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Research Paper

A Simple Reliable Hybrid Method Based on Taylor Series for Generalized Local Fractional Nonlinear Singular Differential Equations

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Abstract. This paper proposes a local fractional derivative of order α when $\alpha > 1$ ([1]). We also propose a simple method and refer to it as the hybrid Taylor series method, which solves a class of generalized local fractional systems of singular differential equations subject to both initial and boundary conditions. This can be used to compute solutions of even complex nonlinear generalized local fractional singular problems and hence is useful to check the validity of the other numerical methods for such problems involving local generalized fractional derivatives.

Keywords: Lane-Emden; Generalized Local Fractional Derivative; Taylor Series; Singular; Nonlinear.

1. Introduction

Fractional derivatives generalize classical derivative to non-integer orders and useful for modelling complex systems in various research areas. Teodoro et al. [2] very effectively summarised most of the existing fractional derivatives in their review paper. They discuss and classify the proposed non-integer order operators and restrict the discussion to the case where the order α is a real number and the order of the operator is fixed as follow.

(i) *Classical derivatives* (Sonin [3], Riemann-Liouville integral (Liouville [4]), Gerasimov [5] and Caputo ([6,7]), Dzhrbashyan-Nersesyan [8])

(ii) *Modified derivatives* (Weyl [9], ψ-Caputo [10], Caputo-Hadamard [11], Caputo type fractional derivative [12], Hilfer-Katugampola [13], (k,ρ)-fractional [14], ψ-Hilfer [15], and ψ-Riemann-Liouville [16])

(iii) *Local operators* (Katugampola [17], Khalil et al. [18], Zhao et al. [19], Elnady et al. [1])

(iv) *Operators with non-singular kernel* (Caputo-Fabrizio ([20,21]), Losada-Nieto [22], Atangana [23], Atangana-Baleanu [24]).

Fractional differential equations FrDEs finds applications in different areas of research in real life [16,25]. Recently Wang [26] studied the dynamics of nonlinear singular heat conduction model of the human head, and presented a new fractional heat conduction model for the human head taking into account the effect of febrifuge. Wang used He's fractal derivative [27,28] which uses fractal geometry. Under the given circumstances in real-life examples exact determination of fractal geometries is very complex.

A unified and generalized fractional derivative definition [1] is presented very recently which takes care of problems reported (see [29,30]) in other existing local derivatives (see [17–19] etc). In this paper, we propose a simple model and a simple method easy to apply for a large class of similar problems. Since the method is very simple this can be used to check the credibility of solutions by other methods. For this, we utilize the unified and generalized fractional derivative definition [1] and propose a new class of unified and generalized fractional singular nonlinear differential equations and solve it by a hybrid Taylor series method. This definition is as simple as the classical one. The results derived in this line of thought are encouraging and in the limiting cases, the results match with classical singular nonlinear differential equations. The results of this paper are motivated and also generalize the results obtained by Chang [31] and He et al. [32,33].

Let $\mathcal{D}^{\alpha}$, $\alpha \in (1/2, 1]$ be a generalized local fractional derivative as defined in [1]. Also, let , $\kappa_i = k_i(1-\alpha),\ i = 0,1,2$, $\tau = 2-2\alpha$. Now, we propose the following class of generalized local fractional singular nonlinear differential equations.

Generalized Local Fractional Nonlinear Singular BVPs

$$\begin{aligned}&\mathcal{D}^{(2\alpha)}y(x)+\frac{k_0}{x^\alpha}\mathcal{D}^{(\alpha)}y(x)+F(x,y,y')=0,\quad 0<x<1,\\&y'(0)=0,\ \beta_1 y(1)+\beta_2 y'(1)=d,\end{aligned}\tag{1}$$

where $F(x,y,y')=g(x,y(x))-\left(\frac{\kappa_0}{x^\alpha}\right)y(x)-\tau y'(x)$.

Generalized Local Fractional System of Nonlinear Singular IVPs

$$\begin{aligned}&\mathcal{D}^{(2\alpha)}u+\frac{k_1}{x^\alpha}\mathcal{D}^{(\alpha)}u+F_1(x,u,v,u',v')=0,\quad 0<x<1,\quad u(0)=1,\ u'(0)=0,\\&\mathcal{D}^{(2\alpha)}v+\frac{k_2}{x^\alpha}\mathcal{D}^{(\alpha)}v+F_2(x,u,v,u',v')=0,\quad 0<x<1,\quad v(0)=1,\ v'(0)=0,\end{aligned}\tag{2}$$

Generalized Local Fractional System of Nonlinear Singular BVPs

$$\begin{aligned}&\mathcal{D}^{(2\alpha)}u+\frac{k_1}{x^\alpha}\mathcal{D}^{(\alpha)}u+F_1(x,u,v,u',v')=0,\quad 0<x<1,\quad u'(0)=0,\ u(1)=a_1,\\&\mathcal{D}^{(2\alpha)}v+\frac{k_2}{x^\alpha}\mathcal{D}^{(\alpha)}v+F_2(x,u,v,u',v')=0,\quad 0<x<1,\quad v'(0)=0,\ v(1)=b_1.\end{aligned}\tag{3}$$

where $F_1(x,u,v,u',v')=f_1(x,u,v)-\left(\frac{\kappa_1}{x^\alpha}\right)u(x)-\tau u'(x)$, $F_2(x,u,v,u',v')=f_2(x,u,v)-\left(\frac{\kappa_2}{x^\alpha}\right)v(x)-\tau v'(x)$.

Here k_0,k_1,k_2 are parameters indicating the dimension of the object (e.g. Human head), and it is α-singular nonlinear differential equations $x^{-\alpha}$ shoots to infinity when $x\to 0$. F,F_1,F_2 are nonlinear source terms and are sufficiently smooth. When $\alpha\to 1$, these generalised local fractional differential equations reduce to classical nonlinear singular differential equations popularly known as Lane-Emden equations (see [34] and Remark 4).

The workflow of the paper is as follows: In Section 2. we discuss some existing results which helped to arrive at the ideas of the current work. In Section 3. we describe the proposed Hybrid Taylor Series Method. In Section 4. we discuss the numerical test cases and in Section 5. we discusse convergence of the method. The paper ends with the conclusion.

2. Preliminary

In this section, we discuss some important results obtained by earlier researchers, which are crucial for the developments achieved in this paper.

2.1 Some Definitions

In this section we define unified generalised local fractional derivative ([1]) for $\alpha>1$ which generalises and corrects some earlier existing definitions (see [18,19,35]). The new unified and generalized local fractional derivative $\mathcal{D}^{(\alpha)}$ ([1]) is defined as follows:

Definition 1. *[1] Given a function $f:[0,\infty)\to\mathbb{R}$, the new unified and generalized local fractional derivative of f of order α, denoted by $\mathcal{D}^{(\alpha)}$ is defined by*

$$\mathcal{D}^{(\alpha)}f(t)=\lim_{\lambda\to 0}\frac{(1+\lambda\mathcal{A}(\alpha))f(t+\lambda\psi(t,\alpha))-f(t)}{\lambda},\tag{4}$$

where α is the order of the derivative, $\mathcal{A}(\alpha)$ is a function of α, $\psi(t,\alpha)$ is a function of t and α and λ is a real positive number.

Remark 1. *When $\mathcal{A}=0$, $\psi(t,\alpha)=t^{1-\alpha}$, it coincides with Khalil's definition [18].*

Remark 2. *From Definition 1, it is easy to deduce that*

$$\mathcal{D}^{(\alpha)}(f)(t)=\mathcal{A}(\alpha)f(t)+\psi(t,\alpha)\frac{df}{dt}\tag{5}$$

(see [1, Theorem 1]).

If we consider $A(\alpha)=1-\alpha$ when $\alpha\in[0,1]$ and $\psi(t,\alpha)=\chi_{\mathbb{R}^+}(\alpha)t^{1-\alpha}$, where $\chi_{\mathbb{R}^+}(\alpha)$ is Characteristic function defined on $\mathbb{R}^+=(0,\infty)$. Hence when $\alpha\in[0,1]$ we arrive at

$$\mathcal{D}^{(\alpha)}(f)(t)=(1-\alpha)f(t)+\chi_{\mathbb{R}^+}(\alpha)t^{1-\alpha}\frac{df}{dt}.\tag{6}$$

In [1] authors gave definition for the case when $\alpha\in(0,1)$. Here, we define the unified generalised local fractional derivative when $\alpha>1$ based on the idea presented in recent work [1,18].

Definition 2. *Let $\alpha\in(n,n+1],\ n\in\mathbb{N}$ and f be an $n-$differentiable at t, where $t>0$. Then the unified genrealized fractional derivative of f of order α is defined as,*

$$\mathcal{D}^{(\alpha)}f(t)=\lim_{\lambda\to 0}\frac{(1+\lambda\mathcal{A}(\alpha))f^{(\lceil\alpha\rceil-1)}(t+\lambda\psi(t,\alpha))-f^{(\lceil\alpha\rceil-1)}(t)}{\lambda}.\tag{7}$$

Remark 3. *From Definition 2 it is easy to deduce that*

$$\mathcal{D}^{(\alpha)}(f)(t)=\mathcal{A}(\alpha)f^{(\lceil\alpha\rceil-1)}(t)+\psi(t,\alpha)f^{(\lceil\alpha\rceil)}(t),\tag{8}$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

If we consider $A(\alpha)=\lceil\alpha\rceil-\alpha$ and $\psi(t,\alpha)=\chi_{\mathbb{R}^+}(\alpha)t^{\lceil\alpha\rceil-\alpha}$, then

$$\mathcal{D}^{(\alpha)}(f)(t)=(\lceil\alpha\rceil-\alpha)f^{(\lceil\alpha\rceil-1)}(t)+\chi_{\mathbb{R}^+}(\alpha)t^{\lceil\alpha\rceil-\alpha}f^{(\lceil\alpha\rceil)}(t).\tag{9}$$

Remark 4. *If α tends to 0 then $\mathcal{D}^{(0)}f(t)=f(t)$, if α tends to 1, $\mathcal{D}^{(1)}f(t)=f'(t)$ and if α tends to 2, $\mathcal{D}^{(2)}f(t)=f''(t)$, in general when $\alpha\to n$ then $\mathcal{D}^{(n)}f(t)=f^{(n)}(t)$. Hence, this definition takes care of the drawbacks reported about Khalil's derivative [18] in papers (see [29,30]).*

3. Hybrid Taylor Series Method

In this section, we discuss the general approach to solve three classes of generalized local fractional nonlinear differential equations defined in section 1.

3.1 Description

Consider the unified generalised local fractional nonlinear α-singular BVPs

$$\mathcal{D}^{(2\alpha)}y(x)+\frac{k}{x^{\alpha}}\mathcal{D}^{(\alpha)}y(x)+F(x,y,y')=0,\quad 0<x<1,\quad y'(0)=0,\ \beta_1 y(1)+\beta_2 y'(1)=d. \tag{10}$$

Let $x^{\alpha}=\xi$, then $\frac{d\xi}{dx}=\alpha x^{\alpha-1}$. By using Remark 2 and 3, we have

$$\mathcal{D}^{(\alpha)}y=(1-\alpha)y+x^{1-\alpha}\frac{dy}{dx}=(1-\alpha)y+\alpha\frac{dy}{d\xi}, \tag{11}$$

and

$$\mathcal{D}^{(2\alpha)}y=(2-2\alpha)y'(x)+x^{2-2\alpha}y''(x)=(2-2\alpha)\alpha x^{\alpha-1}\frac{dy}{d\xi}+\alpha^2\frac{d^2y}{d\xi^2}+\frac{\alpha^2-\alpha}{\xi}\frac{dy}{d\xi}. \tag{12}$$

Substituting (11) and (12) in (10) we arrive at

$$\alpha^2\frac{d^2y}{d\xi^2}+\left(\frac{\alpha^2+(k-1)\alpha}{\xi}\right)\frac{dy}{d\xi}+g(x,y)=0. \tag{13}$$

Assume that $y(0)=C$. Now using the boundary condition $y'(0)=0$ and taking limit in equation (13) and $\lim_{\xi\to 0}\xi g(\xi,y(\xi))=0$ we get that

$$y''(0)=-\frac{g(0,y(0))}{2\alpha^2+(k-1)\alpha}. \tag{14}$$

Similarly, differentiating (13) repeatedly we arrive at the following:

$$y'''(0)=-\frac{2\frac{d}{d\xi}g(\xi,y(\xi))|_{\xi=0}}{3\alpha^2+(k-1)\alpha},\quad y^{(iv)}(0)=-\frac{3\frac{d^2}{d\xi^2}g(\xi,y(\xi))|_{\xi=0}}{4\alpha^2+(k-1)\alpha},\cdots,y^{(m)}(0)=-\frac{(m-1)\frac{d^{(m-2)}}{d\xi^{(m-2)}}g(\xi,y(\xi))|_{\xi=0}}{m\alpha^2+(k-1)\alpha}. \tag{15}$$

The solution y represent by $y(\xi,\alpha,C)$ is given by Taylor series around $\xi=0$ as follows:

$$y(\xi,\alpha,C)=y(0)+\frac{y'(0)}{1!}\xi+\frac{y''(0)}{2!}\xi+\cdots+\frac{y^{(m)}(0)}{m!}\xi^m+\cdots.$$

Now using $\xi=x^{\alpha}$, $y(0)=C$, $y'(0)=0$ and values of the other derivatives of y with respect to ξ at $\xi=0$, we arrive at $y(x,\alpha,C)$ in terms of Taylor series. Now, using the boundary condition at the other end $x=1$, we arrive at a nonlinear equation in C which we solve to get the value of C by using Mathematica Command "FindRoot" in terms of α. By putting the value of C in $y(x,\alpha,C)$, we compute the solution of the proposed generalized local fractional nonlinear singular BVP (10).

Remark 5. *This method can easily of extended to a system of generalized local fractional nonlinear IVPs and BVPs. For brevity, we skip the details for the reader. We shall provide an algorithm in each case, though.*

4. Numerical Experiments

In this section we test the proposed method on the newly proposed system of Generalized Local Fractional problems. All the computations are preformed using Mathematica.

4.1 Initial Value Problems (IVPs)

In this case, we shall discuss the solution of the generalized local fractional system of IVPs.

4.1.1 Algorithm for Generalized Local Fractional (GLF) IVPs (GLF-IVPs)

Step 1. Use $\xi=x^{\alpha}$, Remark 2, Remark 3 and transform the fractional system of IVPs into classical IVPs with ordinary derivatives.

Step 2. Differentiate the differential equation obtained in Step 1 and use ICs to compute $u^{(m)}(0)$, $v^{(m)}(0)$, $m=1,2,3,\cdots$.

Step 3. Substitute $u^{(m)}(0)$, $v^{(m)}(0)$ in Taylor Series of u and v around 0 to arrive at the solution of generalized local fractional system of IVPs.

Here we consider three set of system of Generalized Local Fractional IVPs to test our proposed methodology.

4.1.2 IVPs

GLF-IVP 1 Let us consider the following class of IVPs

$$
\begin{aligned}
&\mathcal{D}^{(2\alpha)}u + \frac{1}{x^\alpha}\mathcal{D}^{(\alpha)}u - v^3(u^2+1) - \frac{1-\alpha}{x^\alpha}u - \tau u' = 0, \quad u(0)=1,\ u'(0)=0,\\
&\mathcal{D}^{(2\alpha)}v + \frac{3}{x^\alpha}\mathcal{D}^{(\alpha)}v + v^5(u^2+3) - \frac{3-3\alpha}{x^\alpha}v - \tau v' = 0, \quad v(0)=1,\ v'(0)=0.
\end{aligned} \tag{16}
$$

Following the algorithm 4.1.1, we deduce that

$$
u(x) = 1 + \frac{x^{2\alpha}}{2\alpha^2} - \frac{(5\alpha-1)x^{4\alpha}}{16\alpha^4(\alpha+1)} + \frac{\left(52\alpha^3+27\alpha^2-8\alpha+1\right)x^{6\alpha}}{96\alpha^6(\alpha+1)^2(2\alpha+1)}, \tag{17}
$$

$$
v(x) = 1 - \frac{x^{2\alpha}}{\alpha(\alpha+1)} + \frac{(19\alpha-1)x^{4\alpha}}{8\alpha^3\left(2\alpha^2+3\alpha+1\right)} - \frac{\left(934\alpha^3+557\alpha^2-54\alpha+3\right)x^{6\alpha}}{96\alpha^5(\alpha+1)^2\left(6\alpha^2+5\alpha+1\right)}. \tag{18}
$$

If we take $\alpha \to 1$ in equation (17)-(18) we observe that $u \to \sqrt{1+x^2}$ and $v \to \frac{1}{\sqrt{1+x^2}}$.

GLF-IVP 2 Let us consider the following class of IVPs

$$
\begin{aligned}
&\mathcal{D}^{(2\alpha)}u + \frac{5}{x^\alpha}\mathcal{D}^{(\alpha)}u + 8\Big(e^{(u-1)} + 2e^{-(v/2)^{-(1/2)}}\Big) - \frac{5-5\alpha}{x^\alpha}u - \tau u' = 0, \quad u(0)=1,\ u'(0)=0,\\
&\mathcal{D}^{(2\alpha)}v + \frac{3}{x^\alpha}\mathcal{D}^{(\alpha)}v - 8\Big(e^{-(v-1)} + e^{(u/2)^{(-1/2)}}\Big) - \frac{3-3\alpha}{x^\alpha}v - \tau v' = 0, \quad v(0)=1,\ v'(0)=0.
\end{aligned} \tag{19}
$$

Following the algorithm 4.1.1 we deduce that

$$
u(x) = 1 - \frac{6x^{2\alpha}}{\alpha(\alpha+2)} + \frac{(5\alpha+7)x^{4\alpha}}{\alpha^2(\alpha+1)^2(\alpha+2)} - \frac{2\left(61\alpha^3+197\alpha^2+212\alpha+70\right)x^{6\alpha}}{3\alpha^3\left(\alpha^2+3\alpha+2\right)^2\left(6\alpha^2+7\alpha+2\right)} \tag{20}
$$

$$
v(x) = 1 + \frac{4x^{2\alpha}}{\alpha(\alpha+1)} - \frac{(7\alpha+11)x^{4\alpha}}{\alpha^2\left(2\alpha^3+7\alpha^2+7\alpha+2\right)} + \frac{\left(74\alpha^3+292\alpha^2+367\alpha+131\right)x^{6\alpha}}{3\alpha^3\left(\alpha^2+3\alpha+2\right)^2\left(6\alpha^2+5\alpha+1\right)}. \tag{21}
$$

If we take $\alpha \to 1$ in equation (20)-(21) we arrive at $u \to 1 + 2\log(1+x^2)$ and $v \to 1 - 2\log(1+x^2)$.

GLF-IVP 3 Let us consider the following class of IVPs

$$
\begin{aligned}
&\mathcal{D}^{2\alpha}u + \frac{8}{x^\alpha}\mathcal{D}^{(\alpha)}u + \Big(18u - 4u\ln v\Big) - \frac{8-8\alpha}{x^\alpha}u - \tau u' = 0, \quad u(0)=1,\ u'(0)=0,\\
&\mathcal{D}^{2\alpha}v + \frac{4}{x^\alpha}\mathcal{D}^{(\alpha)}u + \Big(4v\ln u - 10v\Big) - \frac{4-4\alpha}{x^\alpha}v - \tau v' = 0, \quad v(0)=1,\ v'(0)=0.
\end{aligned} \tag{22}
$$

Following the algorithm 4.1.1 we deduce that

$$
u(x) = 1 - \frac{9x^{2\alpha}}{\alpha(2\alpha+7)} + \frac{(182\alpha+313)x^{4\alpha}}{2\alpha^2\left(16\alpha^3+108\alpha^2+224\alpha+147\right)} - \frac{\left(19088\alpha^3+76524\alpha^2+95188\alpha+34425\right)x^{6\alpha}}{6\alpha^3(2\alpha+3)^2\left(192\alpha^4+1376\alpha^3+3276\alpha^2+3136\alpha+1029\right)}, \tag{23}
$$

$$
v(x) = 1 + \frac{5x^{2\alpha}}{\alpha(2\alpha+3)} + \frac{(86\alpha+229)x^{4\alpha}}{2\alpha^2\left(16\alpha^3+92\alpha^2+144\alpha+63\right)} + \frac{\left(11472\alpha^3+65132\alpha^2+124436\alpha+79625\right)x^{6\alpha}}{18\alpha^3(2\alpha+7)^2\left(64\alpha^4+288\alpha^3+452\alpha^2+288\alpha+63\right)}. \tag{24}
$$

If we take $\alpha \to 1$ in equation (23)-(24) we arrive at $u \to \exp(-x^2)$ and $v \to \exp(x^2)$.

4.2 Boundary Value Problems (BVPs)

Here we consider the BVPs defined by (1).

4.2.1 Algorithm for Generalized Local Fractional BVPs (GLF-BVPs)

Step 1. Use $\xi = x^\alpha$, Remark 2, Remark 3, and transform the fractional BVP into a classical BVP with ordinary derivatives.

Step 2. Assume $y(0) = C$.

Step 3. Differentiate the differential equation obtained in Step 1 and use the $y'(0) = 0$ to compute $y^{(m)}(0)$, $m = 1, 2, 3, \cdots$.

Step 4. Substitute $y^{(m)}(0)$ in the Taylor series of y around 0 and arrive at $y(x, \alpha, C)$.

Step 5. Use the BC at $x = 1$ and solve the resulting nonlinear equation in C & α to get a value of C in terms of α.

Step 6. Substitute the value of C in $y(x, \alpha, C)$ we arrive at the solution of the Generalized Local Fractional BVPs in terms of α.

Now we solve some generalised local fractional nonlinear singular BVPs to test our proposed methodology.

GLF-BVP 1 Consider the following BVP

$$
\mathcal{D}^{(2\alpha)}y(x) + \frac{2}{x^\alpha}\mathcal{D}^{(\alpha)}y(x) - \frac{2-2\alpha}{x^\alpha}y(x) - \tau y'(x) + e^{-y(x)} = 0, \quad 0 < x < 1, \quad y'(0)=0,\ 2y(1)+y'(1)=0. \tag{25}
$$

Following the algorithm 4.2.1 we deduce that (for $\alpha = 0.75$)

$$
y(x) = -5.645518806299298 \times 10^{-6}x^{7.5} - 0.000800538x^{4.5} - 0.186653x^{1.5} - 0.0000652698x^{6.} - 0.0108873x^{3.} + 0.35675. \tag{26}
$$

If we take $\alpha \to 1$ in equation (25) the solutions lie in close proximity to the solution of the corresponding classical case.
GLF-BVP 2 Consider the following BVP

$$\mathcal{D}^{2\alpha}y(x) + \frac{2}{x^\alpha}\mathcal{D}^{(\alpha)} - \frac{2-2\alpha}{x^\alpha}y(x) - \tau y'(x) + y^5(x) = 0, \quad 0 < x < 1, \quad y'(0) = 0,\ y(1) = \frac{\sqrt{3}}{2}. \tag{27}$$

Following the algorithm 4.2.1 for $\alpha = 0.75$ we deduce that

$$y(x) = -0.109236x^{7.5} - 0.198085x^{4.5} - 0.448779x^{1.5} + 0.0837488x^{9.} + 0.145081x^{6.} + 0.283577x^{3.} + 1.10972. \tag{28}$$

If we take $\alpha \to 1$ in equation (27) the solutions lie in close proximity to the solution of the corresponding classical case.
GLF-BVP 3 Consider the following BVP

$$\mathcal{D}^{2\alpha}y(x) + \frac{3}{x^\alpha}\mathcal{D}^{(\alpha)}y(x) - \frac{3-3\alpha}{x^\alpha}y(x) - \tau y'(x) + \Big(-\frac{1}{2} + \frac{1}{8y^2}\Big) = 0, \quad 0 < x < 1, \quad y'(0) = 0.\ y(1) = 0. \tag{29}$$

Following the algorithm 4.2.1 for $\alpha = 0.75$ we deduce that

$$y(x) = -0.022643x^{7.5} - 0.0415276x^{4.5} - 0.137581x^{1.5} - 0.0187892x^{9.} - 0.0291363x^{6.} - 0.0701144x^{3.} + 0.319791. \tag{30}$$

If we take $\alpha \to 1$ in equation (29), the solutions lie in close proximity of the solution of the corresponding classical case.

4.3 System of Generalized Local Fractional BVPs

Here we focus on generalized local fractional System of nonlinear singular BVPs (3).

4.3.1 Algorithm for Generalized Local Fractional System of BVPs (GLF-SBVP)

Step 1. Use $\xi = x^\alpha$, Remark 2, Remark 3 and transform the fractional conformable BVP into a classical BVP with ordinary derivatives.

Step 2. Assume $u(0) = C_1$, $v(0) = C_2$.

Step 3. Differentiate the differential equation obtained in Step 1 and use $u'(0) = 0 = v'(0)$ to compute $u^{(m)}(0), v^{(m)}(0), m = 1, 2, 3, \cdots$.

Step 4. Substitute $u^{(m)}(0), v^{(m)}(0), m = 1, 2, 3, \cdots$ in the Taylor series of u and v around 0 and arrive at $u(x, \alpha, C_1, C_2)$ and $v(x, \alpha, C_1, C_2)$.

Step 5. Use the BCs at $x = 1$ and solve the resulting two nonlinear equations in $C_1,\ C_2$ & α to get a value of C_1 and C_2 in terms of α.

Step 6. Substituting the value of C_1, C_2, in $u(x, \alpha, C_1, C_2)$ and $v(x, \alpha, C_1, C_2)$ we arrive at the solutions u, v of the Generalized Local Fractional system of BVPs in terms of α.

Now we solve some systems of generalized local fractional BVPs to test our proposed methodology.

4.3.2 GLF-SBVP 1

Let us consider the following GLF-SBVPs for $\alpha = 0.75$.

$$\begin{aligned} \mathcal{D}^{(2\alpha)}u + \frac{8}{x^\alpha}\mathcal{D}^{(\alpha)}u - \frac{8-8\alpha}{x^\alpha}u - \tau u' + \Big(18u - 4u\ln v\Big) = 0, \quad u'(0) = 0,\ u(1) = \frac{1}{e}, \\ \mathcal{D}^{(2\alpha)}v + \frac{4}{x^\alpha}\mathcal{D}^{(\alpha)}v - \frac{4-4\alpha}{x^\alpha}v - \tau v' + \Big(4v\ln u - 10v\Big) = 0, \quad v'(0) = 0,\ v(1) = e. \end{aligned} \tag{31}$$

Following the algorithm 4.3.1 we deduce that

$$u(x) = -0.0775763x^{7.5} - 0.898171x^{4.5} - 2.49883x^{1.5} + 0.013308x^{9.} + 0.309173x^{6.} + 1.86087x^{3.} + 1.65911, \tag{32}$$

$$v(x) = 0.0353249x^{7.5} + 0.307319x^{4.5} + 0.874563x^{1.5} + 0.00903699x^{9.} + 0.116573x^{6.} + 0.635228x^{3.} + 0.740237. \tag{33}$$

If we take $\alpha \to 1$ in equation (31) the solution moves closer towards $u \to \exp(-x^2)$ and $\to \exp(x^2)$.

4.3.3 GLF-SBVP 2

Let us consider the following GLF-SBVPs for $\alpha = 0.99$.

$$\begin{aligned} \mathcal{D}^{(2\alpha)}u + \frac{5}{x^\alpha}\mathcal{D}^{(\alpha)}u - \frac{5-5\alpha}{x^\alpha}u - \tau u' + 8\Big(e^{(u-1)} + 2e^{-(v-1)/2}\Big) = 0, \quad u'(0) = 0,\ u(1) = 1 - 2\ln 2, \\ \mathcal{D}^{(2\alpha)}v + \frac{3}{x^\alpha}\mathcal{D}^{(\alpha)}v - \frac{3-3\alpha}{x^\alpha}v - \tau v' - 8\Big(e^{-(v-1)} + e^{(u-1)/2}\Big) = 0, \quad v'(0) = 0,\ v(1) = 1 + 2\ln 2. \end{aligned} \tag{34}$$

Following the algorithm 4.3.1 we deduce that

$$u(x) = -2.21396x^{1.5} + 1.24855x^{3.} + 0.579118, \tag{35}$$
$$v(x) = 2.24266x^{1.5} - 1.26958x^{3.} + 1.41322. \tag{36}$$

If we take $\alpha \to 1$ in equation (34) the solution moves towards the solution of the corresponding classical system of BVP.

5. Remark on Convergence

Let $Y = [u, v]^T$ be solution of the coupled system of differential equations where u, v are real valued sufficiently smooth functions on $[0, 1]$. In case of scalar differential equation $Y = y$. Let $Y(\xi) = \sum_{m=0}^{M} \frac{Y^{(m)}(0)}{m!}\xi^m + R_M$ where $Y = [u, v]^T$ and $Y^{(m)}(0) = [u^{(m)}(0), v^{(m)}(0)]^T$ and $\xi = x^\alpha$. R_M is defined as

$$R_M = [R_M^u, R_M^v]^T$$

such that $R_M^u = \frac{u^{(M+1)}(\bar{\xi}_1)}{(M+1)!}\xi^{M+1}$, $R_M^v = \frac{v^{(M+1)}(\bar{\xi}_2)}{(M+1)!}\xi^{M+1}$, $\bar{\xi}_1, \bar{\xi}_2 \in (0, 1)$. If $x \in [0, 1]$ then $\xi \in [0, 1]$ and $|R_M^u| \le \frac{K_u}{(M+1)!}$ and $|R_M^v| \le \frac{K_v}{(M+1)!}$, where K_u and K_v are some non-negative constants. Then $\|R_M\|_\infty \le \frac{\max\{K_u, K_v\}}{(M+1)!}$. Thus if $Y^{(m)}$ is bounded in $[0, 1]$ then R_M can be made as small as we please by sufficiently increasing the value of M.

6. Conclusion

In this work, we extend the idea of work in [1] and defined unified generalized local fractional derivative definition for $\alpha > 1$ where α is the order of the local fractional derivative. We have also proposed a new class of generalized local fractional order singular nonlinear problems and solved them based on generalized local fractional derivative [1]. Several test examples validate the problem as well as the methodology.

Author Contributions

A.K. Verma initiated the work along with S. Kayenat. L. Verma and L.N. Kannaujiya worked on the test examples in consultation with A.K. Verma, S. Kayenat and R.P. Agarwal. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

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Data Availability Statements

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