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Research Paper

# Numerical Approximation for a Dynamical System of the SIR System

Jafar Ahmadi Shali[1], Shahbaz Mustafa[2]

[1] Faculty of Mathematics, Statistics and Computer Science, University of Tabriz, Tabriz, Iran, Email: J_ahmadishali@tabrizu.ac.ir

[2] Department of Mathematics, Institute of Numerical Sciences, Gomal University, Dera Ismail Khan 29050, KPK, Pakistan, Email: shahbazmustafa5545@gmail.com

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Corresponding author: J. Ahmadi Shali (J_ahmadishali@tabrizu.ac.ir)


**Abstract.** In this paper, we are going to approximate the dynamical system of the SIR mathematical model, numerically. The expression for basic reproduction number $R_0$ is obtained which plays main role in the local and global stability. On the other hand, if $R_0 > 1$ then the equilibrium point is unstable. Our numerical simulations show that our proposed NSFD method provides reliable and consistent, positive and converging results at all step sizes as compared to the traditional RK-4 method and Euler method which diverge at large step sizes and produce negative and unstable results with large oscillations. The local stability of the equilibrium point is proved by Jacobean method; however, Schur-Cohn conditions are used to discuss the local stability for the discrete NSFD scheme. To prove the global stability of equilibria, the Enatsu criterion and Lyapunov function are used. This paper presents theoretical findings and numerical simulations that can be used as a powerful instrument for forecasting the spread of added infectious illnesses.

**Keywords:** Stability analysis, Dynamical system, Global warming, Basic reproduction number.

## 1. Introduction

The mathematical modeling is a helpful tool to distillate on the technique that how a desirable issue can reach out into a population. The dissimilar kinds of recovery and incidence rates play an important role to examine the dynamical system of mathematical model of the health. TB is an issue which cause a high rate of mortality. Studying of the dynamical system of TB is a very interested topics in applied mathematics, because of its structure, equilibrium points, stability properties and etc. World Health Organization (WHO) (2009) states that third of the world rom worldwide. Also, 95% of TB cases from worldwide occur in developing countries. Five of 22 countries with the highest TB cases are located in Southeast Asia, where 35% of all TB cases in the world come from this region. Given these conditions, the WHO declared TB as a global emergency since 2013.

So, the study of the mathematical behavior and also the dynamical system of TB is very important and interested problems in mathematics [1-5]. TB case increased during the years of 2013 to 2015. The case number increased up to 34 percent in India. Globally, there were 4.3 million cases at which the cases happened in India, Indonesia, and Nigeria were counted nearly half of them [6-8]. According to World Health Organization (WHO) 2015, Indonesia ranks second in the number of TB cases after India [9-15]. Most TB patients need to take TB medication for at least 6 months in order to be cured. TB spread through the air in droplets when a person with pulmonary TB coughs, sneezes, spits, laughs, or talks. Only people with active TB can transmit the issue. Most frequently, TB affects our lungs. Coughing, chest pain, shortness of breath, loss of appetite, weight loss, a fever, a cold, and fatigue are all signs of TB [16, 17]. In [8], the authors have constructed the mathematical model of the transmission infections. The consistency of the endemic and infections-free equilibrium for the given model was one of the basic analytical findings that the author examined. The purpose of this paper is to develop multiple schemes using the Euler and Runge–Kutta methods. To demonstrate the model's sustainability and biological viability, an advanced non-standard finite difference (NSFD) scheme was used to verify various model properties. With the aid of step size, the NSFD scheme construct for the model exhibits dynamic consistency. All the computer simulations are the strong evidences that the proposed NSFD scheme is a reliable tool for solving the compartmental systems of infectious infections. So, we can conclude that the proposed NSFD scheme remains convergent for all step sizes and preserves all the essential properties of the continuous model. The NSFD scheme is a numerical method used to solve differential equations, particularly those with non-linear or non- local term [19-21].

This paper is arranged as follows: In infectious modeling, the function that describes the incidence rate is a main factor that determines the dynamics of the model. In Section 2, the SIR mathematical model is provided, and the related parameter is discussed. We calculated the basic reproduction which plays an important role in the investigation of local and global stability of DFE and DEE points. We can use the Jacobean matrix method for the infection free equilibrium (DFE) to find out the local and global stability analysis. In Section 3, the Euler scheme is construct of the given model. In Section 4, RK-4 scheme is developed in our model. In Section 6, the construction of NSFD scheme which is independent of step size ($h$). Our calculations demonstrate that the NSFD scheme outperforms the Euler and RK-4 schemes in terms of accuracy and step size independence. Conversely, Scour-Cohn

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situations are not used to deliberate the homegrown permanence of the widespread symmetry ideas on behalf of the distinct NSFD method. However, monotonic sequences are designed to elaborate the global stability of both the equilibria for the NSFD scheme. The NSFD scheme is shown to be convergent regardless of step size. The numerical simulations are also drawn at each steep which indicate that the discrete NSFD scheme keeps all dynamical properties of the continuous model. Additionally, offered are the numerical simulations, which support our theoretical findings. The final section offers a succinct conclusion.

## 2. Mathematical Model of Dynamical System

A SIR (Susceptible-Infected-Recovered) epidemic splits the entire into three subpopulations to represent how the infection spreads. In other words, the individual sub-population is treated, as are the helpless different people and the sick sub-population.

The SIR ailment organism [8] of non-linear regular discrepancy equalities can be summarized as follows from Fig. 1.

Below is a list of the following differential equations:

$$S' = \frac{ds}{dt} = \alpha - V\frac{1}{N}S - \Delta s \tag{1}$$

$$I' = \frac{dI}{dt} = V\frac{1}{N}S - (\delta + \delta_t + \theta + e)I \tag{2}$$

$$R' = \frac{dR}{dt} = (\theta + e)I - \Delta r \tag{3}$$

With initial conditions $S(0) > 0$, $I(0) \geq 0$ and $R(0) \geq 0$.

## 3. Limitations

The following are explanations of the limitations charity in system (1-3).

$\alpha$ = Number of births in the inhabitants;

$V$ = Incidence of transmission of infection;

$\delta$ = Number of deaths from other factors that cause;

$\delta_t$ = Rate of infectious deaths;

$\theta$ = Size of the cure;

$e$ = Therapy usage rate.

Suppose that each of the parameters $\alpha, v, \delta, \delta_t, \theta, e$ are positive. In the given model, the third equation is independent of the first two equations (1-3). The objective of the subsequent discussion will be the following reduced model because the first two equations are sufficient:

$$S' = \frac{dS}{dt} = \alpha - V\frac{1}{N}S - \delta S \tag{4}$$

$$I' = \frac{dI}{dt} = V\frac{1}{N}S - (\delta + \delta_t + \theta + e)I \tag{5}$$

### 3.1. Steady state of model (1-3) and its stability

We discuss about two equilibrium points of system (1-3).

i. Illness Free Equilibrium (DFE) Point $\mathrm{E}^0(\mathrm{S}^0, \mathrm{I}^0, \mathrm{R}^0) = \frac{\alpha}{\delta}$.

ii. Endemic Equilibrium (EE) Points:

$$E^*(S^*, I^*, R^*) = \frac{\alpha(\delta + \theta + e)}{\delta(v - \delta_t)}, \frac{\alpha(v - \delta - \delta_t - \theta - e)}{\delta + \delta_t + \theta + e)(v - \delta_t)}, \frac{\alpha(\theta + e)(v - \delta - \delta_t - \theta - e)}{\delta(\delta + \delta_t + \theta + e)(v - \delta_t)} \tag{6}$$

### 3.2. Basic reproduction number ($R_0$)

The average number of secondary infections a contagious person causes over the course of their entire period is the basic reproduction number. The basic reproduction number is a crucial non-dimensional quantity in epidemiology because it establishes the cutoff for a sickness study, allowing for both the prediction of an outbreak and the evaluation of control measures [22, 23].

If $\mathrm{R}_0 < 1$ during its communicable date, so contamination does not raise. If $\mathrm{R}_0 > 1$, so, illness grows in the whole world, spreading out of infection. Using the translation matrix $V(x)$ and the transmission matrix $F(x)$, we can determine $(\mathrm{R}_0)$. The matrix expressions are as follows:

$$F(\theta) = \begin{bmatrix} 0 \\ \alpha\frac{I}{N} \end{bmatrix}, V(\theta) = \begin{bmatrix} 0 \\ \delta + \delta_t + \theta + e \end{bmatrix} \tag{7}$$

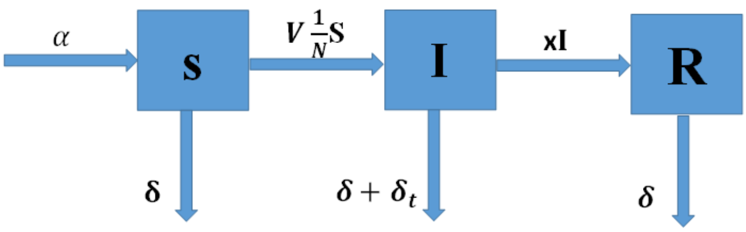


**Fig. 1.** The comprehensive explanation of prevalent typical for TB ailment communication.

**Table 1.** Fixed values of parameters used in system (1-3).

| V | $\alpha$ | $\delta$ | $\delta_t$ | $\theta$ | E | N |
|---|---|---|---|---|---|---|
| 5.6 | 1.5 | 0.5 | 8.89 | 2.9 | 0.02 | 1 |

The quantity of threshold ratios $(R_0)$ is given as:

$$R_0 = \frac{v\alpha}{\delta(\delta + \delta_t + \ \theta + e)} \tag{8}$$

## 4. Forward Euler Finite Scheme

First order differential equations can be solved using the Euler scheme, which is purely numerical. In order to solve the given SIR model, which depends on differential equations, Euler's method is more appropriate [14]. The description of Euler method is given below:

$$\frac{S_{n+1} - S_n}{h} = \alpha - V\frac{1}{N}S_n - \delta S \tag{9}$$

$$\frac{I_{n+1} - I_n}{h} = V\frac{1}{N}S_n - (\delta + \delta_t + \ \theta + e)I_n \tag{10}$$

$$\frac{R_{n+1} - R_n}{h} = (\ \theta + e)I_n - \delta R_n. \tag{11}$$

Finally, we get the Euler scheme:

$$S_{n+1} = S_n + h\left(\alpha - V\frac{1}{N}S_n - \delta S\right) \tag{12}$$

$$I_{n+1} = I_n + h\left(V\frac{1}{N}S_n - (\delta + \delta_t + \ \theta + e)I_n\right) \tag{13}$$

$$R_{n+1} = R_n + h((\ \theta + e)I_n - \delta R_n) \tag{14}$$

It is clear that the Euler's approach for DFE produced positive numerical results. Figure 2 illustrates how the stability of the DFE for the continuous model (1-3) is affected when the step size raises. Thus, we draw the conclusion that the stability and positivity of the Euler scheme cannot be maintained for any high step sizes.

## 5. Runge-Kutta Order 4 (RK-4)

Because it is precise and easy to use, the Runge-Kutta method is a common technique for solving ordinary differential equations (ODEs). According to Roberts [15], not all function derivatives are simple to calculate, which is why the Taylor series method requires them to be find out. Let $S = A_1$, $I = B_1$, and $R = C_1$, then represent the "RK-4," scheme as:

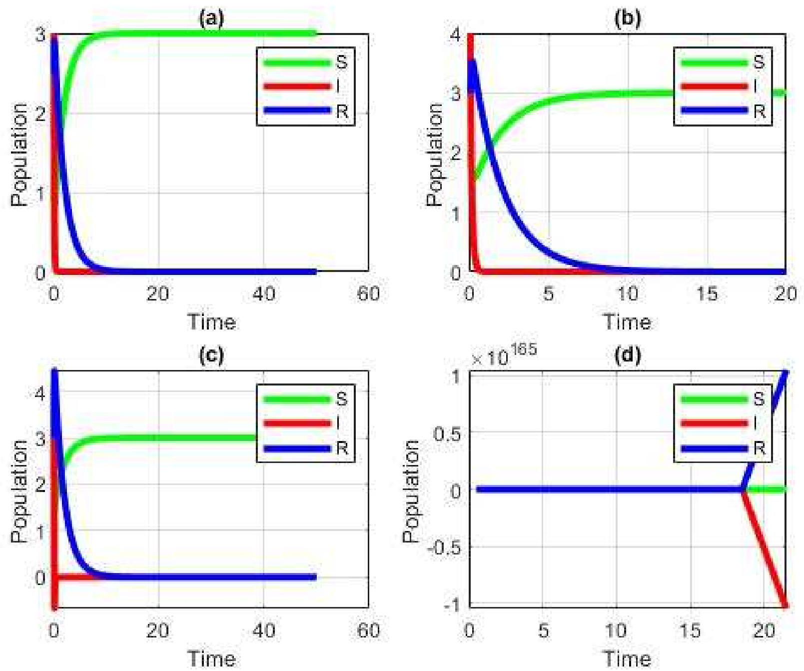


**Fig. 2.** Mathematical results of SIR epidemic system (1-3) gained by Euler pattern through (a) h = 0.005, (b) h = 0.01, (c) h = 0.1, and (d) h = 0.001. All Table 1 lists the values for additional parameters.

Step-1

$$A_1 = h[\alpha - V\frac{1}{N}S_n - \delta S_n] \tag{15}$$

$$B_1 = h\left[V\frac{1}{N}S_n - (\delta + \delta_t + \ \theta + e)I_n\right] \tag{16}$$

$$C_1 = h[(\ \theta + e)I_n - \delta R_n] \tag{17}$$

Step-2

$$A_2 = h[\alpha - V\frac{1}{N}(S_n + \frac{A_1}{2}) - \delta(S_n + \frac{A_1}{2})] \tag{18}$$

$$B_2 = h[V\frac{1}{N}(S_n + \frac{A_1}{2}) - (\delta + \delta_t + \ \theta + e)(I_n + \frac{B_1}{2})] \tag{19}$$

$$C_1 = h[(\ \theta + e)(I_n + \frac{B_1}{2}) - \delta(R_n + \frac{C_1}{2})] \tag{20}$$

Step-3

$$A_2 = h[\alpha - V\frac{1}{N}(S_n + \frac{A_2}{2}) - \delta(S_n + \frac{A_2}{2})] \tag{21}$$

$$B_2 = h[V\frac{1}{N}(S_n + \frac{A_2}{2}) - (\delta + \delta_t + \ \theta + e)(I_n + \frac{B_2}{2})] \tag{22}$$

$$C_1 = h[(\ \theta + e)(I_n + \frac{B_2}{2}) - \delta(R_n + \frac{C_2}{2})] \tag{23}$$

Step-4

$$A_2 = h[\alpha - V\frac{1}{N}(S_n + A_3) - \delta(S_n + A_3)] \tag{24}$$

$$B_2 = h[V\frac{1}{N}(S_n + A_3) - (\delta + \delta_t + \ \theta + e)(I_n + B_3)] \tag{25}$$

$$C_1 = h[(\ \theta + e)(I_n + B_3) - \delta(R_n + C_3)] \tag{26}$$

Finally step,

$$S_{n+1} = S_n + \frac{1}{6}[A_1 + 2A_2 + 2A_3 + A_4] \tag{27}$$

$$I_{n+1} = I_n + \frac{1}{6}[B_1 + 2B_2 + 2B_3 + B_4] \tag{28}$$

$$R_{n+1} = R_n + \frac{1}{6}[C_1 + 2C_2 + 2C_3 + C_4]. \tag{29}$$

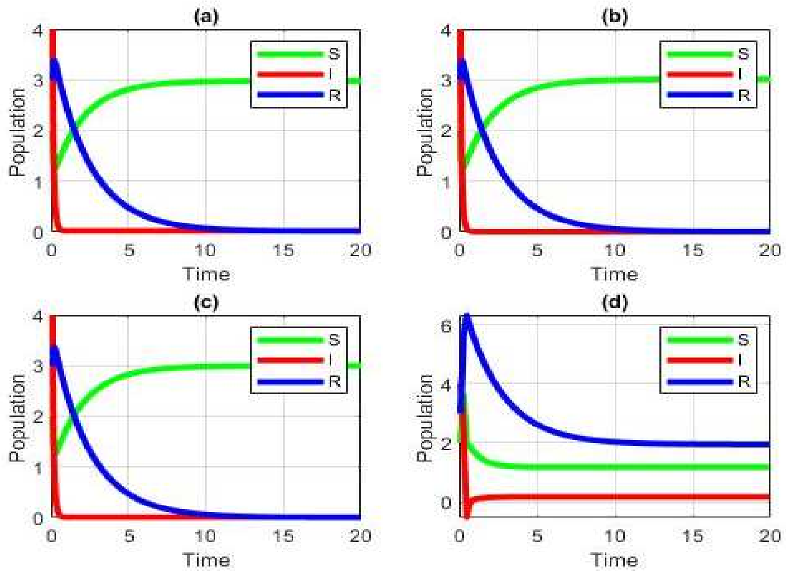


**Fig. 3.** Mathematical results of SIR epidemic system (1-3) gained by Rk-4 structure through (a) h = 0.03, (b) h = 0.01, (c) h = 0.02, and (d) h = 0.2. Table 1 contains the values for every other parameter.

Figure 3 provides a graphic illustration of the"RK-4," system (a-d). According to Fig. 3, the numerical outcomes for the "RK-4," scheme at small step size are favorable (a-c). According to Fig. 5, the stability of the DFE for system (1-3) is destroyed when the step size rises (d). The "RK-4," scheme is divergent for huge step sizes.

## 6. Non-Standard Finite Difference (NSFD)

### 6.1. Construction of the NSFD scheme

Our primary goal in this section, is to suggest an NSFD scheme for this model. Mickens presents a novel finite difference method [16-18]. Their dynamical analysis demonstrates that, regardless of the step size, the NSFD scheme in [19-21] maintains the stability assets of all prevailing equilibrium points. It is possible to construct the NSFD scheme for the continuous dynamical system (1-3) as shown below:

$$\frac{S^{n+1}-S^{n}}{h}=\alpha-\frac{V}{N}S^{n+1}-\delta S^{n+1} \tag{30}$$

$$\frac{I^{n+1}-I^{n}}{h}=\frac{V}{N}S^{n}-(\delta+\delta_t+\ \theta+e)I^{n+1} \tag{31}$$

$$\frac{R^{n+1}-R^{n}}{h}=(\ \theta+e)I^{n}-\delta R^{n+1} \tag{32}$$

The estimated conditions $S_0\geq 0, I_0\geq 0$ and $R_0\geq 0$ of the distinct NSFD, SIR widespread system is likewise expected to be nonnegative. To obtain explicit form, the distinct NSFD above system can be reorganized:

$$S^{n+1}=\frac{S^{n}+h\alpha}{1+h(\frac{V}{N}+\delta)} \tag{33}$$

$$I^{n+1}=\frac{I^{n}+h\frac{V}{N}\delta^{n}}{1+h(\delta+\delta_t+\ \theta+e)} \tag{34}$$

$$R^{n+1}=\frac{R^{n}+h(\ \theta+e)I^{n}}{1+h\delta} \tag{35}$$

Meanwhile entirely parameters in above system are confident, it is proved that is $S^n\geq 0, I^n\geq 0$ and $R^n\geq 0$ aimed at all $n>0$ and in place of some $h>0$.

### 6.2. Local stability analysis for the discrete NSFD Scheme

We deliberate the following to demonstrate that DFE and DEE ideas are (LAS):

$$G_1=\frac{S^{n}+h\alpha}{1+h\left(\frac{V}{N}+\delta\right)} \tag{36}$$

$$G_2=\frac{I^{n}+h\frac{V}{N}\delta^{n}}{1+h(\delta+\delta_t+\ \theta+e)} \tag{37}$$

Refers to the global asymptotic stability (GAS) of the NSFD scheme over the entire domain, ensuring that the numerical solution remains bounded and converges to the exact solution or step size decreases. Concerned with the local asymptotic stability (LAS) of the NSFD scheme at a specific point or region, ensuring that the numerical solutions remain stable and accurate in particular area.

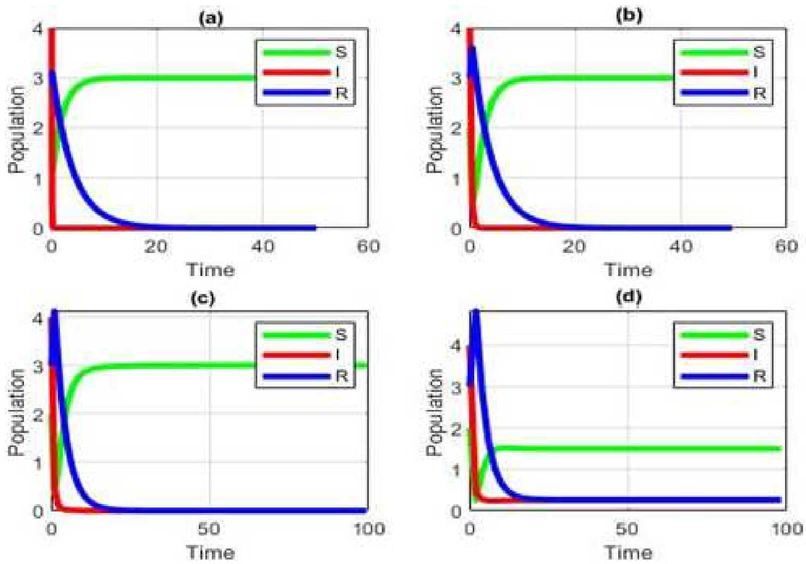


**Fig. 4.** Mathematical results of SIR epidemic system (1-3) obtained by NSFD structure through (a) h = 0.1, (b) h = 0.5, (c) h = 1, and (d) h = 2. All remaining parameter values are given Table 1.

**Theorem 1:** If $\mathrm{R}_0 < 1$, then DFE point $\mathrm{E}^0$ of the "NSFD," structure 3 is LAS for all $\mathrm{h} > 0$.
**Proof:** Let us yield a look at the Jacobian matrix:

$$J(E^0) = \begin{bmatrix} \frac{\partial G_1}{\partial S} & \frac{\partial G_1}{\partial I} \\ \frac{\partial G_2}{\partial S} & \frac{\partial G_2}{\partial I} \end{bmatrix} \tag{38}$$

$$\frac{\partial G_1}{\partial S} = \frac{1}{1 + h\left(\frac{V}{N} + \delta\right)}, \frac{\partial G_1}{\partial I} = 0, \frac{\partial G_2}{\partial S} = 0, \frac{\partial G_2}{\partial I} = \frac{1}{1 + h(\delta + \delta_t + \ \theta + e)}, \tag{39}$$

Putting the all values in Eq. (38):

$$\mathrm{J}(E^0) = \begin{pmatrix} \frac{1}{1 + h\left(\frac{V}{N} + \delta\right)} & 0 \\ 0 & \frac{1}{1 + h(\delta + \delta_t + \ \theta + e)} \end{pmatrix}$$

These eigenvalues are clearly shown by the aforementioned matrix:

$$\Lambda_1 = \frac{1}{1 + h(\frac{V}{N} + \delta)} < 1 \text{ and } \Lambda_2 = \frac{1}{1 + h(\delta + \delta_t + \ \theta + e)} \tag{40}$$

This verify that if $\mathrm{R}_0 < 1$, then the DFE point $E_0$ is locally asymptotically stable otherwise if $\mathrm{R}_0 > 1$.

In the next target, we provide the speech of the Scour-Cohn principle [26, 27], which shows a significant part in the investigation of local stability of DEE point $E^*$.

**Lemma 1:** The solutions of equality $\lambda^2 - \mathrm{T}\lambda + \mathrm{D} = 0$ fulfill $|\lambda_{\mathrm{m}}| < 1$, $\mathrm{m} = 1{,}2$ is unknown quantity and only if the situations itemized below are not seen:

1. $\mathrm{D} < 1$,
2. $1 + \mathrm{D} + \mathrm{T} > 0$,
3. $1 - \mathrm{T} + \mathrm{D} > 0$.

where D and T viewpoint for the Jacobean matrix's determinant and trace, respectively.

**Theorem 2:** If $\mathrm{R}_0 > 1$, then DFE point $E^*$ of the "NSFD," structure is LAS for all $\mathrm{h} > 0$.

**Proof:** Let us consider the Jacobian matrix's:

$$J(E^*) = \begin{bmatrix} \frac{\partial G_1}{\partial S} & \frac{\partial G_1}{\partial I} \\ \frac{\partial G_2}{\partial S} & \frac{\partial G_2}{\partial I} \end{bmatrix} \tag{41}$$

By replacing $\mathrm{G}_1$ and $\mathrm{G}_2$, and then laying DEE point $E^*$ in Eq. (41), it becomes:

$$J(E^*) = \begin{bmatrix} \frac{1}{b} & -\frac{z}{b^2} \\ \frac{t}{l} & \frac{1+z}{l} \end{bmatrix} \tag{42}$$

where $b = 1 + \mathrm{h}(K^* + \delta) > 1$, $q = 1 + \mathrm{h}(\delta + \delta_{\mathrm{t}} + \ \theta + \mathrm{e}) > 1$,$t = h(K^*)$ and $z = \frac{hc(S^* + hg)}{n + I^*}$. Here $K^* = \frac{\mathrm{V}}{\mathrm{N}}$.

The eigenvalues of $\mathrm{J}(E^*)$ is conclude by the previous results:

$$\lambda^2 - T\lambda + D = 0, \tag{43}$$

$T$ = Trace$\left(J(E^*)\right)$= $\frac{1}{b} + \frac{1+z}{l} > 0$ and $D = \frac{b(1+z) + tz}{b^2 l} > 0$. The outcomes that follow are as follows:

1. Due to $b > 1$, we have:

$$D = \frac{b(1+z) + tz}{b^2 l} < \frac{1+z}{bl} + \frac{tz}{b^2 l}, \quad D < 1. \tag{44}$$

2. It is clear that $1 + T + D > 0$.
3. In the simplest scheming, it is shown that if $R_0 > 1$, now:

$$1 - T + D = 1 - 1 + h(K^* + \delta)(1 + h(\delta + \delta_t + \ \theta + e)(1 + \frac{hx}{S^*})[1 - R_0] > 0. \tag{45}$$

If $R_0 > 1$ for any step size $\mathrm{h}$, then the DFE point $E^*$ is locally asymptotically stable according to the Schur-Cohn criterion deliberated in Lemma 1.

### 6.3. Global stability analysis of the discrete NSFD system

Now, using the proper Lyapunov function, we can examine the global stability of illness-free and infection-endemic equilibrium.

**Theorem 3:** If $\mathrm{R}_0 \le 1$, then the (DFE) idea $\mathrm{E}_0$ of the distinct NSFD scheme is globally asymptotically stable.
**Proof:** Let's take a look at the following discrete Lyapunov function:

$$U_n = \frac{1}{h}\left[S_0 p\left(\frac{S_n}{S_0}\right) + I_n\right] + S_0 \Psi_n, \tag{46}$$

where $p(y) = y - 1 - \ln y \geq p(1) = 0$. Using the system's first and second equations, we were able to derive Eq. (46). Now:

$$\Delta U_n = U_{n+1} - U_n \tag{47}$$

$$\begin{aligned}\Delta U_n &= \frac{1}{\phi}\left[S_0\left(\frac{S_{n+1}}{S_0} - 1 - ln\left(\frac{S_{n+1}}{S_0}\right)\right) - \left(\frac{S_n}{S_0} - 1 - ln\left(\frac{S_n}{S_0}\right)\right) + (I_{n+1} - I_n)\right] + S_0(\Psi_{n+1} - \Psi_n)\\ &= \frac{1}{h}\left[S_0\left(\left(\frac{S_{n+1} - S_n}{S_0}\right) - \left(ln\left(\frac{S_{n+1}}{S_0}\right) - ln\left(\frac{S_n}{S_0}\right)\right)\right) + (I_{n+1} - I_n)\right] + S_0(\Psi_{n+1} - \Psi_n)\end{aligned} \tag{48}$$

In consuming Entasu et al.'s criterion [26], $\ln\frac{S_2}{S_1} \geq \frac{S_2 - S_1}{S_2}$, we get:

$$\begin{aligned}&\leq \frac{1}{h}\left[(S_{n+1} - S_n) - S_0\left(\frac{S_{n+1} - S_n}{S_{n+1}}\right) + (I_{n+1} - I_n)\right] + S_0(\Psi_{n+1} - \Psi_n)\\ &= \left[\left(\frac{S_{n+1} - S_0}{S_{n+1}}\right)\frac{(S_{n+1} - S_n)}{h} + \frac{(I_{n+1} - I_n)}{h}\right] + S_0(\Psi_{n+1} - \Psi_n)\\ &= \frac{1}{S_{n+1}}\left[(S_{n+1} - S_0)\frac{(S_{n+1} - S_n)}{h} + \frac{(I_{n+1} - I_n)}{h}\right] + S_0(\Psi_{n+1} - \Psi_n)\\ &= \frac{1}{S_{n+1}}\left[(S_{n+1} - S_0)\alpha - \frac{V}{N}S^{n+1} - \delta S^{n+1} - \frac{V}{N}S^n - (\delta + \delta_t + \theta + e)I^{n+1}\right] + S_0(\Psi_{n+1} - \Psi_n)\\ &\leq \frac{V}{NS_{n+1}}\left[(S_{n+1} - S_0)\left(\frac{\alpha V}{N} - S_{n+1}\right) - (\delta + \delta_t + \theta + e)I^{n+1}\right] + S_0\Psi_{n+1} - S_0\Psi_n\\ &= -\frac{\alpha V}{NS_{n+1}}(S_{n+1} - S_0)^2 - (\frac{V}{N} - \delta)\left(1 - \frac{v\alpha}{\delta(\delta + \delta_t + \theta + e)}\right)I_{n+1}\\ &= -\frac{V\alpha}{NS_{n+1}}(S_{n+1} - S_0)^2 - ((\frac{V}{N} - \delta))(1 - R_0)I_{n+1}\end{aligned} \tag{49}$$

Certainly, if $R_0 \leq 1$, then $U_{n+1} - U_n \leq 0$ for all $n \geq 0$. This shows that the decreasing sequences for $U_n$ are monotonic. By way of $U_n \geq 0$, thus $\lim_{n\to\infty} U_n \geq 0$ and $\lim_{n\to\infty} U_{n+1} - U_n = 0$. Consequently, we get $\lim_{n\to\infty} S_{n+1} = S_0$ and $\lim_{n\to\infty} (\delta + \delta_t)I_{n+1} = 0$. It is clear that if $R_0 \leq 1$, then $\lim_{n\to\infty} I_{n+1} = 0$. Thus, we draw the conclusion that $E_0$ is asymptotically stable everywhere [27].

**Theorem 4:** If $R_0 > 1$, then DEE idea $E^*$ of the "NSFD" system is GAS on behalf of some phase size $h$, as shown in Fig. 1. and Fig. 4.

**Proof:** We apply Enatsu et al.'s criterion [28] to extant necessary situations aimed at the global stability of $E^*$. For this, the distinct Lyapunov function is defined as follows:

$$Z_n = S^* g\left(\frac{S_n}{S^*}\right) + I^* g\left(\frac{I_n}{I^*}\right) + h\Psi^* S^* g\left(\frac{\Psi_n}{\Psi^*}\right). \tag{50}$$

where $\Psi^* = \Psi(I^*)$ and $w(y) = y - 1 - \ln(y) \geq p(1) = 0$.

Equation (12) is created by substituting the first and second equations of Eq. (4):

$$\begin{aligned}\Delta Z_n &= Z_{n+1} - Z_n = S^*\left[p\left(\frac{S_{n+1}}{S^*}\right) - p\left(\frac{S_n}{S^*}\right)\right] + I^*\left[p\left(\frac{I_{n+1}}{I^*}\right)n - p\left(\frac{I_n}{I^*}\right)\right] + h\Psi^* S^*\left[p\left(\frac{\Psi_{n+1}}{\Psi^*}\right) - p\left(\frac{\Psi_n}{\Psi^*}\right)\right]\\ &= \left[(S_{n+1} - S_n) - S^* ln\left(\frac{S_{n+1}}{S^*}\right)\right] + \left[(I_{n+1} - I_n) - I^* ln\left(\frac{I_{n+1}}{I^*}\right)\right] + h\Psi^* S^*\left[\left(\frac{\Psi_{n+1}}{\Psi^*} - \frac{\Psi_n}{\Psi^*}\right) + ln\left(\frac{\Psi_n}{\Psi^*}\right) - ln\left(\frac{\Psi_{n+1}}{\Psi^*}\right)\right]\\ &= \left[S^*\left(\frac{S_{n+1} - S_n}{S^*}\right) - S^* ln\left(\frac{S_{n+1}}{S^*}\right)\right] + \left[I^*\left(\frac{I_{n+1} - I_n}{I^*}\right) - I^* ln\left(\frac{I_{n+1}}{I^*}\right)\right] + h\Psi^* S^*\left[\left(\frac{\Psi_{n+1}}{\Psi^*} - \frac{\Psi_n}{\Psi^*}\right) + ln\frac{\Psi_n}{\Psi^*} - ln\frac{\Psi_{n+1}}{\Psi^*}\ )\right].\\ &\leq \frac{1}{S_{n+1}}(S_{n+1} - S^*)(S_{n+1} - S_n) + \frac{1}{I_{n+1}}(I_{n+1} - I^*)(I_{n+1} - I_n) + h\Psi^* S^*\left[\left(\frac{\Psi_{n+1}}{\Psi^*} - \frac{\Psi_n}{\Psi^*}\right) + ln\left(\frac{\Psi_n}{\Psi^*}\right) - ln\left(\frac{\Psi_{n+1}}{\Psi^*}\right)\right]\\ &= \frac{h}{S_{n+1}}(S_{n+1} - S^*)\frac{(S_{n+1} - S_n)}{h} + \frac{1}{I_{n+1}}(I_{n+1} - I^*)\frac{(I_{n+1} - I_n)}{h} + h\Psi^* S^*\left[\left(\frac{\Psi_{n+1}}{\Psi^*} - \frac{\Psi_n}{\Psi^*}\right) + ln\left(\frac{\Psi_n}{\Psi^*}\right) - ln\left(\frac{\Psi_{n+1}}{\Psi^*}\right)\right]\\ &= \frac{h}{S_{n+1}}(S_{n+1} - S^*\ (\alpha - \frac{V}{N}S^{n+1} - \delta S^{n+1}) + \frac{1}{I_{n+1}}(I_{n+1} - I^*)\frac{V}{N}S^n - (\delta + \delta_t + \theta + e)I^{n+1}\\ &\quad + h\Psi^* S^*\left[\left(\frac{\Psi_{n+1}}{\Psi^*} - \frac{\Psi_n}{\Psi^*}\right) + ln\left(\frac{\Psi_n}{\Psi^*}\right) - ln\left(\frac{\Psi_{n+1}}{\Psi^*}\right)\right]\\ &= \frac{\delta h}{S_{n+1}}(S_{n+1} - S^*)^2 + h\Psi^* S^*\left(1 - \frac{S^*}{S_{n+1}}\right)\left(1 - \frac{\Psi_n S_{n+1}}{\Psi^* S^*}\right) + h\Psi^* S^*\left(1 - \frac{I^*}{I_{n+1}}\right)\left(\frac{\Psi_n S_{n+1}}{\Psi^* S^*} - \frac{I_{n+1}}{I^*}\right) + h\Psi^* S^*\left[\left(\frac{\Psi_{n+1}}{\Psi^*} - \frac{\Psi_n}{\Psi^*}\right) + ln\left(\frac{\Psi_n}{\Psi^*}\right) - ln\left(\frac{\Psi_{n+1}}{\Psi^*}\right)\right].\end{aligned} \tag{51a}$$

Let us denote $\Delta_n = \frac{S_n}{S^*}$, $\Gamma_n = \frac{I_n}{I^*}$ and $\theta_n = \frac{\Psi_n}{\Psi^*}$, then Eq. (51a) can be expressed as:

$$\begin{aligned}\Delta Z_n &\leq \frac{-\delta h S^*}{\Gamma_{n+1}}(\Delta_{n+1} - 1)^2 + h\Psi^* S^*\left(1 - \frac{1}{\Delta_{n+1}}\right)(1 - \theta_n\Delta_{n+1}) + h\Psi^* S^*\left(1 - \frac{1}{\eta_{n+1}}\right)(\theta_n\Delta_{n+1} - \Gamma_{n+1})\\ &\quad + \ h\Psi^* S^*[(\theta_{n+1} - \theta_n) - (ln\,\theta_{n+1} - ln\theta_n\ )\ ]\\ &= \frac{-\delta h S^*}{\Gamma_{n+1}}(\Delta_{n+1} - 1)^2 - p(\frac{1}{\Gamma_{n+1}}) - p(\frac{\Delta_{n+1}\Gamma_n}{\Delta_{n+1}}) + h\Psi^* S^*\big(p(\theta_{n+1}) - p(\Gamma_{n+1})\big).\end{aligned} \tag{51b}$$

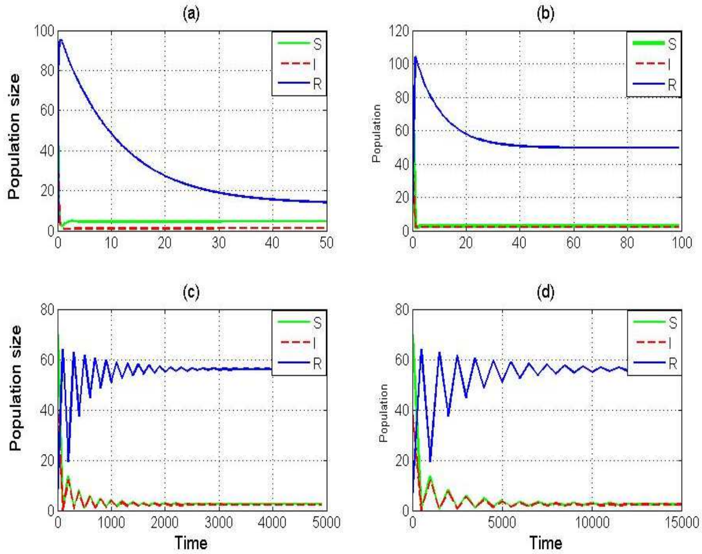


**Fig. 5.** Mathematical results of SIR system (1-3) achieved from NSFD structure through (a) h = 0.5, (b) h = 1, (c) h = 50, (d) h = 500.

Using the definition of $p(z)$, we get:

$$\begin{aligned} p(\theta_{n+1}) - p(\Gamma_{n+1}) &= \frac{\Psi_{n+1}}{\Psi^*} - \frac{I_{n+1}}{I^*} + ln\left(\frac{I_{n+1}}{I^*}\frac{\Psi^*}{\Psi_{n+1}}\right) \\ &\leq \frac{\Psi_{n+1}}{\Psi^*} - \frac{I_{n+1}}{I^*} + \frac{I_{n+1}}{I^*}\frac{\Psi^*}{\Psi_{n+1}} - 1 \\ &= \frac{-\delta\ (I_{n+1} - I^*)^2}{I^*(1+\alpha I^*)(1+\alpha I_{n+1})} \\ &\leq 0. \end{aligned} \tag{52}$$

So, $\Delta Z_n$ is a monotonic decreasing pattern as a result (see Fig. 5). We can demonstrate that $\lim_{n\to\infty}(Z_{n+1} - Z_n) = 0$ by using the same methods as in Theorem 3. As a result, for all $h$, we obtain $\lim_{n\to\infty} S_{n+1} = S^*$ and $\lim_{n\to\infty} I_{n+1} = I^*$. Thus, the proof is conclusive.

## 7. Conclusion

We determined the fundamental reproduction, which is crucial in examining the local and global stability of DFE and DEE points. Less error in convergence is achieved with the Euler and RK-4 scheme. The model is built using the NSFD scheme, which somewhat generates but is unconditionally convergent extra results that make sense both mathematically and biologically. The outcomes demonstrate that the NSFD scheme overcome all the drawbacks of the RK-4 scheme as well as present reliable numerical solutions. In contrast to the NSFD scheme, which produced stable, positive, and convergent results at all time steps because it is independent of time steps, forward Euler and RK-4 methods failed to maintain the positivity property and produced negative results that caused numerical instabilities and diverges at small step sizes. The RK-4 scheme can produce numerical solutions that are somewhat different from the original model's solutions because it is unable to precisely maintain the fundamental characteristic of the original continuous models. The LAS and GAS of both DFE and DEE points are shown to be earned under different conditions and criteria for the NSFD scheme. The benefits of the NSFD scheme are listed, which explains why its results are efficient and precise in terms of quality. The NSFD can be created and examined in similar ways for extra generalized widespread models. In our future study, we intend to investigate additional generalized epidemic models with characteristics comparable to the one under consideration in order to gain a deeper understanding of disease transmission dynamics.

## Author Contributions

J. Ahmadi Shali planned the scheme, initiated the project, developed the mathematical modeling; S. Mustafa analyzed the simulated results. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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## Conflict of Interest

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## Data Availability Statements

All data generated or analyzed during this study are included in the published version.

## References

[1] Colijn, C., Cohen, T.E.D., Murray, M., Mathematical models of tb: accomplishments and future challenges, *BIOMAT*, 2007, 123-148.
[2] Olinky, R., Huppert, A., Stone, L., Seasonal dynamics and thresholds governing recurrent epidemics, *Journal of Mathematical Biology*, 56, 2008, 827-839.
[3] Kamoh, N.M., Dang, B.C., Sunday, J., A numerical block hybrid algorithm for solving systems of first-order initial value problems, *Afrika Matematika*, 35(4), 2024, 72.
[4] World Health Organization, Global TB report 2013. World Health Organization, 2013.
[5] Fredlina, K.Q., Oka, T.B., Dwipayana, I.M., Susceptible, infectious, recovered model for tuberculosis disease transmission, *Journal of Matematika*, 1(1), 2012, 52-58.
[6] Side, S., A susceptible-infected-recovered model and simulation for transmission of tuberculosis, *Advanced Science Letters*, 21(2), 2015, 137-139.
[7] Nasution, H., Marlina, S., Mathematical model susceptible, infected and recovered with therapy of TB transmission, *Journal of Physics*, 1462(1), 2020, 120-156.
[8] Driessche, V.D., Pauline, Watmough, J., Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Mathematical Biosciences*, 180(1-2), 2002, 29-48.
[9] Odo, D., Heesterbeek, J.A.P., Anton, J., Metz, J., On the definition and the computation of the basic reproduction ratio $R_0$ in models for infectious diseases in heterogeneous populations, *Journal of Mathematical Biology*, 28, 1990, 365-382.
[10] Akanbi, M.A., Third order Euler method for numerical solution of ordinary differential equations, *ARPN Journal Engineering and Applied Sciences*, 5, 2010, 42-49.
[11] Ramos, H., Singh, G., Kanwar, V., Bhatia, S., Solving first-order initial-value problems by using an explicit non-standard A-stable one-step method in variable step-size formulation, *Applied Mathematics and Computation*, 268, 2015, 796-805.
[12] Mickens, R.E., *Applications of nonstandard finite difference schemes*, World Scientific, 2000.
[13] He, C.H., Liu, C., He, J.H., Sedighi, H.M., Shokri, A., Gepreel, K.A., A fractal model for the internal temperature response of a porous concrete, *Applied and Computational Mathematics*, 21(1), 2022, 71-77.
[14] Mickens, R.E., Nonstandard finite difference schemes for differential equations, *Journal of Difference Equations and Applications*, 8(9), 2002, 823-847.
[15] Hattaf, Kh., Lashari, A.A., Boukari, B.E., Yousfi, N., Effect of discretization on dynamical behavior in an epidemiological model, *Differential Equations and Dynamical Systems*, 23, 2015, 403-413.
[16] Darti, I., Suryanto, A., Stability preserving non-standard finite difference scheme for a harvesting Leslie–Gower predator–prey model, *Journal of Difference Equations and Applications*, 21(6), 2015, 528-534.
[17] Darti, I., Suryanto, A., Dynamics preserving nonstandard finite difference method for the modified Leslie-Gower predator-prey model with Holling-type II functional response, *Far East Journal of Mathematical Sciences*, 99(5), 2016, 719-733.
[18] He, J., Verma, L., Pandit, B., Verma, A.K., Agarwal, R.P., A new Taylor series based numerical method: simple, reliable, and promising, *Journal of Applied and Computational Mechanics*, 9(4), 2023, 1122-1134.
[19] Elaydi, S., J. Sacker, R.J., *Skew-product dynamical systems: Applications to difference equations*, In: Proceedings of the UAE Math Day, NOVA, 2006.
[20] Mehdizadeh Khalsaraei, M., Shokri, A., Ramos, H., Heydari, S., A positive and elementary stable nonstandard explicit scheme for a mathematical model of the influenza disease, *Mathematics and Computers in Simulation*, 182, 2021, 397-410.
[21] Enatsu, Y., Nakata, Y., Muroya, Y., Izzo, G., Vecchio, A., Global dynamics of difference equations for SIR epidemic models with a class of nonlinear incidence rates, *Journal of Difference Equations and Applications*, 18(7), 2012, 1163-1181.
[22] Chefnaj, N., Taqbibt, A., Hilal, K., Melliani, S., Kajouni, A., Boundary Value Problems Differential Equations Involving the Generalized Caputo-Fabrizio Fractional Derivative in λ Metric Space, *Turkish Journal of Science*, 8(1), 2023, 24-36.
[23] Singh, J., Kumar, D., Hammouch, Z., Atangana, A., A fractional epidemiological model for computer viruses pertaining to a new fractional derivative, *Applied Mathematics and Computation*, 316, 2018, 504-515.
[24] Bayrak, M.A., Demir A., On the challenge of identifying space dependent coefficient in space-time fractional diffusion equations by fractional scaling transformations method, *Turkish Journal of Science*, 7(2), 2022, 132-145.
[25] Dokuyucu, M.A., Analysis of a novel finance chaotic model via ABC fractional derivative, *Numerical Methods for Partial Differential Equations*, 37(2), 2021, 1583-1590.
[26] Ilhan, E., Analysis of the spread of Hookworm infection with Caputo-Fabrizio fractional derivative, *Turkish Journal of Science*, 7(1), 2022, 43-52.
[27] Steven, R., Sarshar, A., Sandu. A., Coupled multirate infinitesimal GARK schemes for stiff systems with multiple time scales, *SIAM Journal on Scientific Computing*, 42(3), 2020, A1609-A1638.
[28] Mehdizadeh Khalsaraei, M., Shokri, A., Ramos, H., Yao, S.W., Molayi, M., Efficient Numerical Solutions to a SIR Epidemic Model, *Mathematics*, 10(18), 2022, 3299.

## ORCID iD

Jafar Ahmadi Shali https://orcid.org/0000-0001-8090-7582
Shahbaz Mustafa https://orcid.org/0009-0005-6417-7221



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