.

λ	Lower solution	Upper solution
0.97	3.55×10^{-15}	1.56×10^{-13}

Table 16. L_∞ error of approximate solutions $h(z)$ for $\alpha = 3$.

λ	Lower solution	Upper solution
1.13	2.13×10^{-14}	1.14×10^{-13}

Table 17. L_∞ error of approximate solutions $h(z)$ for $\alpha = 2$.

λ	Lower solution	Upper solution
0.16	1.78×10^{-15}	3.19×10^{-14}

Table 18. L_∞ error of approximate solutions $h(z)$ for $\alpha = 3$.

λ	Lower solution	Upper solution
0.21	5.32×10^{-15}	7.86×10^{-14}

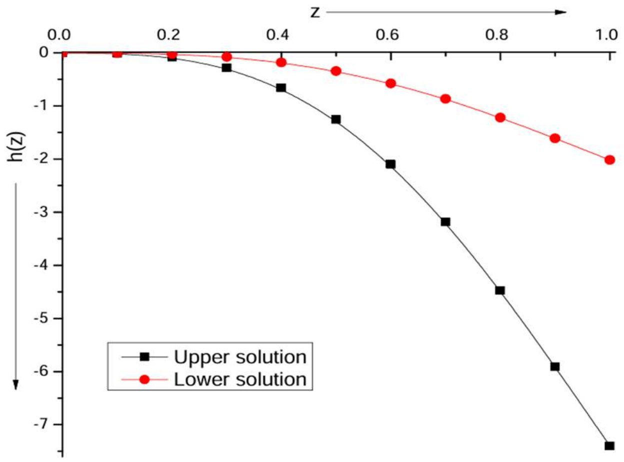

Fig. 8. Approximate solutions $h(z)$ for $\lambda = 23.79$.

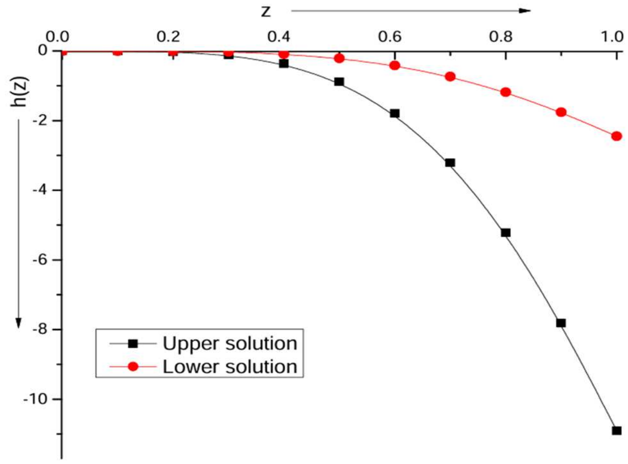

Fig. 9. Approximate solutions $h(z)$ for $\lambda = 47.83$.

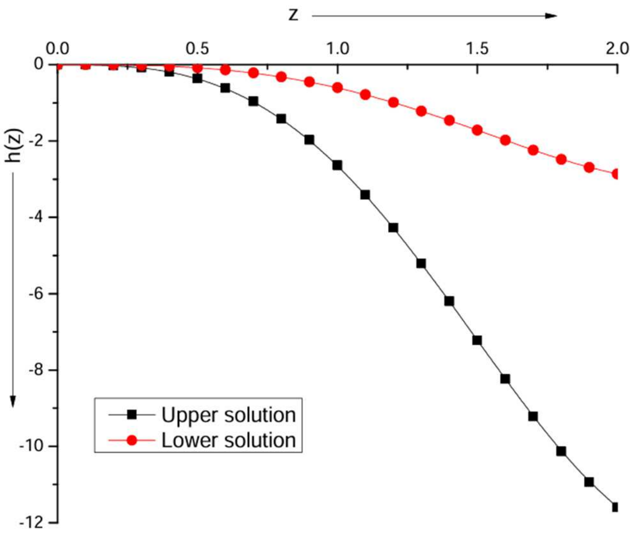

Fig. 10. Approximate solutions $h(z)$ for $\lambda = 0.97$.

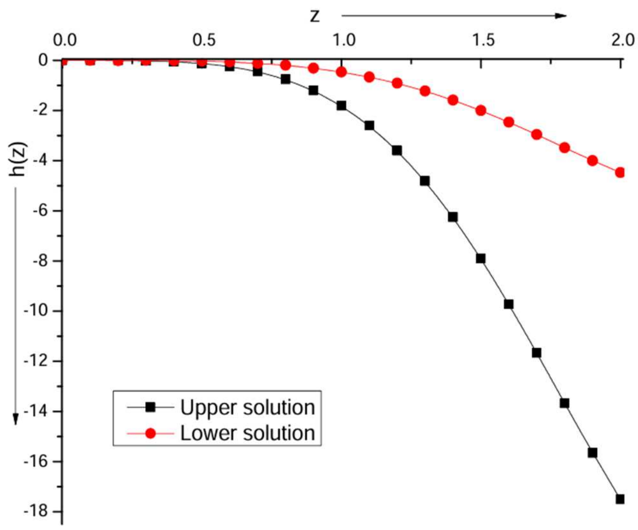

Fig. 11. Approximate solutions $h(z)$ for $\lambda = 1.13$.

Remark 5.3 *We have seen that, throughout the analysis process, the critical values of λ consistently satisfy the theoretical estimations obtained by Pandit et al. [1].*

5.3 Approximate Solutions $h(z)$ for $T > 1$ and $\alpha > 1$

Here, we compute the approximate numerical solutions of $h(z)$ corresponding to equation (3) under the following boundary conditions, whenever $\alpha > 1$ and $T > 1$.

5.3.1 BCs (10)

In this section, we choose $T = \frac{11}{10}$, $p = 1$, and $G(z) = 1$. We have computed the approximate solutions for $h(z) = \sum_{i=0}^{79} h_i(z)$ with the help of iterations (52) and (54). In addition, we note similar findings as in Subsection 5.1.1 and Remark 5.3. The values of λ that correspond to $\alpha = 2$ and $\alpha = 3$ are equivalent to 543.99 and 1671.99. Tables 13 and 14 include numerical data for different values of λ.

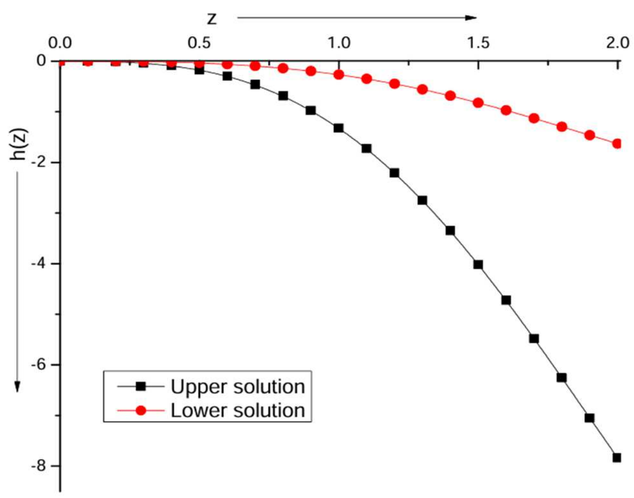

Fig. 12. Approximate solutions $h(z)$ for $\lambda = 0.16$.

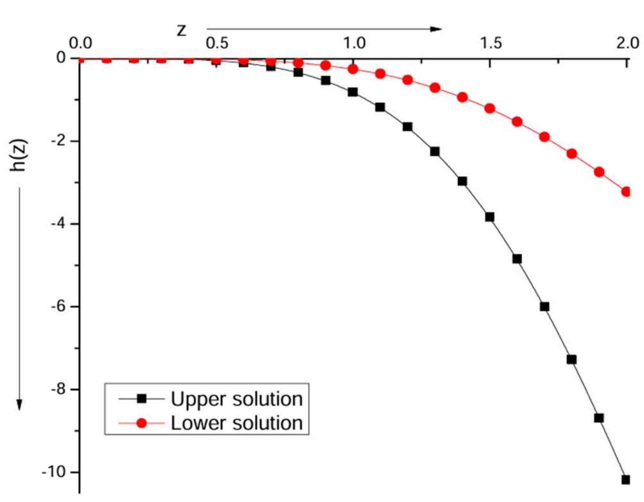

Fig. 13. Approximate solutions $h(z)$ for $\lambda = 0.21$.

5.3.2 BCs (11)

Here, we have defined $\mathrm{G}(z)$ as $1+z$, $\mathrm{p}=1$, and $\mathrm{T}=2$. We have computed the approximate solutions as $\sum_{i=0}^{39} h_i(z)$ by using similar procedure as in Subsection 5.2. The same observations as in Subsection 5.1.1 are made here. The values of λ_c that correspond to $\alpha = 2$ and $\alpha = 3$, respectively, are estimated as 1.55 and 1.74. Numerical values are given in Table 15, Table 16, Figs. 10 and 11, which correspond to a few values of λ.

5.3.3 BCs (12)

We consider $\mathrm{p}=1$, $\mathrm{T}=2$, and $\mathrm{G}(z) = 1+z+z^2$ here. Approximate solutions can be expressed as $\sum_{i=0}^{39} h_i(z)$. We have observed similar results to Subsection 5.1.1. We estimate the value of λ_c to be 0.282 for $\alpha = 2$ and 0.287 for $\alpha = 3$, respectively. Figures 12 and 13, Table 17 and Table 18 are placed concerning the maximum absolute residue error.

Remark 5.4 *Numerical approximations of fourth-order singular BVP (1) is easily computed by using the transformation $h(z) = z^{\alpha}\Psi'(z)$ and boundary condition $\Psi(T) = 0$.*

5.4. Comparison with Existing Results

In this subsection, we compare the numerical results computed by proposed method with the existing results which is derived in [1]. We have seen that there are no numerical results in the literature concerning Subsection 5.1 and Subsection 5.3. Corresponding to Subsection 5.2, we have calculated the numerical approximations of fourth order singular BVP (1) with the help of Remark 5.4. Due to lack of space, here we place few results of maximum absolute residue error in Table 19 and Table 20.

5.5. Order of Convergence

Here, we illustrate the order of the proposed method. Since the problem has no exact solution, therefore we calculate approximated computational order of convergence [6].

Table 19. L_∞ error to the numerical approximations $\Psi(z)$ of the singular BVP (1) and the boundary condition (5) corresponding to $\lambda = 109.92, \alpha = 2$ and $T = 1$.

Proposed Method	n = 17	$n = 19$	$n = 21$	$n = 23$	Method in [1]
Lower Solution	2.42×10^{-7}	9.85×10^{-9}	3.82×10^{-10}	1.45×10^{-11}	9.23×10^{-9}
Upper Solution	2.62×10^{-7}	1.07×10^{-8}	4.17×10^{-10}	1.57×10^{-11}	1.14×10^{-8}

Table 20. L_∞ error to the numerical approximations $\Psi(z)$ of the singular BVP (1) and the boundary condition (5) corresponding to $\lambda = 262.19, \alpha = 3$ and $T = 1$.

Proposed Method	n = 17	n = 19	n = 21	n = 23	Method in [1]
Lower Solution	6.19×10^{-9}	1.45×10^{-10}	3.37×10^{-12}	2.99×10^{-13}	1.31×10^{-10}
Upper Solution	6.48×10^{-9}	1.52×10^{-10}	2.42×10^{-12}	2.82×10^{-13}	1.69×10^{-10}

Table 21. Approximated computational order of convergence for the singular BVP (3) and the boundary condition (11) corresponding to lower approximation, when $\lambda = 91.92, \alpha = 2$ and $T = 1$.

Iterations (n)	ϵ_{n-1}	ϵ_n	ϵ_{n+1}	ACOC ($\bar{A}$)
9	1.21×10^{-5}	1.44×10^{-6}	1.67×10^{-7}	1.01055
10	1.44×10^{-6}	1.67×10^{-7}	1.89×10^{-8}	1.00854
11	1.67×10^{-7}	1.89×10^{-8}	2.12×10^{-9}	1.00704
12	1.89×10^{-8}	2.12×10^{-9}	2.34×10^{-10}	1.00590
13	2.12×10^{-9}	2.34×10^{-10}	2.56×10^{-11}	1.00495
14	2.34×10^{-10}	2.56×10^{-11}	2.77×10^{-12}	1.00447

Table 22. Approximated computational order of convergence for the singular BVP (3) and the boundary condition (11) corresponding to lower approximation, when $\lambda = 137.29, \alpha = 3$ and $T = 1$.

Iterations (n)	ϵ_{n-1}	ϵ_{n}	ϵ_{n+1}	ACOC ($\bar{A}$)
6	3.47×10^{-5}	1.86×10^{-6}	9.55×10^{-8}	1.01757
7	1.86×10^{-6}	9.55×10^{-8}	4.70×10^{-9}	1.01277
8	9.55×10^{-8}	4.70×10^{-9}	2.24×10^{-10}	1.00972
9	4.70×10^{-9}	2.24×10^{-10}	1.04×10^{-11}	1.00750
10	2.24×10^{-10}	1.04×10^{-11}	4.81×10^{-13}	1.00633
11	1.04×10^{-11}	4.81×10^{-13}	2.17×10^{-14}	1.00407

Definition 5.1 Approximated computational order of convergence (ACOC) (say $\bar{A}$) of the proposed iterative method is computed by:

$$\mathrm{ACOC} = \bar{A} = \frac{p}{q}, \qquad p = \ln\left|\frac{\epsilon_{n+1}}{\epsilon_n}\right|, q = \ln\left|\frac{\epsilon_n}{\epsilon_{n-1}}\right|, \text{ where } \epsilon_n = \max_{z\in[0,T]}\left|\sum_{i=0}^{n} h_i(z) - \sum_{i=0}^{n-1} h_i(z)\right|. \tag{102}$$

Due to lack of space, here we place only few data regarding the ACOC of the proposed scheme in Tables 21 and 22. We have seen that order of the convergence of the proposed method is one, i.e., linear.

Below, we plot few numerical approximations of the solutions to visually represent the stability of the proposed method. From Figs. 14 to 17, we have noticed that the numerical approximations coincide with each other for increasing the value of n.

Remark 5.5 *We have used the 8GB random access memory (RAM) of the CPU to run the method using the Mathematica software. It has been found that the CPU computes the data up to n = 10 iterations in less than 30 seconds. However, if we increase the number of iterations, the entire process will take longer than a minute to complete.*

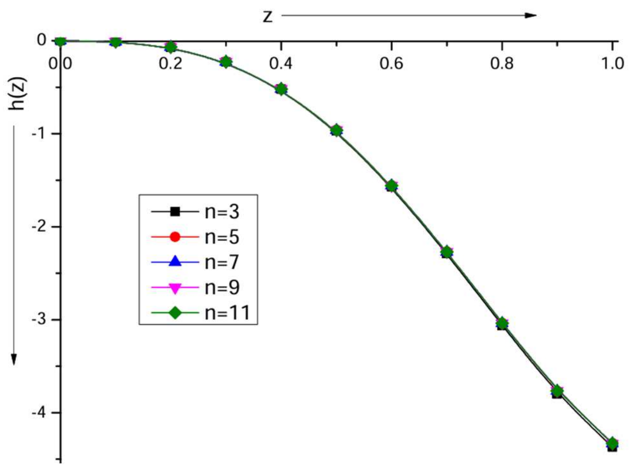

Fig. 14. Approximate lower solution $h(z)$ for $\lambda = 91.92$, $T = 1$ and $\alpha = 2$.

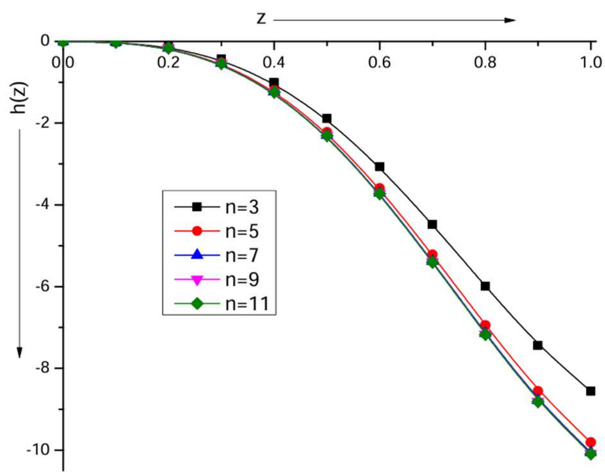

Fig. 15. Approximate upper solution $h(z)$ for $\lambda = 91.92$, $T = 1$ and $\alpha = 2$.

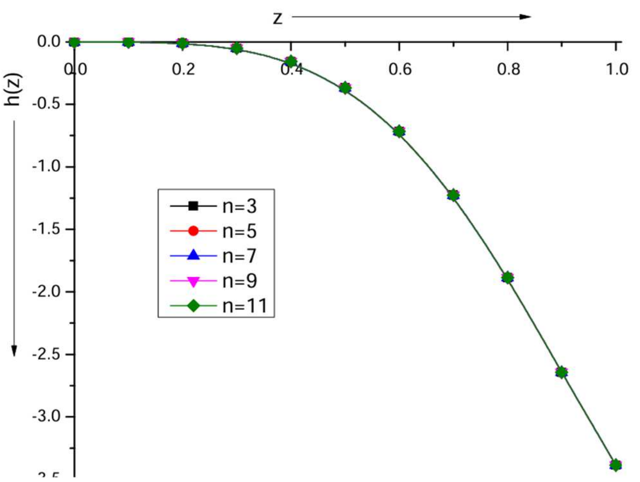

Fig. 16. Approximate lower solution $h(z)$ for $\lambda = 137.29$, $T = 1$ and $\alpha = 3$.

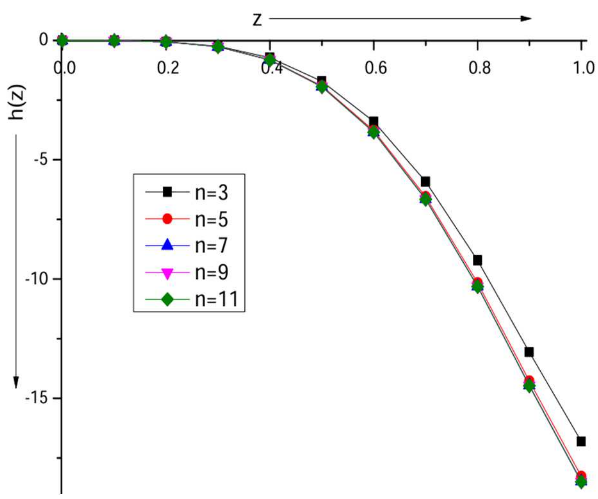

Fig. 17. Approximate upper solution $h(z)$ for $\lambda = 137.29$, $T = 1$ and $\alpha = 3$.

6. Conclusions

We have successfully presented the second-order singular boundary value problem (3) subject to three boundary conditions emerging in the semiconductor industry. We have also classified the characteristics of the solutions. Because there are two solutions to the problem, computing both numerical approximations are challenging. Here, we have successfully developed an iterative scheme and computed numerical approximations that are very close to the exact solutions of the given class of problems. One of the solutions becomes trivial when the value of λ is set to zero, but there is still a second, nontrivial, negative solution. There are two ordered solutions for positive values of λ, and they get closer as λ increase. When λ is adjusted to its critical value, we conjecture that all solutions merge into a single solution that is still nontrivial and negative. Based on the results given in Table 2, we deduce that for all values of $\alpha \geq 1$ and $\mathrm{T} > 0$, the suggested method is unable to capture the upper solution of problem (3) with respect to the boundary conditions $\mathrm{h}'(0) = 0, \mathrm{h}(\mathrm{T}) = 0$. However, the proposed method converges very quickly for determining the numerical approximations of the problem (3) corresponding to boundary conditions (10) and (11) for any values of $\alpha > 1, \mathrm{T} > 0$ and $\lambda \geq 0$ with desired accuracy. Despite differing in their quantitative behavior, the results corresponding to the three different types of boundary conditions we have examined here, demonstrate a qualitative agreement. We anticipate that the current findings will influence how these theoretical models of epitaxial development are understood in the semiconductor sector. As we've seen, the value of the parameter λ affects the characteristics of the solution set. This parameter represents the rate at which new material is deposited on top of the epitaxially developing solid, and its physical meaning is obvious. When λ is small enough, the suggested method yields two stationary solutions for any constant or variable deposition rate $G(z)$. In view of above discussion, for $\alpha = 1$ and $T = 1$, we conclude that lower solution of time dependent problem (8) should be stable whereas the upper solution should be unstable. We draw the conclusion that the numerical and theoretical results support one another. Moreover, here we have successfully extended the literature by developing an iterative method that is capable to determine two solutions of a class of higher order SBVPs. We point out that researcher may try to develop a suitable discrete method to find the numerical approximations of the governing SBVP and the investigations are still open. Lastly, we believe that the proposed method is powerful, promising, and reliable. We still expect it can be applied to higher-order SBVP which has dual solutions. Finally, we conclude that the current work provides a valuable contribution, and the researcher may consider the proposed problem to explore the theoretical and numerical results.

Author Contributions

The authors confirmed the contribution to this article as follows: Conceptualization, B. Pandit and R.P. Agarwal; methodology, B. Pandit and P. Manini; software, B. Pandit, P. Manini and B.R. Priyanka; validation, B. Pandit and P. Manini and B.R. Priyanka; formal analysis, B. Pandit and R.P. Agarwal; investigation, P. Manini and B.R. Priyanka; resources, P. Manini and B.R. Priyanka; data curation, B. Pandit and P. Manini; writing original draft preparation, P. Manini and B.R. Priyanka; writing review and editing, B. Pandit and P. Manini; supervision, R.P. Agarwal. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Acknowledgments

We are thankful to all the reviewers and Editor for their valuable suggestions to improve the quality of the work.

Conflict of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

Funding

The authors received no financial support for the research, authorship, and publication of this article.

Data Availability Statements

Not Applicable.

Nomenclature

BVP	Boundary Value Problem	SPDE	Stochastic Partial Differential Equation
VIM	Variational Iteration Method	SBVP	Singular Boundary Value Problem

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How to cite this article: Manini P., Pandit B., Priyanka B.R., Agarwal R.P. Higher-Order Singular BVP Emerging in the Semiconductor Industry: Analysis and Computation of Solutions, *J. Appl. Comput. Mech.*, 11(3), 2025, 568-584. https://doi.org/10.22055/jacm.2024.47592.4747